M. SZYSZKOWICZ (Wroclaw)

TWO ONE-STEP METHODS WITH A GIVEN PARAMETER

1. Procedure declaration. Two procedures \( sode123, sode567 \) in ALGOL 60 for solving the system of ordinary differential equations
\[
y' = f(x, y), \quad y(x_0) \text{ given},
\]
are presented.

The procedures have the same parameters.

Data:

- \( x \) — value of \( x_0 \) in (1.1),
- \( x_l \) — value of the argument for which the problem (1.1) is solved,
- \( \text{eps} \) — relative error (the given tolerance),
- \( \text{eta} \) — number which is used instead of zero obtained as solution,
- \( h_{\text{min}} \) — minimum allowed step-size,
- \( n \) — number of differential equations,
- \( y[1:n] \) — vector with initial data \( y(x_0) \) in (1.1),
- \( \text{sigma} \) — for the procedure \( sode123 \): \( \text{sigma} = \frac{1}{4} \) or \( \frac{1}{3} \), for the procedure \( sode567 \): \( \text{sigma} = \frac{1}{52} \) or \( \frac{1}{42} \).

Results:

- \( x \) — value of \( x_l \).
- \( y[1:n] \) — vector with the solution at point \( x_l \).

Additional parameters:

- \( \text{steph} \) — label outside of the body of the procedures \( sode123, sode567 \) to which a jump is made if \( |h| < h_{\text{min}} \) (\( h \) is the step-size of integration); increasing \( \text{eps} \) or decreasing \( h_{\text{min}} \) it is possible to continue the computations,
- \( f \) — procedure with the heading: \textbf{procedure} \( f(x, n, y, d); \quad \text{value} \; x, n; \quad \text{real} \; x; \quad \text{integer} \; n; \quad \text{array} \; y, d; \quad \) which computes the values of the functions \( f(x, y) \) in (1.1) and assigns them to \( d[1:n] \).
procedure sode123(x, x1, eps, eta, hmin, n, y, steph, f, sigma);

value x1, eps, eta, hmin, n, sigma;

real x, x1, eps, eta, hmin, sigma;

integer n;

array y;

label steph;

procedure f;

begin

real h, hh, ww, w1, w2, w3;

integer i;

Boolean last;

array d1, y1, y2, y3, y4[1:n];

eps := .5/eps;

h := x1 - x;

last := true;

f(x, n, y, d1);

cont:

hh := .5*h;

w3 := h*sigma;

ww := hh*sigma;

for i := 1 step 1 until n do

begin

w1 := y[i];

w2 := d1[i];

y1[i] := w1 + w3*xw2;

y2[i] := w1 + ww*xw2

end i;
\[ f(x+w_3, r, y_1, y_3); \]
\[ \text{for } i:=1 \text{ step 1 until } n \text{ do} \]
\[ y_1[i]:=y[i]+h\times y_3[i]; \]
\[ f(x+w_w, n, y_2, y_3); \]
\[ \text{for } i:=1 \text{ step 1 until } n \text{ do} \]
\[ y_2[i]:=y[i]+h\times y_3[i]; \]
\[ f(x+h_h, n, y_2, y_3); \]
\[ \text{for } i:=1 \text{ step 1 until } n \text{ do} \]
\[ y_4[i]:=y_2[i]+w_w\times y_3[i]; \]
\[ f(x+h_h+w_w, n, y_4, y_3); \]
\[ w_w:=0; \]
\[ \text{for } i:=1 \text{ step 1 until } n \text{ do} \]
\[ \begin{align*}
& w_2:=y_2[i]+h\times y_3[i]; \\
& w_3:=w_2-y_1[i]; \\
& w_1:=y_3[i]:=w_2+w_3; \\
& w_3:=\text{abs}(w_3); \\
& w_1:=\text{abs}(w_1); \\
& \text{if } w_1<\text{eta} \\
& \quad \text{then } w_1:=\text{eta}; \\
& w_1:=w_3/w_1; \\
& \text{if } w_1>\text{ww} \\
& \quad \text{then } \text{ww}:=w_1 \\
\end{align*} \]
\[ \text{end } i; \]
\[ \text{ww}:=\text{if } \text{ww}=0 \text{ then } \text{eta} \text{ else } \sqrt{\text{eps}\times \text{ww}}\times 1.25; \]
\[ h_h:=h/\text{ww}; \]
\[ \text{if } w_w>1.25 \]
\[ \quad \text{then} \]
\[ \begin{align*}
& \text{begin} \\
\end{align*} \]
if abs(hh)<hmin
    then go to steph;
last:=false
end

