On non-linear Volterra integral-functional equations in several variables

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Abstract. Let $B$ be an arbitrary Banach space and $G \subset R^p_+$ be a compact set, where $R_+ = [0, +\infty)$. Assume that the functions $F \in C(G \times B^n \times B)$, $f_i \in C(G \times B)$, $i = 1, \ldots, m$, $\beta \in C(G, G)$, $\alpha_i \in C(G, G)$, $i = 1, \ldots, m$, are given and $\beta(x) \leq x$, $\alpha_i(x) \leq x$ for $x \in G$, $i = 1, \ldots, m$. $(C(X, Y)$ denotes the class of continuous functions defined on $X$ with range in $Y$).

In the paper the non-linear Volterra integral-functional equation

\[ u(x) = F\left(x, \int_{H_{1}(x)} f_1(x, s, u(\alpha_1(s))) (ds) \right)_{p_1}, \ldots, \int_{H_m(x)} f_m(x, s, u(\alpha_m(s))) (ds) \right)_{p_m}, u(\beta(x)) \right) \quad x \in G, \]

with $H_j(x) \subset E(x) = \{ \xi : \xi \in G, \xi \leq x \}$ for $x \in G$, $j = 1, \ldots, m$, is considered.

In the first part of the paper equation (V) is discussed by means of a comparative method. If $F$ and $f_i$ satisfy the Lipschitz condition with respect to all variables except $x$ or $x$, respectively, then, under certain additional assumptions concerning the functions $\beta$, $\alpha_j$ and the Lipschitz coefficients, it is proved that there exists exactly one (in a certain class of functions) continuous solution of (V). This solution is the limit of the sequence of successive approximations. It is not assumed that the Lipschitz coefficient $k$ of the function $F$ with respect to the last variable satisfies the condition $k < 1$.

The second part of the paper deals with equation (V) considered in a finite dimensional Banach space. A theorem on the existence of at least one solution of equation (V) is proved. Also in this case conditions milder than $k < 1$ are assumed.

Introduction. Let $B$ be an arbitrary Banach space with norm $\| \cdot \|$. Denote by $C(X, Y)$ the set of all continuous functions defined in $X$ taking values in $Y$, $X$, $Y$ being arbitrary metric spaces. For $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in R^n$ ($R^n$ - real Euclidean space of dimension $n$) we define $x \leq y$ as $x_i \leq y_i$ for $i = 1, \ldots, n$. We denote by $| \cdot |$ the Euclidean norm in $R^n$. Let $G \subset R^p_+$ be a compact set, where $R_+ = [0, +\infty)$. Let

\[ E(x) = \{ \xi : \xi \in G, \xi \leq x \}. \]

Assume that the functions $F \in C(G \times B^n \times B)$, $f_i \in C(G \times B)$, $i = 1, \ldots, m$, $\beta \in C(G, G)$, $\alpha_i \in C(G, G)$, $i = 1, \ldots, m$, are given and $\beta(x) \leq x$, $\alpha_i(x) \leq x$ for $x \in G$, $i = 1, \ldots, m$. 
We shall consider the non-linear Volterra integral-functional equation

\[(1) \quad u(x) = F\left(x, \int_{H_{1}(x)} f_{1}(x, s, u(\alpha_{1}(s))) (ds)_{p_{1}}, \ldots \right.
\]

\[\ldots, \int_{H_{m}(x)} f_{m}(x, s, u(\alpha_{m}(s))) (ds)_{p_{m}}, u(\beta(x)) \right) \quad x \in G,
\]

where \(H_{j}(x) \subset E(x)\) for \(x \in G, j = 1, \ldots, m\).

We assume further that \(H_{j}(x)\) is contained in a \(p_{j}\)-dimensional hyperplane \((1 \leq p_{j} \leq n)\), parallel to the coordinate axes, and it is Lebesgue measurable, considered as a \(p_{j}\)-dimensional set. Let \(L_{p_{j}}(H_{j}(x))\) denotes the \(p_{j}\)-dimensional Lebesgue measure of \(H_{j}(x)\). We assume that \(p_{j}\) does not depend on \(x\).

