Extended rotation of the covariant vector density

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Abstract. In the present note a notion of extended rotation $G_{ab}$ of an arbitrary vector density $w_i$ of weight $-p$, $p \neq 0$ is defined axiomatically (Definition 1). Under assumption $1^o$, $2^o$, $3^o$ it is proved that such a rotation has the form

$$G_{ab} = C_{ab}^i w_i + w_{(a,b)},$$

where $C_{ab}^i$ is an object with transformation formula

$$C_{ab}^i = A_{a}^i A_{b}^j A_{c}^k C_{cd}^l + p (\ln |J|)_{a} (\delta_{b}^{c} \\ \delta_{c}^{d}).$$

The object $C_{ab}^i$ can be prescribed in the following form:

$$C_{ab}^i = W_{ab}^i - \frac{2}{n-1} C_{(a} \delta_{b)}^{i},$$

where $W_{ab}^i$ is a tensor and $C_a$ has transformation rule:

$$C_{a'} = A_{a}^c C_{c} - \frac{n-1}{2} p (\ln |J|)_{c} a'.$$

Some further properties of the extended rotation are investigated by means of objects $W_{ab}^i, C_a, D_a = \frac{2}{n-1} C_a$.

Introduction. Let $w_i$ be a covariant vector and let $G_{ab}$ be a differential concomitant of the first order of $w_i$.

We assume that the $G_{ab}$ satisfies the following conditions:

(i) $G_{ab}$ is additive with respect to $(w_i, \partial_j w_i)$,
(ii) $G_{ab}$ satisfies a certain Leibniz rule concerning the product of $w_i$ by a scalar $\sigma$ and
(iii) $G_{ab}$ is a covariant tensor of valence $(2, 0)$.

In [4] the notion of rotation has been derived from the above hypotheses.

In the present note we define axiomatically the extended rotation of an arbitrary covariant vector density $w_i$ of weight $-p$, $p \neq 0$, but we do not assume that it is a differential concomitant of the first order of $w_i$. 
Let $X^n$ be an $n$-dimensional manifold. If transformation of the coordinate system has the form
\[ x^{i'} = x^{i'}(x^i), \quad i, i' = 1, 2, \ldots, n, \]
then we put
\[ A_i^{i'} = \frac{\partial x^{i'}}{\partial x^i}, \]
\[ A_i^i = \frac{\partial x^i}{\partial x^{i'}}, \quad J = \text{Det} ||A_i^i||. \]

1. Let $w_i$ be a covariant vector density of weight $-p$, $p \neq 0$. Then the transformation rule of $w_i$ has the following form:

\[ w_i = \varphi(J) A_i^i w_i, \quad (1.1) \]

where
\[ \varphi(J) = \begin{cases} |J|^p & \text{for a } W\text{-density}, \\ (\text{sgn } J)|J|^p & \text{for a } G\text{-density}. \end{cases} \]

If we denote by $U, j$ the partial derivatives of a function $U$, then we have from (1.1)

\[ w_{i,j'} = \varphi(J)_{,i} A_i^i w_i + \varphi(J) A_i^i_{,j'} w_i + \varphi(J) A_i^i A_j^j w_{i,j}. \quad (1.2) \]

The object $(w_i, w_{i,j})$ is called the differential extension of the first order of $w_i$. Its transformation rule is defined by (1.1) and (1.2).

Now we formulate the following definition:

**DEFINITION 1.** A system of $n^2$ functions
\[ G_{ab}, \quad a, b = 1, 2, \ldots, n, \]
is called the extended rotation of the covariant $W$- (or $G$-) vector density (1.1) of weight $-p$, $p \neq 0$, if it satisfies:

1° $G_{ab}$ depends on $w_i, w_{i,j}$:

\[ G_{ab} = G_{ab}(w_i, w_{i,j}); \quad (1.3) \]

2° $G_{ab}$ is additive with respect to $(w_i, w_{i,j})$:

\[ G_{ab}(w_i + w_i', w_{i,j} + w_{i,j}') = G_{ab}(w_i, w_{i,j}) + G_{ab}(w_i', w_{i,j}'); \quad (1.4) \]

($w_i$ and $w_{i,j}$ are covariant vector densities of weight $-p$);

3° $G_{ab}$ satisfies the following Leibniz rule for the product of $w_i$ by a scalar $\sigma$:

\[ G_{ab}[\sigma w_i, (\sigma w_i)_j] = \sigma G_{ab}(w_i, w_{i,j}) + w_i [\sigma, b]; \quad (1.5) \]

