The difference method for non-linear elliptic differential equations with mixed derivatives

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Abstract. In this paper we consider a difference method for elliptic differential equation \( F(x, u, \nabla u, u_{xx}) = 0 \) with Dirichlet's boundary conditions in an arbitrary bounded domain within \( \mathbb{R}^n \). A convergence theorem is proved and the error estimate is given.

1. Introduction. Let us consider the elliptic differential equation

\[
F(x, u, \nabla u, u_{xx}) = 0,
\]

where \( x = (x^1, \ldots, x^n) \) is a point of the open and bounded subset \( \Omega \) of space \( \mathbb{R}^n \), \( u \) is a function defined in \( \Omega \), \( \nabla u \) is the gradient of \( u \) and \( u_{xx} \) denotes the \( n \times n \) symmetric matrix of second order derivatives with respect to \( x \).

Along with (1.1) we consider the boundary condition

\[
u(x) = \bar{u}(x) \quad \text{for} \quad x \in \partial \Omega,
\]

where \( \bar{u} \) is a function defined on \( \partial \Omega \).

Malec in his paper [2] uses the well-known seven-point scheme where \( F_{w_{ij}} \) are of constant signs (here \( w_{ij} \) is the argument of \( F \) replaced in the equation by \( u_{xi}u_{xj} \)). The constant signs of \( F_{w_{ij}} \) does not allow us to consider e.g. the equation \( u_{xx} + xyu_{xy} + u_{yy} = 0 \) for \( (x, y) \in (0, 1) \times (-1, 1) \).

Fitzke [1] has proposed a nine-nodal point difference scheme (for \( n = 2 \)), however, because of the assumption adopted, the study does not even cover the equation \( u_{xx} + u_{xy} + u_{yy} = 0 \).

In the papers mentioned above only the cubics \( n \)-dimensional and the square nets are allowed.

A difference scheme on an arbitrary rectangular net but without mixed derivatives is considered in [3].

There are no restrictions on the sign of \( F_{w_{ij}} \) in the method exposed by Voigt [4]. Yet, the existence of a certain matrix is assumed there that
is hard to find (particularly for \( n > 2 \)) in the case of the non-linear equation. In the example cited by Voigt, with a Minkowski matrix, also adequate assumptions are fulfilled, warranting the convergence of the method in my paper.

Let me recall Professor Szarski in this place. I participated in his seminar for many years and it was with his discreet support that I worked in the present paper.

2. Assumptions. Let the function \( F \) of arguments \( x \in \Omega, \ z \in \mathbb{R}, \ q = (q_1, \ldots, q_n) \in \mathbb{R}^n, \ w = (w_i) \in \mathbb{R}^{n^2} \) be of class \( C^1 \) with respect to \( z, \ q, \ w \) and satisfy the assumptions

\[
(2.1) \quad F_{w_{ij}}(x, z, q, w) = F_{w_{ji}}(x, z, q, w) \quad (i, j = 1, \ldots, n),
\]

\[
(2.2) \quad F_z(x, z, q, w) \leq -L, \quad L > 0
\]

for all \( x, z, q, w \). Let us assume further that there exists a bounded symmetric matrix \( G(x) = (G^{ij}(x)) \) for \( x \in \Omega \) such that

\[
(2.3) \quad |F_{w_{ij}}(x, z, q, w)| \leq G^{ij}(x) \quad (i, j = 1, \ldots, n; \ i \neq j)
\]

and there is a \( \varepsilon_0 \in (0, 1) \) such that

\[
(2.4) \quad \frac{H}{2} |F_{w_i}(x, z, q, w)| \leq F_{w_{ii}}(x, z, q, w) - \frac{1}{\varepsilon_0} \sum_{j \neq i} G^{ij}(x)
\]

\( (i = 1, \ldots, n) \)

for all \( x, z, q, w \) and \( H > 0 \) sufficiently small.

It follows from (2.3) and (2.4) that \( F_w = (F_{w_{ij}}) \) is diagonally dominant, which ensures the ellipticity of (1.1).

3. Examples.

(a) If \( F(x, z, q, w) = \sum_{i,j=1}^n a_{ij}(x) w_{ij} + b(x, z, q, w_{11}, \ldots, w_{nn}) \), where \( a = (a_{ij}) \) is a symmetric diagonally dominant matrix and \( b_{w_{ii}} \geq 0 \), then we put \( G^{ij}(x) = |a_{ij}(x)| \). Since in this case \( F_{w_{ij}} = a_{ij} \ (i \neq j) \), (2.3) is satisfied. Condition (2.4) remains in the form

\[
\frac{H}{2} |b_{w_i}(x, z, q, w_{11}, \ldots, w_{nn})| \leq a_{ii}(x) - \frac{1}{\varepsilon_0} \sum_{j \neq i} |a_{ij}(x)| \quad (i = 1, \ldots, n)
\]

since \( F_{w_{ii}} = a_{ii} + b_{w_{ii}} \geq a_{ii} \).

