AN EXAMPLE OF 2-DIMENSIONAL QUASI-HOMEOMORPHIC SETS

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1. Introduction. We shall consider only compacta, i.e. compact metric spaces. As defined by Mardešić and Segal in [7], a compactum \( X \) is like to a compactum \( Y \) — or, simply, \( Y \)-like — if for every \( \varepsilon > 0 \) there is an \( \varepsilon \)-mapping of \( X \) onto \( Y \). A map \( f: X \to Y \) is said to be an \( \varepsilon \)-mapping if \( \text{diam} f^{-1}(y) < \varepsilon \) for every \( y \in f(X) \). Two compacta \( X \) and \( Y \) are quasi-homeomorphic if \( X \) is \( Y \)-like and \( Y \) is \( X \)-like (this notion was given by Kuratowski and Ulam in 1933 in the paper [6]).

In [3] K. Borsuk found a famous example of a 3-dimensional acyclic locally connected continuum, which admits a homeomorphism onto itself without any fixed point. He next used this example in the paper [4] to answer in the negative the question of Kuratowski and Ulam whether the fixed point property is a quasi-homeomorphism invariant. Namely, he proved that his example mentioned above is quasi-homeomorphic with the 3-cell. In the same paper it is proved that the fixed point-property is a quasi-homeomorphism invariant in the class of compact ANR's (and therefore the counter-example cannot be an ANR). In [1] (cf. also [2]) R. H. Bing has found a 2-dimensional version of the Borsuk's example [3], i.e. an example of a 2-dimensional acyclic locally connected continuum being the intersection of a sequence of 3-cells, which admits a homeomorphism onto itself without any fixed point. It is easy to see (cf. section 2), that the Bing's example is \( Z \)-like, where \( Z \) is a 2-dimensional AR-set obtained as the union of two disks \( D_1 \) and \( D_2 \) stucked together in such a way that some arcs \( I_1 \subset \hat{D}_1 \) and \( I_2 \subset \hat{D}_2 \) are identified by a homeomorphism. (Here, as usual, \( \hat{D} \) denotes the interior of the disk \( D \).) However, it can probably be proved that the Bing's example is not quasi-homeomorphic with any ANR (which, of course, must be 2-dimensional). By modifying the Bing's example, we prove in this note that there is another 2-dimensional acyclic locally connected continuum without the fixed point-property, which is quasi-homeomorphic with a 2-dimensional AR-set.

It is easy to see that a 1-dimensional version of the Borsuk's example [3] cannot exist, because any compactum quasi-homeomorphic with a local dendron is a local dendron. It has been proved by Lê Xuân Biên [5] that a
compactum is quasi-homeomorphic with a disk iff it is a 2-dimensional AR-set embeddable into the plane $E^2$, and therefore there cannot exist an example of the considered sort, which is quasi-homeomorphic with a disk. However, it is not known if there is such an example, which is quasi-homeomorphic with a 2-dimensional polyhedron. The above-mentioned result of Lê Xuân Binh suggests that the answer to this question is probably negative. On the other hand, since our example does not admit a homeomorphism onto itself without fixed points, it is neither known if there is a compactum quasi-homeomorphic with a 2-dimensional AR-set, which admits such a homeomorphism\(^{(1)}\).

