THE PSEUDO-ARC AS AN INVERSE LIMIT
WITH SIMPLE BONDING MAPS*

BY

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G. W. Henderson [4] has stated that in describing the pseudo-arc "inverse limits has seen little use as the description of the bonding functions were restatements of a chain construction, involved infinitely many different functions and thus at least as complicated as chains to use". In his paper, Professor Henderson proceeded to reduce the complexity by reducing the number of different bonding maps from infinitely many to one. The purpose of this paper is to reduce the complexity in a different manner by defining simple bonding maps without relying on a chain construction. These simple bonding maps have the property that each has a graph that looks like one "N" has been inserted into the graph of the identity function. It is hoped that the simple nature of these bonding maps will aid the reader in obtaining a better mental picture of the pseudo-arc.

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A simple function $F$ is defined to be a function from the closed interval $[0, 1]$ onto $[0, 1]$ with the property that there exists a closed interval $[r, s]$ of $[0, 1]$ such that $F(x) = x$ if $x$ is not in $[r, s]$ and the graph of $F$ over $[r, s]$ is an "N" within $[r, s] 	imes [r, s]$, i.e.

$$F(x) = \begin{cases} 
3x - 2r & \text{if } r \leq x \leq (2r + s)/3, \\
3x - 2s & \text{if } (2r + s)/3 < x \leq r + 2s/3, \\
-3x + 2(r + s) & \text{if } (r + 2s)/3 < x \leq s.
\end{cases}$$

If $f_i$ is a function from $X_{i+1}$ onto $X_i$ for $n \leq i < m$, then the notation $f_n^m$ will denote the function from $X_m$ onto $X_n$ which is equal to the composite function $f_n \circ f_{n+1} \circ \ldots \circ f_{m-1}$.

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Theorem. The pseudo-arc is the inverse limit of an inverse limit system which has each coordinate space equal to [0, 1] and has simple functions as bonding maps.

Proof. Define $X_i$ for each natural number $i$ to be the $i$th coordinate space which is equal to [0, 1]. Let $\{[r_1, s_1], [r_2, s_2], [r_3, s_3], \ldots\}$ denote the collection of all the closed intervals within [0, 1] which have rational endpoints. Define $f_i$ from $X_2$ onto $X_1$ so that $f_i(x) = x$ if $x$ is not in $[r_1, s_1]$ and the graph of $f_i$ over $[r_1, s_1]$ is an “N” within $[r_1, s_1] \times [r_1, s_1]$ as defined in the definition above. The bonding function $f_2$ from $X_3$ onto $X_2$ is defined so that it is the same as the identity function except over components $C$ of $f_1^{-1}([r_2, s_2])$ with the property that $f_1(C)$ is $[r_2, s_2]$ in which case the function $f_2$ is defined so that its graph within the Cartesian product $C \times C$ is an “N”. The function $f_3$ from $X_4$ onto $X_3$ is defined in a similar manner by requiring that $f_3(x) = x$ for all $x$ which are not in components $C$ of $f_2^{-1}([r_1, s_1])$ that have $f_2(C) = [r_1, s_1]$ and requiring that the graph of $f_3$ within $C \times C$ for such $C$ be an “N”. Through the use of mathematical induction it can be proven that there exists an infinite sequence of functions $f_1, f_2, f_3, f_4, \ldots$ such that (1) $f_1, f_2, f_3$ are as defined above and (2) if $i$ is a natural number such that $n(n+1)/2 < i \leq (n+1)(n+2)/2$ for some $n \geq 2$, the function $f_i$ is the function from $X_{i+1}$ onto $X_i$ possessing the property that, if $k = i - n(n+1)/2$, $f_i$ is the same as the identity function except over components $C$ of $(f_k)^{-1}([r_{n+2-k}, s_{n+2-k}])$ which have $f_k(C) = [r_{n+2-k}, s_{n+2-k}]$ in which case the function $f_i$ has an “N” as its graph within $C \times C$. The following table probably best describes how the functions are defined.