(b) If in the general non-linear case there is a symmetric, bounded matrix \( A(x) = (A_{ij}(x)) \) such that

\[
(3.1) \quad F_{w_{ii}}(x, z, q, w) \geq A_{ii}(x), \quad |F_{w_{ij}}(x, z, q, w)| \leq |A_{ij}(x)|
\]

\( (i, j = 1, \ldots, n; \ i \neq j) \)
and

\[(3.2) \quad 0 < g \leq A_{ii}(x) - \frac{1}{\varrho} \sum_{j \neq i} |A_{ij}(x)| \quad (i = 1, \ldots, n); \quad \varrho \in (0, 1],\]

then by putting \(G^{ij}(x) = |A_{ij}(x)|\) it is simple to verify that assumptions (2.3) and (2.4) are fulfilled as soon as \(F_{\varrho_i}\) are bounded (for \(F_{\varrho_i} = 0, g\) might equal 0). If there is a Minkowski matrix which satisfies besides (3.1) other additional assumptions (see [4]), the condition (3.2) is satisfied as well and hence (2.3) and (2.4).

**4. Discretization.** Let \([h^i_{j}]_{i \in \mathbb{Z}}\) \((i = 1, \ldots, n)\) be a sequence of numbers such that

\[(4.1) \quad h = \max \{\sup_{1 \leq i \leq n} h^i_j\} < \infty, \quad \chi = \min \{\inf_{1 \leq i \leq n} h^i_j\} > 0\]

and let us put \(x^i_0 = 0, x^i_{j+1} = x^i_j + h^i_{j+1}, h^i_j = \frac{1}{2}(h^i_{j+1} + h^i_j)\) for \(j \in \mathbb{Z} (i = 1, \ldots, n)\) (cf. Fig. 1).

![Fig. 1. The space intervals \(h^i_j\) (for \(n = 2\)](image)

We denote by \(\mathcal{M}\) the set of multiindices \(m = (m_1, \ldots, m_n)\), where \(m_i \in \mathbb{Z}\) and we put \(x_m = (x^i_{m_1}, \ldots, x^i_{m_n})\). Let the set of nodal points \(x_m\) be denoted by \(\Sigma\).

For \(x_m \in \Sigma\) we put

\[(4.2) \quad F(x_m) = \bigcup_{i,j=1}^{n} \{x \in \mathbb{R}^n: x^i_{m_i-1} \leq x^i_j \leq x^i_{m_i+1}; x^j_{m_j-1} \leq x^j \leq x^j_{m_j+1}; x^k = x^k_{m_k}, k = 1, \ldots, n, k \neq i, j\}.\]
We put $\hat{\Sigma} = \Sigma \cap \Omega$ and

$$H = \max_{1 \leq i \leq n} \{ h_j^i : h_j^i = x_j^i - x_{j-1}^i \}, \text{ where } x_j^i, x_{j-1}^i \text{ are coordinates of nodal points in } \hat{\Sigma} \}, \eqno(4.3)$$

$$X = \min_{1 \leq i \leq n} \{ h_j^i : h_j^i = x_j^i - x_{j-1}^i \}, \text{ where } x_j^i, x_{j-1}^i \text{ are coordinates of nodal points in } \hat{\Sigma} \}.$$

It is obvious that $\chi \leq X \leq H \leq h$.

We denote by $\varrho$ ($0 < \varrho \leq 1$) the number $X/H$. For $\varrho = 1$ we have the square net.

The nodal point $x_m \in \hat{\Sigma}$ is called: (i) internal nodal point when $P(x_m) \in \Omega$; (ii) type I boundary nodal point when $x_m \in \partial \Omega$ and $P(x_m) \notin \Omega$ (there exists then an $x' \in P(x_m) \cap \Phi$, such that the segment $[x_m, x']$ is contained in $\Omega$); (iii) type II boundary nodal point when $x_m \in \partial \Omega$.

We introduce the sets of multiindices

$$\mathcal{M}_W = \{ m \in \mathcal{M} : x_m \text{ is an internal point} \},$$

$$\mathcal{M}_{B_1} = \{ m \in \mathcal{M} : x_m \text{ is a type I boundary point} \},$$

$$\mathcal{M}_{B_2} = \{ m \in \mathcal{M} : x_m \text{ is a type II boundary point} \}$$

and we put $\mathcal{M}_B = \mathcal{M}_{B_1} \cup \mathcal{M}_{B_2}$ and $\mathcal{M} = \mathcal{M}_W \cup \mathcal{M}_B$.