2. The example. First, let us describe as in [2] the Bing's example mentioned above. It will be called $X_0$. Consider first a cylinder given by the equation $x^2 + y^2 = 4$ with two bases $B^-$ and $B^+$ lying respectively on the planes $z = -\frac{1}{2}$ and $z = \frac{1}{2}$. Next, form two cones $C^-$ and $C^+$, whose bases respectively are $B^-$ and $B^+$ and whose vertices respectively are the points $v^+ = (2, 0, \frac{1}{2})$ and $v^- = (-2, 0, -\frac{1}{2})$. Then, we construct the set $X_0$ from the set $C^- \cup C^+$ by the rotation of the intersection of this set with any horizontal plane $z = z_0$, where $-\frac{1}{2} < z_0 < \frac{1}{2}$, about the $z$-axis by the angle $\tan \pi z_0$. Thus $X_0 = \tilde{C}^+ \cup \tilde{C}^-$, where $\tilde{C}^+$ ($\tilde{C}^-$) is the image of $C^+ \setminus \{v^+\}$ ($C^- \setminus \{v^-\}$) under this function of $C^+ \cup C^-$ onto $X_0$ (the function is the identity on $B^+ \cup B^-$). For each $\theta$, where $0 < \theta < 2\pi$, there is a homeomorphism $h_\theta$ of the set $X_0$ onto itself without any fixed point. This homeomorphism $h_\theta$ rotates the bases $B^+$ and $B^-$ about the $z$-axis by the angle $\theta$ and on the intersections of the set $X_0$ with other horizontal planes it acts by the same rotation, suitable lift and stretch (i.e., contracting of one circle and expanding of the other so that they agree with the lifted ones). Under this homeomorphism the tangent spiral $S$ of the rotated cones $\tilde{C}^+$ and $\tilde{C}^-$ is moved onto itself by these rotation and lift.

Now, we shall describe the AR-set $Y$ and after that the compactum $X$ without the fixed-point property, which is quasi-homeomorphic with $Y$. The set $Y$ will be the union of AR-sets $Y_n$ $n = 0, 1, 2, \ldots$, constructed as follows:

Let $I$ denote the tangent segment of the cones $C^+$ and $C^-$, i.e. the segment joining the vertices $v^+$ and $v^-$. Find a sequence $I_0, I_1^+, I_1^-, I_2^+, I_2^-$, $I_2^-, \ldots$ of disjoint segments lying on $I$ such that $I_j^+$ and $I_j^-$ are symmetric with respect to the center 0 of $I$, $\text{diam } I_j^+ = \text{diam } I_j^- < 1/j$ for $j \neq 0$, the segments $I_j^+$ ($I_j^-$), $j = 1, 2, \ldots$, converge to $v^+$ ($v^-$). Assume that the endpoints $w^+$ and $w^-$ of $I_0$ are also symmetric with respect to 0 with $w^-$ having the negative $z$-coordinate and that the ordering of the segments $I_j^-$'s on $I$ agrees with the ordering of their indices. Moreover, find another sequence $J_1^+, J_1^-, J_2^+, J_2^-, \ldots$ of segments such that $J_j^+ \subset I_j^+$, $J_j^- \subset I_j^-$, $j = 1, 2, \ldots$

\(^{(1)}\) See T. Maćkowiak, A continuum without the fixed point property which is quasi-homeomorphic with an AR-set, this fascicle, p. 79–80. [Note of the Editors]
(cf. Fig. 1). We shall assume that \( \alpha_n^*([w^+, w^-]) \supset \beta_n^*( \bigcup_{j \leqslant n} I_j^+ \cup I_j^-) \), where \( \alpha_n^* (\beta_n^*) \) is a homothety of \( I \) onto \( I_n^* (J_n^*) \) and \( \varepsilon = \mp \).

![Diagram](image)

**Fig. 1.** (Added in proof: \( J_0 \) should be read \( I_0 \).)

Now, define \( Y_0 \) as the union of two cones \( D^- \) with the base \( B^- \) and the vertex \( w^+ \) and \( D^+ \) with the base \( B^+ \) and the vertex \( w^- \). Notice that \( Y_0 \) is topologically the set \( Z \) such that \( X_0 \) is \( Z \)-like, mentioned in the Introduction.

Next, we shall construct the set \( Y_1 \) as the union \( Y_0 \cup \tilde{Y}_0^+ \cup \tilde{Y}_0^- \), where the sets \( \tilde{Y}_0^+ \) and \( \tilde{Y}_0^- \) are constructed as follows: Construct two homothetic images \( Y_{00} \) and \( Y_0^+ \) of \( Y_0 \) such that \( \text{diam } Y_{00}^+ = \text{diam } I_1^+ \), \( \text{diam } Y_0^+ = \text{diam } J_1^+ \) and let \( L_{00}^+ \), \( L_0^+ \) denote the respective images of the segment \( I \), where \( L_{00}^+ \subset Y_{00}^+ \), \( L_0^+ \subset Y_0^+ \). Then identify the segment \( L_{00}^+ \) with the segment \( I_1^+ \subset Y_0 \) and the segment \( L_0^+ \) with the segment \( J_1^+ \subset Y_0 \) by linear homeomorphisms. Now, the set obtained from the disjoint union \( Y_{00}^+ \cup Y_0^+ \) under these identifications is \( \tilde{Y}_0^+ \). The second set \( \tilde{Y}_0^- \) is constructed in the symmetric manner.