If a \(p_{j}\)-dimensional hyperplane containing the set \(H_{j}(x)\) and being parallel to the coordinate axes is defined by the equations

\[x_{r_{1}} = \hat{x}_{r_{1}}, \quad x_{r_{2}} = \hat{x}_{r_{2}}, \quad x_{r_{r}} = \hat{x}_{r_{r}}, \quad r = n - p_{j},\]

then \(\int_{H_{j}(x)} g(x, s) (ds)_{p_{j}}, \) where \(s = (s_{1}, \ldots, s_{n})\), denotes the \(p_{j}\)-dimensional Lebesgue integral in the space \(Ox_{m_{1}}x_{m_{2}} \ldots x_{m_{p_{j}}}\), \(m_{i} \in \{1, \ldots, n\} - \{t_{1}, \ldots, t_{r}\}\), and \(s_{t_{1}} = \hat{x}_{t_{1}}, s_{t_{2}} = \hat{x}_{t_{2}}, \ldots, s_{t_{r}} = \hat{x}_{t_{r}}\).

Let \(A' = \{i: p_{i} = n\}, B' = \{i: 1 \leq p_{i} < n\}\). By changing notation, if necessary, we may assume that \(A' = \{1, \ldots, k_{0}\}, B' = \{k_{0} + 1, \ldots, m\}\).

We define the sets \(\sigma_{j} \subset \{1, \ldots, n\}, j = 1, \ldots, m,\) in the following way: if the axis \(Ox_{i}\) is parallel to the \(p_{j}\)-dimensional hyperplane in which the set \(H_{j}(x)\) is contained, then \(i \in \sigma_{j}\). Put \(\bar{\sigma}_{j} = \{1, \ldots, n\} - \sigma_{j}\).

For each \(x \in G\) and \(j = 1, \ldots, m\) we introduce the set \(G_{j}(x)\) by

\[G_{j}(x) = \{s: s = (s_{1}, \ldots, s_{n}), s_{t_{i}} = \hat{x}_{t_{i}} \quad \text{for} \quad t_{i} \in \sigma_{j}, \quad 0 \leq s_{t_{i}} \leq \varphi_{i}^{(j)}(x) \quad \text{for} \quad t_{i} \in \bar{\sigma}_{j}\},\]

where \((\varphi_{1}^{(j)}, \ldots, \varphi_{m}^{(j)}) = \varphi_{j} \in C(G, R_{m})\), \(t_{i} \in \bar{\sigma}_{j}\), and \(H_{j}(x) \subset G_{j}(x) \subset E(x)\). The \(p_{j}\)-dimensional Lebesgue-measure of \(G_{j}(x)\) satisfies \(L_{p_{j}}(G_{j}(x)) = \prod_{a \in \sigma_{j}} \varphi_{a}^{(j)}(x)\).

We adopt the following notations:

\[
\int_{H_{(x)}} f(x, s, z(\alpha(s))) ds = \left( \int_{H_{1}(x)} f_{1}(x, s, z(\alpha_{1}(s))) (ds)_{p_{1}}, \ldots \right.
\]

\[\ldots, \int_{H_{m}(x)} f_{m}(x, s, z(\alpha_{m}(s))) (ds)_{p_{m}} \right)
\]

\[L(G(x)) = \left( L_{p_{1}}(G_{1}(x)), \ldots, L_{p_{m}}(G_{m}(x)) \right);\]

if \(K = (K_{1}, \ldots, K_{m}) \in C(G, R_{m})\), then

\[K(x) \int_{H_{(x)}} f(x, s, z(\alpha(s))) ds = \sum_{j=1}^{m} K_{j}(x) \int_{H_{j}(x)} f_{j}(x, s, z(\alpha_{j}(s))) (ds)_{p_{j}}, \quad x \in G,\]
and

\[ K(x) \int_{H(x)} z(x(s))ds = \sum_{j=1}^{m} K_j(x) \int_{H_j(x)} z(x_j(s))(ds)_{p_j}, \quad x \in G. \]

For \( K \in C(G, R^m) \) we define

\[ K(x)L(G(x)) = \sum_{j=1}^{m} K_j(x)L_{p_j}(G_j(x)), \quad x \in G. \]

Equation (1) will be written briefly

\[ (2) \quad u(x) = F\left(x, \int_{H(x)} f(x, s, u(x(s)))ds, u(\beta(x))\right), \quad x \in G. \]

There are various problems which lead to Volterra integral-functional equations of type (2). Perhaps the simplest problem in the theory of differential equations which leads to such an equation with \( n = 1 \) is the initial-value problem for the ordinary differential-functional equation of the neutral type

\[ u'(t) = F\left(t, u(\alpha_1(t)), \ldots, u(\alpha_n(t)), u'(\beta(t))\right), \quad t \in [0, a], \quad u(0) = u_0. \]

Therefore equation (2) is a generalization of equations which have been considered in paper [3] and also of some cases of equations considered in [1], [2], [5], [7], [15].