Let us denote for $m \in \mathcal{M}$

$$i(m) = (m_1, \ldots, m_{i-1}, m_i + 1, m_{i+1}, \ldots, m_n) \quad (i = 1, \ldots, n), \eqno(4.4)$$

$$-i(m) = (m_1, \ldots, m_{i-1}, m_i - 1, m_{i+1}, \ldots, m_n).$$

5. Auxiliary lemmas. For $m, \overline{m} \in \mathcal{M}$ let us put

$$d(m, \overline{m}) = \sum_{i=1}^{n} (m_i - \overline{m_i})^2, \quad S_m = \{ \overline{m} \in \mathcal{M} : 0 < d(m, \overline{m}) \leq 2 \}.$$

If $m \in \mathcal{M}_W$, then $S_m \subset \mathcal{M}$.  

LEMMA 1. Let us assume

$$\beta : \mathcal{M}_W \to \mathbb{R}, \quad \eta : \mathcal{M}_W \to \mathbb{R}, \quad \varepsilon : \mathcal{M}_B \to \mathbb{R}, \quad y : \mathcal{M} \to \mathbb{R},$$

$$a^m : S_m \to \mathbb{R} \quad \text{for } m \in \mathcal{M}_W, \quad \|y\| = \max \{|y(m)| : m \in \mathcal{M} \}.$$

If

$$a^m \geq 0 \quad \text{for } m \in \mathcal{M}_W, \quad \beta \geq \beta_0 > 0, \quad \eta \leq \eta_0, \quad \eta_0 > 0, \quad \varepsilon \leq \varepsilon_0,$$

where $\beta_0, \eta_0, \varepsilon_0$ are constants and

$$\sum_{m \in S_m} a^m(\overline{m})y(\overline{m}) - \left( \sum_{m \in S_m} a^m(\overline{m}) + \beta(m) \right)y(m) \geq -\eta(m) \quad \text{for } m \in \mathcal{M}_W, \eqno(5.2)$$

$$y(m) \leq \varepsilon(m) \quad \text{for } m \in \mathcal{M}_B,$$
then

(5.5) \[ y(m) \leq \max\{\epsilon_0, \eta_0/\beta_0\} \quad \text{for } m \in \mathcal{M}_\pm. \]

The simple proof of Lemma 1 will be omitted.

**Lemma 2.** Under the assumptions of Lemma 1 and

(5.6) \[ \sum_{m \in \mathcal{S}_m} a^m(\bar{m})y(\bar{m}) - \left( \sum_{m \in \mathcal{S}_m} a^m(\bar{m}) + \beta(m) \right)y(m) \leq \eta(m) \]

for \( m \in \mathcal{M}_W \),

(5.7) \[ y(m) \geq -\epsilon(m) \quad \text{for } m \in \mathcal{M}_B \]

we have

(5.8) \[ y(m) \geq \min\{-\epsilon_0, -\eta_0/\beta_0\} \quad \text{for } m \in \mathcal{M}_\pm. \]

The simple proof of Lemma 2 will be omitted.

From Lemma 1 and Lemma 2, as a simple conclusion, we obtain

**Lemma 3.** Under the assumption of Lemma 1 if \( \eta \geq 0 \), \( \epsilon \geq 0 \) and

(5.9) \[ \left| \sum_{m \in \mathcal{S}_m} a^m(\bar{m})y(\bar{m}) - \left( \sum_{m \in \mathcal{S}_m} a^m(\bar{m}) + \beta(m) \right)y(m) \right| \leq \eta(m) \]

for \( m \in \mathcal{M}_W \),

(5.10) \[ |y(m)| \leq \epsilon(m) \quad \text{for } m \in \mathcal{M}_B, \]

then

(5.11) \[ ||y|| \leq \max\{\epsilon_0, \eta_0/\beta_0\}. \]

6. **Difference expressions.** For a function \( \hat{y}: \hat{\Sigma} \to \mathbb{R} \) we put \( y_m = y(x_m) \) and introduce the following difference expressions:

\[ (y_m)_i = \frac{1}{2h_{m_i}^+} (y_{i(m)} - y_{-i(m)}), \]

(6.1a)

\[ (y_m)_{ii} = \frac{1}{h_{m_i}^+} \left( \frac{y_{i(m)} - y_{m}}{h_{m_i+1}^-} - \frac{y_{m} - y_{-i(m)}}{h_{m_i}^-} \right), \]

(6.1b)

\[ (y_m)^+ = \frac{1}{h_{m_{i+1}} h_{m_{j+1}^-}} (y_{i(j(m))} - y_{i(m)} - y_{j(m)} + y_m), \]

(6.1c)

\[ (y_m)^- = \frac{1}{h_{m_i} h_{m_{j+1}}^-} (y_{j(m)} - y_{-(j(m))} - y_m + y_{-i(m)}), \]

(6.1d)

\[ (y_m)^- = \frac{1}{h_{m_{i+1}} h_{m_{j+1}}^+} (y_{j(m)} - y_{i(-j(m))} - y_{-i(m)} + y_{j(m)}), \]

(6.1e)

\[ (y_m)^{---} = \frac{1}{h_{m_{i+1}} h_{m_{j}}^-} (y_{j(m)} - y_{m} - y_{-(j(m))} + y_{j(m)}), \]