4° $G_{ab}$ is a $W$- (or $G$-) tensor density of weight $-p$ and valence $(0, 2)$:

\[ G_{a'b'} = \varphi(J) A_a^{a'} A_{b'}^b G_{ab}. \quad (1.6) \]
From (1.4) and (1.5) it follows that $G_{ab}$ is a linear function with respect to $(w_i, w_{i,j})$. In fact, putting $\sigma = a = \text{const}$ into (1.5) we have

$$G_{ab}(aw_i, aw_{i,j}) = aG_{ab}(w_i, w_{i,j}).$$

Hence we obtain the following form of $G_{ab}$:

$$(1.7) \quad G_{ab}(w_i, w_{i,j}) = C^i_{ab}w_i + C^j_{ab}w_{i,j},$$

where

$$C^i_{ab} = \text{const}, \quad C^j_{ab} = \text{const}, \quad a, b, i, j = 1, \ldots, n.$$  

Putting (1.7) into (1.5) we find

$$(C^j_{ab} - \delta^j_{[a} \delta^i_{b]})w_i \sigma_j = 0,$$

where $\delta^i_a$ is the general Kronecker delta. Since $w_i$ and $\sigma_j$ are arbitrary, we have

$$(1.8) \quad C^j_{ab} = \delta^j_{[a} \delta^i_{b]}.$$

Inserting (1.8) into (1.7) we obtain

$$(1.9) \quad G_{ab} = C^i_{ab}w_i + w_{[a,b]}.$$

Now we determine the transformation rule of the $C^i_{ab}$. From (1.6) and (1.9) it follows that

$$(1.10) \quad C^\nu_{\alpha\beta}w_{\nu} + w_{[\alpha^*, \beta]} = \varphi(J)A^\alpha_{\alpha'}A^\beta_{\beta'}(C^i_{ab}w_i + w_{[a,b]}).$$

Putting (1.1) and (1.2) into (1.10) it is easily seen that

$$C^\nu_{\alpha\beta}A^\mu_{\nu}w_i = [A^\alpha_{\alpha'}A^\beta_{\beta'}C^i_{ab} + p(\ln |J|), [\alpha^*, \beta]]w_i.$$  

Since here $w_i$ are arbitrary, we get

$$(1.11) \quad C^\nu_{\alpha\beta} = A^\nu_{\alpha'}(A^\mu_{\beta'}C^i_{ab} + p(\ln |J|), [\alpha^*, \beta]).$$

We have thus obtained the following

**Theorem 1.** Every extended rotation of a covariant $W$- (or $G$-) vector density of weight $-p$, $p \neq 0$, has the form

$$G_{ab}(w_i, w_{i,j}) = C^i_{ab}w_i + w_{[a,b]},$$

where $C^i_{ab}$ is an object with transformation formula (1.11).

Let $C^i_{ab}$ be the object with transformation formula (1.11) and

$$(1.12) \quad C_{a} = \frac{dt}{dt}C^i_{ia}.$$  

From (1.11) it follows that $C_{a}$ is an object with transformation rule

$$(1.13) \quad C'_{a} = A^a_{\alpha'}C_{a} - \frac{n-1}{2}p(\ln |J|)_{,a'}.$$
Now we assume that \( n \geq 2 \) and put
\[
W_{ab}^i \overset{dt}{=} C_{ab}^i + \frac{2}{n-1} C_{[a}^i \delta_{b]}^i.
\]

It is easily seen that \( W_{ab}^i \) is a tensor of valence \( 1, 2 \). Thus we have the following corollaries:

**Corollary 1.** The object \( C_{ab}^i \) has the form
\[
C_{ab}^i = W_{ab}^i - \frac{2}{n-1} C_{[a}^i \delta_{b]}^i,
\]
where \( W_{ab}^i \) is a tensor of valence \( 1, 2 \) and \( C_a \) is an object with transformation \( (1.13) \).

**Corollary 2.** \( C_{[ab]}^i \overset{dt}{=} \frac{1}{2}(C_{ab}^i + C_{ba}^i) \) is a tensor, \( C_{[ab]}^i \overset{dt}{=} \frac{1}{2}(C_{ab}^i - C_{ba}^i) \) has precisely the same transformation rule as that of \( C_{ab}^i \).

**Corollary 3.** The extended rotation is an antisymmetric tensor density if and only if
\[
W_{ab}^i = W_{[ab]}^i.
\]

2. Let us consider a field of an object \( C_{ab}^i \) of class \( C^1 \) on \( X^n, n \geq 2 \). The object
\[
D_a \overset{dt}{=} \frac{2}{n-1} C_a,
\]
where \( C_a = C_{ia}^i \) has the following transformation formula:
\[
D_{a'} = A^a_a D_a - p (\ln |J|)_{a}.
\]

M. Kucharzewski has proved in [2] that every covariant derivative of a density \( g \) of weight \( -p \), \( p \neq 0 \), has the form
\[
F_i(g, g, j) = g, i + K_i g,
\]
where \( K_i \) is an object with transformation rule \( (2.2) \).