The various initial value problems for the partial hyperbolic differential-functional equation of the neutral type

\[ z_{xy}(x, y) = F\left(x, y, z(\alpha_1^{(0)}(x, y), \alpha_2^{(0)}(x, y)), z_x(\alpha_1^{(1)}(x, y), \alpha_2^{(1)}(x, y)), z_y(\alpha_1^{(2)}(x, y), \alpha_2^{(2)}(x, y))\right), \]

can be reformulated in terms of Volterra integral-functional equations. Let us consider as an example the Darboux problem, where the domain is a rectangle \( \{(x, y): x \in [0, a], y \in [0, b]\} \), and where initial values \( u(x, 0) = \sigma(x), \quad x \in [0, a], \quad u(0, y) = \tau(y), \quad y \in [0, b] \) are prescribed. The Volterra integral-functional equation corresponding to that problem is

\[ u(x, y) = F\left(x, y, \sigma(\alpha_1^{(0)}(x, y)) + \tau(\alpha_2^{(0)}(x, y)) - \sigma(0) + \int_{H_0(x, y)} u(s, t)ds dt, \sigma'(\alpha_1^{(1)}(x, y)) + \int_{H_1(x, y)} u(s, t)ds dt, \tau'(\alpha_2^{(2)}(x, y)) + \int_{H_2(x, y)} u(s, t)ds, u(\beta_1(x, y), \beta_2(x, y))\right), \quad (x, y) \in [0, a] \times [0, b], \]

where

\[ H_0(x, y) = \{(s, t): s \in [0, \alpha_1^{(0)}(x, y)], t \in [0, \alpha_2^{(0)}(x, y)]\}, \]

\[ H_1(x, y) = \{(s, t): s = \alpha_1^{(1)}(x, y), t \in [0, \alpha_2^{(1)}(x, y)]\}, \]

\[ H_2(x, y) = \{(s, t): s \in [0, \alpha_1^{(2)}(x, y)], t = \alpha_2^{(2)}(x, y)\}. \]
Therefore our equation is a generalization of the equation which was considered in paper [6] and of an adequate case of the equation discussed in [4].

The Cauchy problem and the Goursat problem for hyperbolic differential-functional equations leads to a Volterra integral-functional equation of type (2) (see [13]).

Similar initial value problems for equations in more than two variables and problems for equations of higher order can be reformulated in terms of Volterra integral-functional equations.

As a particular case of equation (2) we can obtain the system of Volterra integral equations which was considered by W. Walter in paper [12] and monograph [13]. These papers contain the extensive bibliography concerning Volterra integral equations.

In the case when \( u \) is a function of several variables equation (2) is a generalization of equations which have been considered in [8]–[11].

In this paper we give theorems concerning the existence and uniqueness of continuous solutions of (2) in a certain class of functions.

The paper is divided into two parts. In the first part we investigate equation (2) by means of the comparative method. A general formulation of this method can be found in paper [14]. If we assume that \( F \) and \( f_i \) satisfy the Lipschitz condition with respect to all variables except \( x \) or \( x, s \), respectively, then we prove, under certain additional assumptions concerning the functions \( \beta \), \( \alpha_j \) and the Lipschitz coefficients, that there exists exactly one (in a certain class of functions) continuous solution of (2). This solution is the limit of a sequence of successive approximations. This result is obtained by means of the comparative method.

The essential fact in our considerations is that we do not assume that the Lipschitz coefficient \( k \) of the function \( F \) with respect to the last variable satisfies the condition \( k < 1 \) (see Lemmas 4–9). If \( k < 1 \), then we have a theorem on the existence and uniqueness of solutions of (2), which can be obtained by means of the Banach fixed-point theorem.

The second part of the paper concerns equation (2) considered in a finite dimensional Banach space. We prove here a theorem on the existence of at least one solution of equation (2). In this case it is an important fact that we also do not assume that the Lipschitz coefficient \( k \) of the function \( F \) with respect to the last variable satisfies the condition \( k < 1 \) (see Lemma 14). This part of the paper is an extension of the result contained in paper [3], where an equation of type (2) with the unknown function of one variable was considered.

Remark 1. Let

\[
G^*(x) = \{\xi : \xi \leq x\}, \quad \tilde{G} = \bigcup_{s \in G} G^*(s).
\]
(We do not assume that \( G^* < G \).) Suppose that the functions \( F \in C (G \times B^m \times B, B) \), \( f_i \in C (G^2 \times B, B) \), \( \alpha_i \in C (G, \tilde{G}) \), \( i = 1, \ldots, m \), \( \beta \in C (G, \tilde{G}) \), \( \varphi \in C (\tilde{G} - G, B) \) are given and \( \beta (x) \leq x \), \( \alpha_i (x) \leq x \) for \( x \in G \), \( i = 1, \ldots, m \).