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$X_5$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[r_1, s_1]$</td>
<td>$f_1$</td>
<td>$f_3$</td>
<td>$f_6$</td>
<td>$f_{10}$</td>
<td>$f_{15}$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$[r_2, s_2]$</td>
<td>$f_2$</td>
<td>$f_5$</td>
<td>$f_9$</td>
<td>$f_{14}$</td>
<td></td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$[r_3, s_3]$</td>
<td>$f_4$</td>
<td>$f_8$</td>
<td>$f_{13}$</td>
<td></td>
<td></td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$[r_4, s_4]$</td>
<td>$f_7$</td>
<td>$f_{12}$</td>
<td></td>
<td></td>
<td></td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$[r_5, s_5]$</td>
<td>$f_{11}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\ldots$</td>
</tr>
<tr>
<td></td>
<td>$\ldots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

The variable $n$ refers to the $n$th diagonal of the table that runs from the lower left to the upper right, and $k$ refers to the $k$th position of the function $f_i$ in the $(n+1)$st diagonal. The closed interval at the left of the row containing $f_i$ is the interval $[r_{n+2-k}, s_{n+2-k}]$ which is viewed as a subset of the coordinate space $X_k$ which heads the column that contains $f_i$.

In order to see that the pseudo-arc is the inverse limit of the system that has $f_1, f_2, f_3, \ldots$ as bonding maps, one need only observe that the inverse
limit is hereditarily indecomposable since R. H. Bing [2] has established that all chainable hereditarily indecomposable nondegenerate continua are topologically equivalent to the pseudo-arc as defined by E. E. Moise [6], to the homogeneous plane continuum as defined by Bing [1] and to the hereditarily indecomposable continuum as defined by Knaster [5].