ww>1.25
else
begin
    x:=x+h;
    for i:=1 step 1 until n do
        y[i]:=y3[i];
    if last
        then go to endp;
    f(x,n,y,d1);
    w1:=x1-x;
    if (w1-hh)*sign(h)<0
        then
            begin
                hh:=w1;
                last:=true
            end
    end
    (w1-hh)*h<0
    end
    ww<1.25;
    h:=hh;
    go to contn;
endp;
end sode123;
procedure sode567(x,x1,eps,eta,hmin,n,y,steph,f,sigma);
value x1,eps,eta,hmin,n,sigma;
real x,x1,eps,eta,hmin,sigma;
integer n;
array y;
label steph;
procedure f;
begin
real h,hh,ww,w1,w2,w3;
integer i;
boolean last;
array k,k1,k2,k3,k4,yh,y1,y2,y3[1:n];
procedure steprk5(h,x,k1,y,df);
value h,x;
real h,x;
array k1,y,df;
begin
w1:=.5*h;
for i:=1 step 1 until n do
yh[i]:=y[i]+w1*k1[i];
f(x+.5*h,n,yh,k2);
w1:=.0625*h;
for i:=1 step 1 until n do
yh[i]:=y[i]+w1*(3.0*k1[i]+k2[i]);
f(x+.25*h,n,yh,k3);
w1:=$.25-16.0\times \text{sigma}$;
w2:=32.0$\times$ sigma;
for i:=1 step 1 until n do
    y=[i]+h*(w1*k[i]+w2*k2[i]+w3*k3[i]+.5625*k4[i]);
    f(x+h,n,yh,k4);
    w1:=12.0*sigma-.1875;
    w2:=12.0*sigma-.375;
    w3:=.75-24.0*sigma;
    for i:=1 step 1 until n do
        yh[i]:=y[i]+h*(w1*k[i]+w2*k2[i]+w3*k3[i]+.5625*k4[i]);
        f(x+.75*h,n,yh,df);
        w1:=h/7.0;
        w2:=4.0-192.0*sigma;
        w3:=7.0-192.0*sigma;
        ww:=384.0*sigma;
    for i:=1 step 1 until n do
        y[i]:=y[i]+w1*(w2*k1[i]+w3*k2[i]+ww*k3[i]-12.0*k4[i]+8.0*df[i]);
        f(x+h,n,yh,k2);
        w1:=h/90.0;
    for i:=1 step 1 until n do
        df[i]:=y[i]+w1*(7.0*(k1[i]+k2[i])+32.0*(k3[i]+df[i])+12.0*k4[i])
end steprk5;

h:=x1-x;
eps:=1.0/(eps*62.0);
last:=true;
f(x,n,y,k1);
contb:
    hb=-.5*h;
steprk5(h,x,k1,y,y1);
steprk5(hh,x,k1,y,y2);
f(x+hh,n,y2,k);
steprk5(hh,x+hh,k,y2,y3);
ww:=.0;
for i:=1 step 1 until n do
begin
w3:=y3[i];
w1:=w3-y1[i];
w3:=y3[i]:=w3+w1/31.0;
w1:=abs(w1);
w3:=abs(w3);
if w3<eta
then w3:=eta;
w1:=w1/w3;
if w1>ww
then ww:=w1
end i;
ww:=if ww=0 then eta else (eps×ww)+.16666666666666666×1.25;
hh:=h/ww;
If ww>1.25
then
begin
if abs(hh)<hmin
then goto steph;
last:=false
end ww>1.25
else
begin
x:=x+hh;
for i:=1 step 1 until n do
y[i]:=y3[i];
if last
    then go to endp;

f(x,n,y,k1);
w1:=x1-x;
If (w1-hh)x sign(h)<0
    then
    begin
    hh:=w1;
    last:=true
    end (w1-hh)x h<0
end

ww<1.25;

h:=hh;
go to contb;
endp:
end sode567;

2. Method used. To solve the initial value problem

\[ y' = f(x, y), \quad y(x_0) \text{ given,} \]

we consider the one-step method \( \Phi \) which satisfies the following property:

When applied to the test equation

\[ y' = \lambda y, \quad y(x_0) = y_0, \quad \lambda \in \mathbb{C} \]

with a constant step \( h \) the \( m \)-stage method \( \Phi \) yields a numerical solution \( \{y_n\} \)

which satisfies a recurrence relation of the form

\[ y_{n+1} = w(z) y_n, \]

where \( z = h\lambda \) and

\[
(2.1) \quad w(z) = 1 + z + \ldots + \frac{z^p}{p!} + \sum_{i=p+1}^{m} \frac{a_i z^i}{i!}.
\]

The coefficients \( a_i \) \((i = p+1, p+2, \ldots, m)\) are depending on the parameters of the method \( \Phi \). Moreover, if the method \( \Phi \) has an error of order \( p \), then one has

\[ w(z) - e^z = O(z^{p+1}). \]
In this paper we propose the choice of the coefficients \( a_i \) \((i = p + 1, p + 2, \ldots, m)\) in (2.1) to be

\[(2.2) \quad w^*(z) - e^z = O(z^{m+2}),\]

where

\[w^*(z) = w^2(z/2) + \frac{w^2(z/2) - w(z)}{2^p - 1}\]

and

\[y_{n+1}^* = w^*(z) y_n^* .\]

The new numerical solution \( \{y_n^*\} \) \((y_k := y_k^*, 1 \leq k \leq n)\) is obtained by using Richardson's extrapolation applied to the solutions obtained with step \( h \) and step \( h/2 \).

Here we present two one-step methods \( \Phi_1 \) and \( \Phi_2 \) with \( m > p \). The method \( \Phi_1 \) has \( p = 1, m = 2 \), the method \( \Phi_2 \) has \( p = 5, m = 6 \). The method \( \Phi_1 \) is given by the following formulae

\[
\Phi_1: \quad y_{n+a} = y_n + ahf_n, \\
y_{n+1} = y_n + hf_{n+a},
\]

where \( a \) is the parameter ([2]).

For the method \( \Phi_1 \) the polynomial (2.1) has the form

\[w(z) = 1 + z + az^2.\]

In this paper the method \( \Phi_1 \) is realized with two values of the parameter \( a \), \( a = \frac{1}{6} \) and \( a = \frac{1}{3} \). For \( a = \frac{1}{6} \) the method \( \Phi_1 \) has the interval of absolute stability \((-7.0, 0)\) and there holds

\[w^*(z) - e^z = O(z^3).\]

For \( a = \frac{1}{3} \) the method \( \Phi_1 \) has the interval of absolute stability \((-3.0, 0)\) but there holds (2.2), i.e.

\[w^*(z) - e^z = O(z^4).\]

The method \( \Phi_1 \) in this paper is realized in the procedure sode123 with the parameter \( \text{sigma} \) \((a = \text{sigma})\).

The method \( \Phi_2 \) was given by Lawson [1] and has the following form
For the method $\Phi_2$ the polynomial (2.2) has the form

$$w(z) = \sum_{i=0}^{5} \frac{z^i}{i!} + \frac{36\sigma}{6!} z^6.$$

Lawson [1] has used this method with $\sigma = \frac{1}{4}$, with this value of the parameter $\sigma$ the method $\Phi_2$ has the interval of absolute stability $(-5.62, 0)$. For $\sigma = \frac{1}{4}$ ([2]) there holds (2.2), i.e.

$$w^*(z) - e^z = O(z^8)$$

and the interval of absolute stability is $(-4.25, 0)$. The method $\Phi_2$ is realized in the procedure sode567 with the parameter sigma ($\sigma = \text{sigma}$).

3. Certification. The procedures sode123 and sode567 were tested on the following problems.

Problem A

$$y'_1 = 10 \text{sgn} \sin(20x) y_2, \quad y_1(0) = 0,$$

$$y'_2 = -10 \text{sgn} \sin(20x) y_1, \quad y_2(0) = 1$$

with the exact solution

$$y_1(x) = |\sin 10x|, \quad y_2(x) = |\cos 10x|.$$

Problem B

$$y'_1 = \frac{1}{y_2}, \quad y_1(0) = 1,$$

$$y'_2 = -\frac{1}{y_1}, \quad y_2(0) = 1$$

with the exact solution

$$y_1(x) = e^x, \quad y_2(x) = e^{-x}.$$

Problem C

$$y' = \lambda y, \quad y(0) = 1$$

with the exact solution

$$y(x) = e^{\lambda x}.$$

Below the relative errors $(y_n - y(x_n))/y(x_n)$ and the numbers of function evaluations $f([f])$ are given. The results presented here were obtained by the procedures sode123, sode567 with automatic step size control and with sigma as the parameter in the procedures.