(6.1f)

\[ (y_m)^{++} = \frac{1}{h_{m_{i+1}} h_{m_{j+1}}^+} (y_{j(m)} - y_{i(m)} - y_{j(m)} + y_{-i(m)}), \]

(6.1g)

\[ (y_m)_{ij} = \frac{1}{2} [(y_m)^{++} + (y_m)^{--} + (y_m)^{+-} + (y_m)^{-+}]. \]
(cf. Fig. 2),

\[
(y_m)_d = \sum_{i,j=1 \atop i \neq j}^n G_{ij}^m \left\{ \frac{1}{h_{m_i+1} h_{m_j}} y_{i(j(m))} + \frac{1}{h_{m_i} h_{m_j+1}} y_{i-1(j(m))} \right\} \\
+ \frac{1}{h_{m_i+1} h_{m_j}} y_{i(-j(m))} + \frac{1}{h_{m_i} h_{m_j}} y_{i-1(-j(m))} \\
- \left[ h_{m_i} h_{m_j} \left( \frac{1}{h_{m_i+1}} y_{i(m)} + \frac{1}{h_{m_i}} y_{i-1(m)} \right) \right] + \frac{h_{m_i} h_{m_j}}{h_{m_i+1} h_{m_j+1}} y_m \right\}
\]

(cf. Fig. 3),

\[(y_m)_I = \{(y_m)_1, \ldots, (y_m)_n\}, \quad (y_m)_I = \{(y_m)_{ij}\}.
\]

**Lemma 4.** If \( u \) is a function of the class \( C^2 \) in \( \Omega \), then

\[(6.2) \quad \max_{1 \leq i \leq n, \ \text{me.} \ A} |u_{x_i}(x_m) - (u_m)_i| \leq \eta_1(H), \quad \max_{1 \leq i, j \leq n, \ \text{me.} \ A} |u_{x_i x_j}(x_m) - (u_m)_{ij}| \leq \eta_2(H), \]

\[(6.3) \quad \max_{\text{me.} \ A} |(u_m)_d| \leq \eta_3(H),
\]

where

\[(6.4) \quad \lim_{H \to 0} \eta_i(H) = 0 \quad (i = 1, 2, 3).
\]

**Proof.** We shall prove (6.4) for \( i = 3 \) only because the proofs for \( i = 1, 2 \) are easy. For simplicity let us assume that \( h_j = h \) for \( i = 1, \ldots, n; \ j \in Z \) (cf. (4.1)). Then from the symmetry of the matrix \( G \) and (6.1h) we
have

\[
|u_{m \alpha}| \leq \sum_{i \neq j} G^i_m \left| \frac{1}{2h^2} \left( u_{i(j(m))} + u_{i(-j(m))} + u_{i(-i)(-j(m))} \right) - \frac{1}{h^2} \left( u_{i(m)} + u_{i(-i)(m)} + u_{j(m)} + u_{-j(m)} \right) + \frac{2}{h^2} u_m \right|.
\]

Fig. 3. Let us write \( h \) in place of \( h_{m_{i+1}}^i, h_{m_{i+1}}^j, h_{m_{i+1}}^j, h_{m_{j+1}}^i \). Then the Laplacian \( u_{xixj} + u_{xjxj} \) can be approximated with the aid of the difference expression

\[
\frac{u_{i(m)} - 2u_{m} + u_{-i(m)}}{(h)^2} + \frac{u_{j(m)} - 2u_{m} + u_{-j(m)}}{(h)^2},
\]

as well as with the aid of the quantity \( u_{i(j(m))} + u_{i(-j(m))} + u_{i(-i)(j(m))} + u_{i(-j)(-i)(m)} - 4u_m \) divided by \( 2h^2 \). In a similar way we obtained formula (6.1h)

Using Taylor polynomials we obtain (because \( u \) is a function of the class \( C^2 \) in \( \Omega \)):