If the set \( Y_n \), \( n \geqslant 1 \), is defined, we define the set \( Y_{n+1} = Y_n \cup \tilde{Y}_n^+ \cup \tilde{Y}_n^- \) in a similar way: Construct a homothetic image \( Y_{0n}^+ \) of \( Y_0 \) such that \( \text{diam } Y_{0n}^+ = \text{diam } I_{n+1}^+ \) and a homothetic image \( Y_n^+ \) of \( Y_n \) such that \( \text{diam } Y_n^+ = \text{diam } J_{n+1}^+ \). Then define \( \tilde{Y}_n^+ \) as the set obtained from the disjoint union \( Y_{0n}^+ \cup Y_n^+ \) under the identifications of the segments \( L_{0n}^+ \) and \( I_{n+1}^+ \subset Y_n \) as well as the segments \( L_n^+ \) and \( J_{n+1}^+ \) by linear homeomorphisms. (Here, \( L_{0n}^+ \subset Y_{0n}^+ \) and \( L_n^+ \subset Y_n^+ \) are the respective homothetic images of the segment \( I \).) The sets \( Y_{0n}^+ \) and \( Y_n^+ \) will be non-distinguished from their images under these identifications. The second set \( \tilde{Y}_n^- \) is constructed in the symmetric way (cf. Fig. 2).

It is obvious that the set \( Y \) being the union of all \( Y_n \)'s, \( n = 0, 1, 2, \ldots \) is an AR-set.

Finally, we construct the above-mentioned compactum \( X \). Consider
the tangent spiral $S$ contained in $X_0$ and construct a sequence $K_1^+, K_1^-$, $K_2^+, K_2^-$, ... of disjoint arcs contained in $S$ with diam $K_i^+ = \text{diam } K_i^- < 1/i$ for $i = 1, 2, \ldots$, which are placed on $S$ similarly as the segments $I_1^+, I_1^-, I_2^+, I_2^-$, ... are placed on $I$. We can assume that the arcs $K_i^+$ ($K_i^-$) converge to a point belonging to the base $B^+ \subset X_0$ ($B^- \subset X_0$). Then consider the disjoint union of $X_0$ and of all sets $\tilde{Y}_n^+$, $\tilde{Y}_n^-$, where $n = 0, 1, 2, \ldots$, constructed previously under the definition of $Y$. The desired set $X$ is defined as the set obtained from this disjoint union by the identifications of the arcs $K_1^+_{n+1}$ and $I_1^+_{n+1}$ as well as $K_1^-_{n+1}$ and $I_1^-_{n+1}$, $n = 0, 1, 2, \ldots$, where $I_1^+_{n+1} \subset \tilde{Y}_n^+$ and $I_1^-_{n+1} \subset \tilde{Y}_n^-$ are defined as above. It is obvious that $X$ is an acyclic locally connected continuum, which is not an ANR-set.

As mentioned at the beginning of this section, for each $\theta$, $0 < \theta < 2\pi$, there is a homeomorphism $h_\theta$ of the set $X_0$ onto itself, which has no fixed point. With the aid of $h_\theta$ we can define a map $f_\theta$ of the set $X$ into itself, which has no fixed point. Namely, we define $f_\theta$ as the composition $h_\theta \circ r$, where $r$ is a retraction of $X$ onto $X_0$.

It remains to prove that the sets $X$ and $Y$ are quasi-homeomorphic.