For problem A

<table>
<thead>
<tr>
<th>$\sigma = \frac{1}{4}$</th>
<th>$\sigma = \frac{1}{4}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$\text{eps} = 10^{-4}$</td>
</tr>
<tr>
<td>1.0</td>
<td>$-6.66_{10}^{-4}$</td>
</tr>
<tr>
<td>$-1.46_{10}^{-4}$</td>
<td></td>
</tr>
</tbody>
</table>
### sode567

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sigma = \frac{1}{64}$</th>
<th>$\sigma = \frac{1}{62}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{eps = } \frac{1}{10} \cdot 4$</td>
<td>$\text{eps = } \frac{1}{10} \cdot 4$</td>
</tr>
<tr>
<td>1.0</td>
<td>5.70_{10} \cdot -5</td>
<td>8756 &amp; -2.86_{10} \cdot -5 &amp; 9020</td>
</tr>
<tr>
<td>2.64_{10} \cdot -5</td>
<td>&amp; -2.21_{10} \cdot -5</td>
<td></td>
</tr>
</tbody>
</table>

For problem B

### sode123

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sigma = \frac{1}{4}$</th>
<th>$\sigma = \frac{1}{3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{eps = } \frac{1}{10} \cdot 6$</td>
<td>$\text{eps = } \frac{1}{10} \cdot 6$</td>
</tr>
<tr>
<td>.5</td>
<td>2.61_{10} \cdot -7</td>
<td>939 &amp; -2.12_{10} \cdot -7 &amp; 644</td>
</tr>
<tr>
<td>10.0</td>
<td>-2.61_{10} \cdot -7</td>
<td>&amp; -2.11_{10} \cdot -7</td>
</tr>
<tr>
<td>4.95_{10} \cdot -6</td>
<td>17763 &amp; 3.97_{10} \cdot -6 &amp; 12143</td>
<td></td>
</tr>
<tr>
<td>4.95_{10} \cdot -6</td>
<td>&amp; -3.96_{10} \cdot -6</td>
<td></td>
</tr>
</tbody>
</table>

### sode567

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sigma = \frac{1}{36}$</th>
<th>$\sigma = \frac{1}{42}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{eps = } \frac{1}{10} \cdot 3$</td>
<td>$\text{eps = } \frac{1}{10} \cdot 3$</td>
</tr>
<tr>
<td>10.0</td>
<td>-3.33_{10} \cdot -3</td>
<td>216 &amp; -1.39_{10} \cdot -2 &amp; 198</td>
</tr>
<tr>
<td>4.06_{10} \cdot -3</td>
<td>&amp; 1.83_{10} \cdot -2</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sigma = \frac{1}{64}$</th>
<th>$\sigma = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{eps = } \frac{1}{10} \cdot 3$</td>
<td>$\text{eps = } \frac{1}{10} \cdot 3$</td>
</tr>
<tr>
<td>10.0</td>
<td>9.32_{10} \cdot -3</td>
<td>234 &amp; 1.85_{10} \cdot -2 &amp; 252</td>
</tr>
<tr>
<td>-1.22_{10} \cdot -2</td>
<td>&amp; -2.28_{10} \cdot -2</td>
<td></td>
</tr>
</tbody>
</table>

For problem C

### sode567

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\sigma = \frac{1}{64}$</th>
<th>$\sigma = \frac{1}{42}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{eps = } \frac{1}{10} \cdot 9$</td>
<td>$\text{eps = } \frac{1}{10} \cdot 9$</td>
</tr>
<tr>
<td>-6</td>
<td>1.10_{10} \cdot -9</td>
<td>628 &amp; -2.98_{10} \cdot -10 &amp; 509</td>
</tr>
<tr>
<td>-1</td>
<td>1.38_{10} \cdot -10</td>
<td>118 &amp; -1.97_{10} \cdot -11 &amp; 101</td>
</tr>
<tr>
<td>1</td>
<td>-1.92_{10} \cdot -10</td>
<td>118 &amp; -4.28_{10} \cdot -11 &amp; 101</td>
</tr>
<tr>
<td>6</td>
<td>-1.20_{10} \cdot -9</td>
<td>610 &amp; -2.40_{10} \cdot -10 &amp; 525</td>
</tr>
</tbody>
</table>
References


INSTITUTE OF COMPUTER SCIENCE
UNIVERSITY OF WROCLAW
51-151 WROCLAW

Received on 1983.05.20