\[
\begin{align*}
    u_{i(j(m))} &= u_m + u_{xi}(x_m) h + u_{xj}(x_m) h + \frac{1}{2} u_{xixi}(x_m) h^2 + u_{xjxj}(x_m) h^2 + h^2 u_{xixj}(x_m) h^2 + \frac{1}{2} u_{xjxj}(x_m) h^2 + \delta_{ij}^+(x_m) h^2, \\
    u_{-i(j(m))} &= u_m - u_{xi}(x_m) h + u_{xj}(x_m) h + \frac{1}{2} u_{xixi}(x_m) h^2 - u_{xjxj}(x_m) h^2 + \frac{1}{2} u_{xjxj}(x_m) h^2 + \delta_{ij}^-(x_m) h^2, \\
    u_{i(-j(m))} &= u_m + u_{xi}(x_m) h - u_{xj}(x_m) h + \frac{1}{2} u_{xixi}(x_m) h^2 - u_{xjxj}(x_m) h^2 + \frac{1}{2} u_{xjxj}(x_m) h^2 + \delta_{ij}^-(x_m) h^2, \\
    u_{-i(-j(m))} &= u_m - u_{xi}(x_m) h - u_{xj}(x_m) h + \frac{1}{2} u_{xixi}(x_m) h^2 + u_{xjxj}(x_m) h^2 + \frac{1}{2} u_{xjxj}(x_m) h^2 + \delta_{ij}^+(x_m) h^2, \\
    u_{i(m)} &= u_m + u_{xi}(x_m) h + \frac{1}{2} u_{xixi}(x_m) h^2 + \delta_i^+(x_m) h^2, \\
    u_{-i(m)} &= u_m - u_{xi}(x_m) h + \frac{1}{2} u_{xixi}(x_m) h^2 + \delta_i^-(x_m) h^2.
\end{align*}
\]
(for all \(i, j = 1, \ldots, n; \ i \neq j\), where there exists a function \(\delta(h)\) such that

\[
\begin{align*}
|\delta_{ij}^{++}(x_m)| &\leq \delta(h), & |\delta_{ij}^{--}(x_m)| &\leq \delta(h), & |\delta_{ij}^{-+}(x_m)| &\leq \delta(h), \\
|\delta_{ij}^{+-}(x_m)| &\leq \delta(h), & |\delta_{ij}^{+ -}(x_m)| &\leq \delta(h), & |\delta_{ij}^{- -}(x_m)| &\leq \delta(h)
\end{align*}
\]

(6.7)

(for all \(i, j = 1, \ldots, n; \ i \neq j\) and \(x \in \mathcal{M}_W\)) and

\[
\lim_{h \to 0} \delta(h) = 0.
\]

Putting (6.6) in (6.5) and reducing the expressions, we get

\[
|(u_{m})_d| \leq \sum_{i < j} G_{m}^{ij} 6 \delta(h).
\]

Since the matrix \(G\) is bounded, there exists a constant \(\Gamma\) such that

\[
G_{m}^{ij} \leq \Gamma \quad \text{for} \ m \in \mathcal{M}_W \ (i, j = 1, \ldots, n),
\]

whence in (6.9) we have

\[
|(u_{m})_d| \leq \sum_{i < j} 6 \Gamma \delta(h) = 3n(n - 1) \Gamma \delta(h).
\]

(6.11)

By (6.8) from (6.11) we obtain (6.3) and (6.4).

This ends the proof of Lemma 4.

7. Difference problem. Let a function \( \mathbf{v}: \hat{\Sigma} \to \mathbb{R} \) be a solution of the difference equations system

\[
\begin{align*}
F(x_m, v_m, (v_{m})_1, (v_{m})_2) + (v_{m})_d &= 0 \quad \text{for} \ m \in \mathcal{M}_W, \\
v_m &= \bar{u}(x_m) \quad \text{for} \ m \in \mathcal{M}_{B_2}, \\
v_m &= \bar{u}(x) \quad \text{for} \ m \in \mathcal{M}_{B_1},
\end{align*}
\]

(7.1)

(7.2)

(7.3)

where \(x \in \partial \Omega \cap P(x_m)\) (cf. (4.2) and Fig. 4) and \([x_m, x] \subset \overline{\Omega}\).

Fig. 4. The point \(x_m\) is a type I boundary nodal point and \(x \in \partial \Omega\) is a point such that the segment \([x_m, x] \subset \overline{\Omega}\)
Remark 1. Normally the difference equations (7.1) associated with the differential equation (1.1) could be written in the form

\[(7.4) \quad F(x_m, v_m, (v_m)_1, (v_m)_{11}) = 0 \quad \text{for } m \in \mathcal{M}_W.\]

The term \((v_m)_d\) in the difference equations (7.1) has appeared because without it the adequate assumptions of Lemmas 1, 2, 3 (cf. (5.2)) which we use in the proof of Theorem 1 are not satisfied (cf. Part 8, Convergence).

**Lemma 5.** If the function \(F\) has bounded derivatives \(F_{w_{ii}}\) and \(F_{q_i}\), i.e. there are constants \(g, D\) such that

\[(7.5) \quad F_{w_{ii}}(x, z, q, w) \leq D, \quad |F_{q_i}(x, z, q, w)| \leq g \quad (i = 1, \ldots, n)\]

for all \(x, z, q, w,\) and \(u\) is a solution of the differential problem (1.1), (1.2) of the class \(C^2\) in \(Q\), then

\[(7.6) \quad F(x_m, u_m, (u_m)_1, (u_m)_{11}) + (u_m)_d = \eta_m(H) \quad \text{for } m \in \mathcal{M}_W,\]

\[(7.7) \quad u_m = \bar{u}(x) + \varepsilon_m(H) \quad \text{for } m \in \mathcal{M}_B_1\]

where

\[(7.8) \quad \lim_{H \to 0} \eta(H) = 0, \quad \lim_{H \to 0} \varepsilon(H) = 0\]

and

\[(7.9) \quad \eta(H) = \max_{m \in \mathcal{M}_W} |\eta_m(H)|, \quad \varepsilon(H) = \max_{m \in \mathcal{M}_B_1} |\varepsilon_m(H)|.\]