Now we introduce the following notation:
\[
w_{a;b}^i \overset{dt}{=} C_{ab}^i w_i + w_{[a,b]}^i,
\]
\[
g_{,a} \overset{dt}{=} g_{,a} + D_a g,
\]
\[
V_{ab} \overset{dt}{=} D_{[a,b]}.
\]

It is known (cf. [3], p. 83) that the following equality is true:
\[
\text{RotGrad} \sigma = 0,
\]
where \( \sigma \) is a scalar field of class \( C^1 \). We shall require that the extended rotation in the sense of Definition 1 satisfies

\[
\mathbf{g}_{[a|b]} = 0,
\]

for any density \( \mathbf{g} \) of weight \(-p\), \( p \neq 0 \).

We prove the following

**Theorem 2.** The extended rotation fulfills (2.5) if and only if

\[
\mathbf{W}^i_{ab} = 0, \quad \mathbf{V}_{ab} = 0.
\]

**Proof.** From (1.9), (2.1), (2.3) and (2.4) it follows that

\[
\mathbf{g}_{[a|b]} = (\mathbf{C}^i_{ab} + \mathbf{D}_{[a} \delta^i_{b]} \mathbf{g})_{,i} + (\mathbf{C}^i_{ab} \mathbf{D}_i + \mathbf{V}_{ab}) \mathbf{g}.
\]

If \( \mathbf{g}_{[a|b]} = 0 \) for every density \( \mathbf{g} \) of weight \(-p\), \( p \neq 0 \), then we obtain

\[
\mathbf{C}^i_{ab} + \mathbf{D}_{[a} \delta^i_{b]} = 0, \quad \mathbf{C}^i_{ab} \mathbf{D}_i + \mathbf{V}_{ab} = 0.
\]

Thus we have (2.8).

If (2.6) holds, then it is easily seen that \( \mathbf{g}_{[a|b]} = 0 \).

This completes the proof.

**Remarks.**

I. Let \( \mathbf{X}^n \) be an \( L^n \) and let \( \Lambda^i_{ab} \) be an object of the linear displacement in \( L^n \). Then we can define the extended rotation as follows:

\[
\mathbf{W}^i_{ab} \overset{\text{at}}{=} \mathbf{S}^i_{ab}, \quad C^i_a \overset{\text{at}}{=} \mathbf{n} \frac{1}{2} p \Lambda^i_a,
\]

where \( \mathbf{S}^i_{ab} = \Lambda^i_{[ab]}, \mathbf{A}_a = \Lambda^i_{ai} \).

If \( w_i \) is a covariant vector density of weight \(-p\), then we put

\[
w_{[a|b]} \overset{\text{at}}{=} (\mathbf{S}^i_{ab} - p \Lambda^i_{[a} \delta^i_{b]}) w_i - w_{[a|b]}.
\]

Such an extended rotation coincides with the alternation of the covariant derivative of \( w_i \) with respect to \( \Lambda^i_{ab} \).

II. Now we assume that in \( \mathbf{X}^n \) there is given a field of an object \( \mathbf{A}_a \) with transformation formula

\[
\mathbf{A}_{a'} = \mathbf{A}^a_a \mathbf{A}_a - (\ln |J|)_{,a'}.
\]

Then we can define the following extended rotation of a covariant vector density of weight, \(-p\), \( p \neq 0 \):

\[
w_{[a|b]} \overset{\text{at}}{=} -p \Lambda^i_{[a} \delta^i_{b]} w_i + w_{[a|b]}.
\]

In this case we have \( \mathbf{W}^i_{ab} = 0 \).

(1) In \( \mathbf{X}^n \) a field of an object \( \Lambda^i_{ab} \) with transformation rule

\[
\Lambda^i_{a'b'} = \mathbf{A}^i_a \mathbf{A}^a_a \mathbf{A}^i_b \mathbf{A}^b_b \Lambda^i_{ab} + \mathbf{A}^i_a \mathbf{A}^i_a
\]

is given.
We notice that $W_{ab}^i = 0$ if and only if $g_{,ab} = V_{ab}g$ for every density of weight $-p$, $p \neq 0$.

III. Let $w_i$ be a covariant vector density of weight $p = 0$ (i.e. $w_i$ is a covariant vector or a $J$-vector). It is easily seen (1°, 2°, 3°, (1.9) and (1.11)) that

$$G_{ab}(w_i, w_{i,j}) = C_{ab}^i w_i + w_{[a,b]}^i,$$

where $C_{ab}^i$ is an arbitrary tensor of valence (1.2).

If we assume (cf. [4]) that $G_{ab}$ is a differential concomitant of the first order of $w_i$, then we have

$$C_{ab}^i = 0,$$

i.e.

$$G_{ab}(w_i, w_{i,j}) = w_{[a,b]}.$$

References


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