Suppose \( M \) is a nondegenerate subcontinuum of the inverse limit such that \( M \) is the union of two proper subcontinua \( H \) and \( K \) of \( M \). There is a natural number \( k_0 \) having the property that neither of the \( k_0 \)th projections \( \pi_{k_0}(H) \) nor \( \pi_{k_0}(K) \) is a subset of the other. There is a closed interval \([r_j, s_j]\) in the collection mentioned at the beginning of this proof and there is a closed interval \([a, b]\) with \( a < r_j < s_j < b \) and with \( \{a, r_j\} \) being a subset of one of the sets \( \pi_{k_0}(H) - \pi_{k_0}(K) \) or \( \pi_{k_0}(K) - \pi_{k_0}(H) \) and \( \{s_j, b\} \) being a subset of the other. It may be assumed without loss of generality that \( \{a, r_j\} \) is a subset of \( \pi_{k_0}(H) - \pi_{k_0}(K) \) and \( \{s_j, b\} \) is a subset of the difference \( \pi_{k_0}(K) - \pi_{k_0}(H) \). Through solving the equation \( j = n + 2 - k_0 \) for \( n \), one finds that in the \( (k_0 + j - 2) + 1 \) diagonal of the table there is a function \( \phi_{k_0} \) which was defined in terms of \([r_j, s_j]\) being a subset of \( X_{k_0} \), namely \( i_0 \) is equal to \( k_0 + (k_0 + j - 2)(k_0 + j - 1)/2 \). There must be a component \( C_1 \) of \((\phi_{k_0}^{-1}(i_0, s_j))\) within \( \pi_{i_0+1}(M) \) such that \([a, b]\) is the image of \( C_1 \) under the function \( \phi_{k_0}^{-1} \). There must also be a component \( C_2 \) of \((\phi_{k_0}^{-1}(r_j, s_j))\) containing a point of \( \phi_{k_0}(C_1) \) such that \([r_j, s_j]\) is the image of \( C_2 \) under the function \( \phi_{k_0} \). Choose \( a_1 \) in \( C_1 \cap (\phi_{k_0}^{-1}(a)) \) and \( b_1 \) in \( C_1 \cap (\phi_{k_0}^{-1}(b)) \). Notice that \( C_2 \) cannot contain the images of \( a_1 \) and \( b_1 \) using the function \( \phi_{i_0} \); otherwise, if for example \( \phi_{i_0}(a_1) \) were in \( C_2 \), then \( a = \phi_{k_0}^{i_0+1}(a_1) = \phi_{i_0}^{i_0}(\phi_{i_0}(a_1)) = \phi_{i_0}^{i_0}(C_2) = [r_j, s_j] \) which would deny that \( a < r_j \). Also notice that \( \phi_{i_0}(a_1) \) is in \( \pi_{i_0}(H) - \pi_{i_0}(K) \) because \( a = \phi_{k_0}^{i_0+1}(a_1) \) which is in \( \pi_{k_0}(H) - \pi_{k_0}(K) \) and that \( \phi_{i_0}(b_1) \) is in \( \pi_{i_0}(K) - \pi_{i_0}(H) \) for a similar reason. With these properties and the additional property that \( C_2 \) contains \( \pi_{i_0}(H) \cap \pi_{i_0}(K) \) as a subset, it can be seen that \( \phi_{i_0}(C_1) \) must be a closed interval containing \( C_2 \) within its interior. This shows that \( C_2 \) is a subset of the closed interval \([a_1, b_1]\) (or \([b_1, a_1]\) if \( b_1 < a_1 \)) in \( X_{i_0+1} \) due to the manner in which \( \phi_{i_0} \) was defined in terms of \( C_2 \). Since \( C_2 \subseteq [a_1, b_1] \) (or \([b_1, a_1]\)) \( C_1 \subseteq \pi_{i_0+1}(M) \), the set \( C_2 \) is a subset of \( \pi_{i_0+1}(M) \). With \( C_2 \) being a subset of the union of the two intervals \( \pi_{i_0+1}(H) \) and \( \pi_{i_0+1}(K) \), either \( \pi_{i_0+1}(H) \) or \( \pi_{i_0+1}(K) \) contains a subinterval of \( C_2 \) with length at least half the length of \( C_2 \). Suppose \( \pi_{i_0+1}(H) \) has this property. Since the graph of \( \phi_{i_0} \) is an "N" within the square \( C_2 \times C_2, \phi_{i_0}(\pi_{i_0+1}(H)) \) must have \( C_2 \) as a subset. With \([r_j, s_j] = \phi_{k_0}(C_2) \subseteq \phi_{k_0}(\pi_{i_0+1}(H))) = \pi_{k_0}(H) \), the point \( s_j \) must be in \( \pi_{k_0}(H) \) which is a contradiction. If \( \pi_{i_0+1}(K) \) contained at least half of the interval \( C_2 \), another contradiction would be arrived at by a similar argument. This proves that the inverse limit is hereditarily indecomposable; thus, the inverse limit is the pseudo-arc.

In order to see that the pseudo-arc can be obtained in terms of simple bonding maps, one needs only to observe that, for each \( i \), the function \( f_i \) has
the property there exist finitely many disjoint subintervals \([u_{i1}, v_{i1}],
[u_{i2}, v_{i2}], \ldots, [u_{im_i}, v_{im_i}]\) such that \(f_i(x) = x\) if \(x\) is not in \([u_{ij}, v_{ij}]\) for any \(j \leq m_i\) and the graph of \(f_i\) over \([u_{ij}, v_{ij}]\) is an "N" for each \(j \leq m_i\). Now for each \(i\) and each \(j \leq m_i\), define \(f_{ij}\) to be the simple function from \([0, 1]\) onto \([0, 1]\) such that \(f_{ij}(x) = x\) if \(x\) is not in \([u_{ij}, v_{ij}]\) and \(f_{ij}(x) = f_i(x)\) if \(x\) is in \([u_{ij}, v_{ij}]\). Notice that \(f_i = f_{i1} \circ f_{i2} \circ \ldots \circ f_{im_i}\). Define \(F_1\) to be the simple function \(f_i\). In general, for each natural number \(n > 1\), \(F_n\) is defined to be \(f_{ij}\) if \(n = m_1 + m_2 + \ldots + m_{i-1} + j\). It is easy to see that the inverse limit of the inverse limit system that has \(F_1, F_2, F_3, \ldots\) as simple bonding maps is topologically equivalent to the inverse limit that we have shown to be the pseudo-arc. This completes the proof of the theorem.

REFERENCES


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