**Proof.** From (7.6), (1.1) applying the mean value theorem, we have

\[(7.10) \quad |\eta_m(H)|\]

\[= |F(x_m, u_m, (u_m)_1, (u_m)_{11}) + (u_m)_d - F(x_m, u_m, u_x(x_m), u_{xx}(x_m))|\]

\[\leq \sum_{i=1}^{n} |F_{q_i}(|\sim|)(u_m)_i - u_{x_i}(x_m)| + \sum_{i,j=1}^{n} |F_{w_{ij}}(|\sim|)||u_m||_i - u_{x_ix_j}(x_m)| + |(u_m)_d|.\]

Introducing (7.5), (2.3) and (6.10) in (7.10) and using (6.2), (6.3), we obtain

\[(7.11) \quad |\eta_m(H)| \leq \sum_{i=1}^{n} g_i(H) + \left[ \sum_{i,j=1}^{n} G_{ij}(x_m) + nD \right] \eta_2(H) + \eta_3(H)\]

\[\leq n g_1(H) + [I_{n}(n-1) + nD] \eta_2(H) + \eta_3(H)\]

for \(m \in \mathcal{M}_W\), whence by (6.4) and (7.9) it follows that

\[(7.12) \quad \lim_{H \to 0} \eta(H) = 0.\]
Now we shall take into consideration (7.7) and (1.2). We get

\[(7.13) \quad |e_m(H)| = |u_m - \bar{u}(x)| = |u(x_m) - u(x)| \leq \sum_{i=1}^{n} |u_{x_i}(\sim)| |x_{m_i}^i - x^i| \]

for \( m \in \mathcal{M}_1 \).

Since \( x \in P(x_m) \) (cf. (4.2) and Fig. 4), by definition \( H \) (cf. (4.3)) we have

\[(7.14) \quad \sum_{i=1}^{n} |x_{m_i}^i - x^i| \leq \sqrt{n} H, \]

and since the function \( u \) is of the class \( C^2 \) in \( \Omega \), there exists a constant \( C \) such that

\[(7.15) \quad |u_{x_i}(\sim)| \leq C \quad (i = 1, \ldots, n). \]

Introducing (7.14) and (7.15) in (7.13), we infer that

\[(7.16) \quad |e_m(H)| \leq C \sqrt{n} H \quad \text{for} \ m \in \mathcal{M}_1, \]

whence

\[(7.17) \quad \varepsilon(H) \leq C \sqrt{n} H, \]

which with (7.13) gives Lemma 5.

Remark 2. The difference problem (7.1), (7.2), (7.3) and the differential problem (1.1), (1.2) are consistent because \((u_m)_\delta \) in the difference equations (7.6) satisfy the estimates (6.3), (6.4) and \( \eta_m(H) \) in (7.4) satisfy (7.6).

3. Convergence.

Theorem 1. We suppose that \( u \) is a solution of the class \( C^2 \) in \( \Omega \) of the differential problem (1.1), (1.2), \( v \) is a solution of the difference equations system (7.1), (7.2), (7.3), the function \( F \) satisfies the assumption specified in Part 2, Assumption, and the derivatives \( F_{w_i}; F_{q_i} \) are bounded. Then

1° the difference method is convergent, i.e.

\[(8.1) \quad \lim_{H \to 0} \|u - v\|_H = 0, \]

where \( \|y\|_H = \max \{|y(m)|: m \in \mathcal{M}_2\} \) and

2° we have the error estimate

\[(8.2) \quad \|u - v\|_H \leq \max \{\varepsilon(H), \eta(H)/L\} \quad (\varepsilon \geq \varepsilon_0). \]

Proof. Let us put \( r = u - v \). For \( m \in \mathcal{M}_H \) from (7.7), (1.2), (7.2) and (7.3) we have

\[
\begin{align*}
    u_m - v_m = \begin{cases}
    \bar{u}(x_m) - \bar{u}(x_m) = 0 & \text{for} \ m \in \mathcal{M}_B, \\
    \bar{u}(x) + e_m(H) - \bar{u}(x) = e_m(H) & \text{for} \ m \in \mathcal{M}_B, 
    \end{cases}
\end{align*}
\]
whence

\begin{equation}
|r_m| \leq |\varepsilon_m(H)| \leq \varepsilon(H) \quad \text{for } m \in \mathcal{M}. \tag{8.3}
\end{equation}