First, let us prove that $X$ is $Y$-like. Given an $\varepsilon > 0$, we must construct an $\varepsilon$-mapping of $X$ onto $Y$. For this purpose, we first define a space $X_n$ and a map $f_n$ of $X$ onto $X_n$ for any $n = 1, 2, \ldots$. Given an $n$, let $L_n$ be an arc with the end-points $q_n^+$, $q_n^-$ contained in the spiral $S \subset X_0$ such that $K_i^+, K_i^- \subset L_n$ for $i < n$, $K_i^+ \cap L_n = \emptyset = K_i^- \cap L_n$ for $i \geq n$, $q_n^+, q_n^- \in I$ and $q_n^+$ has a positive $z$-coordinate. (This arc can be found if the arcs $K_i^+, K_i^-$ are suitably placed in $S$.) Hence $S = L_n \cup S_n^+ \cup S_n^-$, where $S_n^+$ and $S_n^-$ are the closures of the respective components of $S \setminus L_n$. Recall that $X_0 = \tilde{C}^+ \cup \tilde{C}^-$, where $\tilde{C}^+$ and $\tilde{C}^-$ are (topological) infinite cones without vertices. Divide $\tilde{C}^-$ by the horizontal plane passing through $q_n^+$ into the union of two closed (in
\( \mathcal{C}^- \) sets \( C_n^- \) and \( E_n^- \), where \( C_n^- \cap \mathcal{C}^+ = S_n^- \cup L_n, E_n^- \cap \mathcal{C}^+ = S_n^+ \). Then do the same for \( \mathcal{C}^+ \). Hence \( C_n^- \) and \( C_n^+ \) are closed cylinders, but \( E_n^- \) and \( E_n^+ \) are infinite cones without vertices. Now, consider two geometric cones \( F_n^+ \), \( F_n^- \), whose bases respectively are \( B^+ \) and \( B^- \) and whose vertices respectively are \( q_n^- \) and \( q_n^+ \). Then define \( f_n \) on the set \( X_0 \subset X \) so that \( f_n(X_0) = F_n^+ \cup F_n^- \), \( f_n(C_n^-) = F_n^- \), \( f_n(C_n^+) = F_n^+ \), \( f_n(B^+ \cup B^-) = \text{id} \), \( f_n((C_n^+ \setminus E_n^+ \cup C_n^- \setminus E_n^-)) \) is a homeomorphism and each intersection of \( E_n^+ \) (\( E_n^- \)) with a horizontal plane is mapped by \( f_n \) to a point. Next, we extend this map to a map (denoted also by \( f_n \)) of \( X \) into the Euclidean 3-space \( E^3 \) such that \( f_n \mid \bigcup_{i<n} (\tilde{Y}_i^+ \cup \tilde{Y}_i^-) \) is a homeomorphism, \( f_n(\tilde{Y}_i^+) \) intersects \( F_n^+ \cup F_n^- \) on the set \( f_n(\tilde{Y}_i^+ \cap X_0) \) and the same holds for the set \( f_n(\tilde{Y}_i^-) \), and moreover \( f_n(\tilde{Y}_i^+) = f_n(\tilde{Y}_i^+ \cap X_0) \), \( f_n(\tilde{Y}_i^-) = f_n(\tilde{Y}_i^- \cap X_0) \) for \( i \geq n \). Then define \( X_n \) to be equal to \( f_n(X) \) (cf. Fig. 3).

![Diagram](image-url)  

Fig. 3. The scheme of the set \( X_n \)

Now, given an \( \varepsilon > 0 \), fix \( n \) so large that

(1) \( f_n \) is a \( \frac{1}{8} \varepsilon \)-mapping.

To find an \( \varepsilon \)-mapping of \( X \) onto \( Y \), we shall construct one more space \( X'_n \) and two maps \( g: X_n \rightarrow X'_n, h: X'_n \rightarrow Y \) where \( \rightarrow \rightarrow \) denotes the surjection. The desired \( \varepsilon \)-mapping will be the composition \( h \circ g \circ f_n \).