For \( m \in \mathcal{M}_W \) from (7.6) and (7.1) and from the mean value theorem, applying the definition of the difference expressions and grouping the terms, we get successively:

\begin{equation}
\eta_m(H) = \left[ F(x_m, u_{m,1} (u_m)_{11}, (u_m)_{11}) - F(x_m, v_{m,1} (v_m)_{11}, (v_m)_{11}) \right] + \\
+ \left[(u_m)_d - (v_m)_d\right], \tag{8.4}
\end{equation}

\begin{equation}
\eta_m(H) = F(x, \sim) r_m + \sum_{i=1}^{n} F_q(\sim)(r_m)_i + \sum_{i,j=1}^{n} F_{w_{ij}}(\sim)(r_m)_{ij} + (r_m)_d, \tag{8.5}
\end{equation}

\begin{equation}
\eta_m(H) = F(x, \sim) r_m + \sum_{i=1}^{n} F_q_i \frac{1}{2h_m^i} (r_i(m) - r_{-i(m)}) + \\
+ \sum_{i,j=1}^{n} F_{w_{ij}} \frac{1}{h_m^i h_m^j} \left[ (r_m)_{ij}^+ + (r_m)_{ij}^- + (r_m)_{ij}^+ + (r_m)_{ij}^- \right] + \\
+ \sum_{i=1}^{n} F_{w_{ii}} \frac{1}{h_m^i} \left( \frac{r_i(m) - r_m}{h_m^i} - \frac{r_m - r_{-i(m)}}{h_m^i} \right) + (r_m)_d
\end{equation}

\begin{equation}
= F(x, \sim) r_m + \sum_{i=1}^{n} F_q_i \frac{1}{2h_m^i} (r_i(m) - r_{-i(m)}) + \\
+ \sum_{i,j=1}^{n} F_{w_{ij}} \frac{1}{h_m^i h_m^j} \left[ \frac{1}{h_m^i + 1 h_m^j + 1} (r_{i(j(m))} - r_{i(m)} - r_{j(m)} + r_m) + \\
+ \frac{1}{h_m^i h_m^j + 1} (r_{j(m)} - r_{-i(\sim)(m)} - r_m + r_{-i(m)}) + \\
+ \frac{1}{h_m^i + 1 h_m^j} (r_i(m) - r_m - r_{-j(m)} + r_{-j(m)}) + \\
+ \frac{1}{h_m^i h_m^j} (r_m - r_{-i(m)} - r_{-j(m)} + r_{-i(\sim)(m)}) \right] + \\
+ \sum_{i=1}^{n} F_{w_{ii}} \frac{1}{h_m^i} \left( \frac{r_i(m) - r_m}{h_m^i} - \frac{r_m - r_{-i(m)}}{h_m^i} \right) + \\
+ \sum_{i,j=1}^{n} G_{m} \left( \frac{1}{h_m^i + 1 h_m^j + 1} r_{i(j(m))} + \frac{1}{h_m^i h_m^j + 1} r_{-i(\sim)(m)} \right) + \\
+ \sum_{i,j=1}^{n} G_{m} \left( \frac{1}{h_m^i + 1 h_m^j + 1} r_{i(j(m))} + \frac{1}{h_m^i h_m^j + 1} r_{-i(\sim)(m)} \right) + \\
+ \sum_{i,j=1}^{n} G_{m} \left( \frac{1}{h_m^i + 1 h_m^j + 1} r_{i(j(m))} + \frac{1}{h_m^i h_m^j + 1} r_{-i(\sim)(m)} \right) + \\
+ \sum_{i,j=1}^{n} G_{m} \left( \frac{1}{h_m^i + 1 h_m^j + 1} r_{i(j(m))} + \frac{1}{h_m^i h_m^j + 1} r_{-i(\sim)(m)} \right) + \\
+ \sum_{i,j=1}^{n} G_{m} \left( \frac{1}{h_m^i + 1 h_m^j + 1} r_{i(j(m))} + \frac{1}{h_m^i h_m^j + 1} r_{-i(\sim)(m)} \right). \tag{8.6}
\end{equation}
\[
\eta_m(H) = \sum_{i<j} \left\{ \frac{1}{2} (G_m^{ij} + E_{wij}) \frac{1}{h_{mj+1}^i h_{mj+1}^j} \right. \\
\left. - \frac{1}{h_{mj+1}^i h_{mj+1}^j} \left( \frac{1}{h_{mj+1}^i} r_{i} - \frac{1}{h_{mj+1}^j} r_{j} \right) \right\} + \\
\frac{1}{2} (G_m^{ij} - E_{wij}) \frac{1}{h_{mj+1}^i h_{mj+1}^j} r_{i} - (r_{j(m)} \right) + \\
\frac{1}{2} (G_m^{ij} - E_{wij}) \frac{1}{h_{mj+1}^i h_{mj+1}^j} r_{j} - (r_{i(m)} \right) + \\
+ \frac{1}{2} \left( \sum_{i=1}^{n} \left( \frac{F_{u_{ij}}}{h_{mj+1}^i h_{mj+1}^j} + \sum_{j \neq i} \left[ \frac{F_{wij}}{2 h_{mj+1}^i} \left( \frac{1}{h_{mj}^i} - \frac{1}{h_{mj+1}^j} \right) \right] + \\
\frac{G_m^{ij}}{2 h_{mj+1}^i} \left( \frac{1}{h_{mj}^i} + \frac{1}{h_{mj+1}^j} \right) \right) \right)_{r_{i(m)}}^+ \\
\left( - \frac{F_{z} + \sum_{i=1}^{n} \frac{F_{wij}}{h_{mj+1}^i} \left( \frac{1}{h_{mj+1}^i} + \frac{1}{h_{mj}^i} \right) \right) - \\
- \frac{1}{2} \sum_{i \neq j} \left( - \frac{1}{h_{mj+1}^i h_{mj+1}^j} \left( \frac{1}{h_{mj+1}^i} - \frac{1}{h_{mj+1}^j} \right) \right) + \\
\frac{1}{h_{mj+1}^i h_{mj+1}^j} - \sum_{i \neq j} \left( \frac{G_m^{ij}}{h_{mj+1}^i h_{mj+1}^j} \right) \right) \right\} r_m.
\]