We shall construct the set \( X'_n \) by attaching some segments to the set \( X_n \). Observe that \( f_n(L_n) = f_n(C_n^+ \cap C_n^-) = F_n^+ \cap F_n^- \) is the segment with the endpoints \( q_n^+ \) and \( q_n^- \) contained in the segment \( I \) joining \( v^+ \) and \( v^- \). Now, divide the segment \( I \) into the union of smaller segments \( I'_1, I'_2, \ldots, I'_{p} \) (successively ordered on \( I \)) such that

(2) \( \text{diam} \left[ f_n^{-1}(I_j) \cap (C_n^+ \cup C_n^-) \right] < \frac{1}{8} \varepsilon \) for each \( j \leq p \).
Next, find a segment $M_j$ going from the center $m_j$ of the segment $I_j$ and disjoint from $X_n$ except for this point. Then $X'_n$ is defined by the formula

\[ X'_n = X_n \cup \bigcup_{j \leq p} M_j. \]

To construct the above-mentioned map $g: X_n \to X'_n$, for any point $m_j$ we find a disk $Q_j \subset F^+_n \cup F^-_n \subset X_n$ such that $Q_j \cap I = \{m_j\}$, that for any point $x \in Q_j \setminus \{m_j\}$ the set $f^{-1}_n(x)$ consists of one point, and so small that

\[ \text{diam} f^{-1}_n(Q_j \setminus \{m_j\}) < \frac{1}{2} \varepsilon. \]

Then define the map $g$ so that it maps homeomorphically the set $X_n \setminus \bigcup_{j \leq p} Q_j$ onto the set $X'_n \setminus \bigcup_{j \leq p} M_j$, $g|I = \text{id}$, $g(Q_j) = \{m_j\}$ and $g(Q_j) = M_j$.

Now, we shall construct the second above-mentioned map $h: X'_n \to Y$. First, observe that for $n > 0$ the set $X_n$ is (naturally) homeomorphic with a subset of the set $\tilde{Y}_n^+ = Y^+_0 \cup Y^+_n \subset Y$ being the union of $Y^+_0$ and of all homothetic images of $\tilde{Y}_i^+ = \tilde{Y}_i^-$ for $i < n$ contained in $Y^+_n$ (under the homothetic transformation of $Y_n$ onto $Y^+_n$). Let $h'$ denote such a linear homeomorphism. The desired map $h$ will be constructed by extending $h'$ onto the set $X'_n$. For this purpose, consider the segment $h'(I)$ and divide the set $Y \setminus h'(X_n)$ into the union of $p$ closed slices by means of $p-1$ horizontal planes passing through the images under $h'$ of the points dividing $I$ into the union $\bigcup_{j \leq p} I'_j$. Obviously the slices are locally connected continua. For any point $h'(m_j)$, where $1 \leq j \leq p$, there is exactly one slice containing this point, which we shall denote by $S_j$. Since $S_j$ is a locally connected continuum, there is a map $h'_j$ of the segment $M_j$ onto $S_j$. Modifying the maps $h'_j$, $j \leq p$, if necessary, we can assume that they agree with the map $h'$, i.e. that $h'(m_j) = h'_j(m_j)$ for $j \leq p$. Then the maps $h'$ and $h'_j$, $j \leq p$, define a map of $X'_n$ onto $Y$, which is the required map $h$.

It remains to verify that the composition $h \circ g \circ f_n$ is an $\varepsilon$-mapping. Consider a point $y \in Y$. If the set $h^{-1}(y)$ consists of more than one point, then $y$ belongs to one slice $S'_j$ or two consecutive slices $S_j$, $S_{j+1}$. Then $h^{-1}(y)$ is contained in $I'_j \cup M_j$ or in the union $I'_j \cup M_j \cup I'_{j+1} \cup M_{j+1}$ for some $j$. By the construction of $g$, in the last case the set $g^{-1} h^{-1}(y)$ is contained in $I'_j \cup Q_j \cup I'_{j+1} \cup Q_{j+1}$. When $h^{-1}(y)$ is not contained in the set $I \cup \bigcup_{j \leq p} M_j$, $g^{-1} h^{-1}(y)$ is a point. Since $f_n$ is a $\frac{1}{2} \varepsilon$-mapping by (1), in this case $\text{diam}(f^{-1}_n g^{-1} h^{-1}(y)) < \frac{1}{2} \varepsilon$. Thus to prove that $\text{diam} f^{-1}_n g^{-1} h^{-1}(y) < \varepsilon$ for each $y \in Y$, it suffices to observe that