From equality (8.7) we shall obtain two inequalities of the form (5.3) and (5.6), the quantity \( y(m) \) being replaced by the error \( r_m \), \( y(m) = r_m \).

It remains to verify the estimates for the coefficients (5.2).
For this purpose let us write for \( m \in \mathcal{M}_B \):

\[
\alpha^m(\overline{m}) = \begin{cases} 
\frac{1}{2} (G_{m}^{ij} + F_{wij}) \frac{1}{h_{m+1}^{i} h_{m+1}^{j}}, & \text{when } \overline{m} = i(j(m)), \\
\frac{1}{2} (G_{m}^{ij} - F_{wij}) \frac{1}{h_{m+1}^{i} h_{m+1}^{j}}, & \text{when } \overline{m} = -i(j(m)), \\
\frac{1}{2} (G_{m}^{ij} - F_{wij}) \frac{1}{h_{m}^{i} h_{m}^{j}}, & \text{when } \overline{m} = i(-j(m)), \\
\frac{1}{2} (G_{m}^{ij} + F_{wij}) \frac{1}{h_{m}^{i} h_{m}^{j}}, & \text{when } \overline{m} = -i(-j(m)).
\end{cases}
\]

We shall prove that

\[
\alpha^m \geq 0.
\]

For \( \overline{m} = \pm i(\pm j(m)) \) by (1.5) we have \( |F_{wij}| \leq G^{ij} \), whence \( G^{ij} \leq \pm F_{wij} \geq 0 \); therefore \( \alpha^m(\overline{m}) \geq 0 \). For \( \overline{m} = i(m) \), applying (1.5), (1.6) and the definitions \( H, X, \varepsilon \), we simply verify that \( \alpha^m(\overline{m}) \geq 0 \). For \( \overline{m} = -i(m) \) we reason analogously and hence obtain finally

\[
\alpha^m(\overline{m}) \geq 0 \quad \text{for } \overline{m} \in S_m.
\]

This ends the proof of (8.9).

Now we write:

\[
\beta(m) = -F_{m}, \quad \beta_0 = L, \quad \eta(m) = \eta_m(H), \quad \eta_0 = \eta(H)
\]

and

\[
\varepsilon(m) = \varepsilon_m(H), \quad \varepsilon_0 = \varepsilon(H) \quad \text{for } m \in \mathcal{M}_B,
\]

\[
y(m) = r_m \quad \text{for } m \in \mathcal{M}_B.
\]

From (8.3)

\[
|y(m)| = |r_m| \leq \varepsilon(H) = \varepsilon_0 \quad \text{for } m \in \mathcal{M}_B.
\]
and from (2.2)  
(8.14) \[ \beta(m) = -F_x \geq L > 0, \quad L = \beta_0 \quad \text{for } m \in \mathcal{M} \]
and according to (7.9)  
(8.15) \[ |\eta(m)| = |\eta_m(H)| \leq \eta(H) = \eta_0. \]

It is easy to verify that the expression \[ \sum_{m \in S_m} \alpha^m(m) + \beta(m) \] (cf. (8.8), (8.11)) is equal to the coefficient of \( r_m \) in (8.7) (cf. the last line in formula (8.7)).

From (8.10), (8.13), (8.14), (8.15) it follows that the assumptions of Lemma 3 are satisfied, whence

(8.16) \[ \|r\|_H \leq \max \{\varepsilon(H), \eta(H)/L\} \quad (\varepsilon \geq \varepsilon_0), \]

and hence on the basis of Lemma 5

\[ \lim_{H \to 0 \atop (\varepsilon \geq \varepsilon_0)} \|u - v\|_H = 0 \]

which completes the proof of Theorem 1.

References


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