\[ \text{diam} f^{-1}_n(I'_j \cup Q_j) < \frac{1}{2} \varepsilon \quad \text{for any } j \leq p. \]
But the set \( f_n^{-1}(I_j \cup Q_j) \) is the union of the following sets:
\( f_n^{-1}(I_j) \cap (C_n^+ \cup C_n^+) \), \( f_n^{-1}(Q_j \setminus \{m_j\}) \) and of some non-degenerate sets of
the form \( f_n^{-1}(x) \), where \( x \in I_j \). Because of (1), (2) and (3), each of these sets
has diameter less than \( \frac{1}{6} \varepsilon \). By the construction of the disks \( Q_j \), \( j \leq p \), and of
the map \( f_n \), the closure of the second set, as well as any one of the sets of the
third kind have a common point with the first set. It follows that the
inequality (4) holds, which completes the proof that the map \( h \circ g \circ f_n \) is an
\( \varepsilon \)-mapping.

Now, we must show that \( Y \) is \( X \)-like, i.e. given an \( \varepsilon > 0 \), we must
construct an \( \varepsilon \)-mapping of \( Y \) onto \( X \). The idea of the construction is the
same as in the previous case, so we shall only sketch it. Denote by \( r_n \) a
retraction of \( Y \) onto \( Y_n \) which maps each of the sets \( \bar{Y}_i^+ \), \( \bar{Y}_i^- \), \( i \geq n \) contained
in \( Y \) onto its common segment with \( Y_0 \subset Y_n \). Fix \( n \) so large that \( r_n \) is a \( \frac{1}{6} \varepsilon \)-
mapping. Divide the segment \( I \subset Y_0 \subset Y_n \) into \( q \) segments \( I''_1, \ldots, I''_q \) such
that \( \text{diam } I''_j < \frac{1}{6} \varepsilon \) for \( j \leq q \) and construct a space \( Y_n' \) by attaching to \( Y_n \) of \( q \)
segments going from the centers of the segments \( I''_j \), \( j \leq q \). Then construct a
\( \frac{1}{6} \varepsilon \)-mapping \( \varphi \) of \( Y_n \) onto \( Y_n' \) in the same way as we previously constructed
the map \( g \) of \( X_n \) onto \( X_n' \). Finally, we construct a map \( \psi \) of \( Y_n' \) onto \( X \)
similarly as we previously constructed the map \( h \) of \( X_n' \) onto \( Y \). Namely,
observe that the set \( Y_n \) is (naturally) homeomorphic with a subset of the set
\( \bar{Y}_n^+ \) (considered as a subset of \( X \)), because \( \bar{Y}_n^+ = Y_0^+ \cup Y_n^+ \), where \( Y_n^+ \) is a
homothetic image of \( Y_n \). Denote this homeomorphism of \( Y_n \) into \( \bar{Y}_n^+ \subset X \) by
\( \psi' \) and extend \( \psi' \) to a map \( \psi \) of \( Y_n' \) onto \( X \) by dividing the rest of \( X \) (i.e. of
the closure of the set \( X \setminus \psi'(Y_n) \)) into \( q \) closed slices and by proceeding as
above. It is clear that the composition \( \psi \circ \varphi \circ r_n \) is the desired \( \varepsilon \)-mapping of
\( Y \) onto \( X \).

This completes the proof of the following theorem:

**Theorem.** There are two quasi-homeomorphic 2-dimensional compacta \( X \)
and \( Y \) such that \( Y \) is an AR-set and \( X \) does not have the fixed-point property.

**References**

64.
[5] Lê Xuân Binh, *Compacts which are quasi-homeomorphic with a disk*, Colloquium Mathe-


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