Let us consider the equation

\[
u (x) = F \left( x, \int_{H(x)} f_i \left( x, s, u (\alpha_i (s)) \right) \right) \frac{d\nu_1, \ldots, \int_{H(x)} f_m \left( x, s, u (\alpha_m (s)) \right) \right) \frac{d\nu_m^m, u (\beta (x))}{x \in G,} \]

\[
(1')

\[
\nu (x) = \varphi (x) \text{ for } x \in \tilde{G} - G,
\]

where \( H(x) \subset \tilde{G} \). (We do not assume that the sets \( H(x) \) satisfy the condition \( H(x) \subset G \).

We want to point out that equation \((1')\) is equivalent to some equation of type \((1)\). We shall prove this only for the case \( m = 1 \), i.e., for the equation

\[
u (x) = F \left( x, \int_{H(x)} f \left( x, s, u (\alpha (s)) \right) \right) \frac{d\nu, u (\beta (x))}{x \in G,} \]

\[
(1'')

\[
u (x) = \varphi (x) \text{ for } x \in \tilde{G} - G.
\]

We define for \( x \in G \)

\[
\tilde{H} (x) = \{s: s \in H(x) \cap G \text{ and } \alpha (s) \in G\}, \quad \tilde{H} (x) = H(x) - \tilde{H} (x).
\]

Then we have

\[
\int_{H(x)} f \left( x, s, u (\alpha (s)) \right) \frac{d\nu}{x \in \tilde{H}(x)} = \int_{H(x)} f \left( x, s, u (\alpha (s)) \right) \frac{d\nu}{x \in \tilde{H}(x)} + \int_{\tilde{H}(x)} f \left( x, s, \varphi (\alpha (s)) \right) \frac{d\nu}{x \in \tilde{H}(x)}.
\]

Let

\[
\tilde{\alpha} = \{x \in G: \beta (x) \in G\}, \quad \tilde{\beta} = \{x \in G: \beta (x) \in \tilde{G} - G\}.
\]

Let \( \tilde{\beta} \) be a function satisfying the following conditions:

(a) \( \tilde{\beta} \in C (G, G) \),

(b) \( \tilde{\beta} (x) = \beta (x) \) for \( x \in \tilde{\alpha} \), \( \tilde{\beta} (x) \leq x \) for \( x \in G \).

Put

\[
\tilde{F} (x, u, v) = \begin{cases} F (x, u, v) & \text{for } x \in \tilde{\alpha}, \\ F (x, u, \varphi (\beta (x))) & \text{for } x \in \tilde{\beta}.
\end{cases}
\]

Now equation \((1'')\) is equivalent to the equation

\[
u (x) = F \left( x, \int_{H(x)} f \left( x, s, u (\alpha (s)) \right) \right) \frac{d\nu, \int_{\tilde{H}(x)} f \left( x, s, \varphi (\alpha (s)) \right) \right) \frac{d\nu, u (\beta (x))}{x \in G,}
\]

which is of type \((1)\).
PART I

1. Assumptions. Let \( x \in G, \ h \in R^n, \ x + h \in G, \ i \in B' \). Suppose that the set \( H_i(x) \) is contained in a \( p_i \)-dimensional hyperplane \( (1 \leq p_i < n) \) parallel to the \( n-p_i \) coordinate axes. We denote this hyperplane by \( S_i(x) \). Let the set \( H_i(x + h) \) be contained in a \( p_i \)-dimensional hyperplane \( S_i(x + h) \) parallel to the hyperplane \( S_i(x) \). There exists a vector \( t_i(x, h) \in R^n \) such that the set \( -t_i(x, h) + H_i(x + h) \) is contained in \( S_i(x) \).

We introduce

Assumption H₁ (see [12], p. 970; [13], p. 134). Suppose that:

1° for \( i \in A' \) we have \( \lim_{h \to 0} L_n[H_i(x) \Delta H_i(x + h)] = 0 \) uniformly with respect to \( x \in G \) (the sign \( \Delta \) denotes the symmetric difference of two sets),

\( Z_2 \) for \( i \in B' \) we have, uniformly with respect to \( x \in G \),

(a) \( \lim_{h \to 0} t_i(x, h) = 0 \),

(b) \( \lim_{h \to 0} L_n[H_i(x) \Delta (-t_i(x, h) + H_i(x + h))] = 0 \).

Remark 2. If \( x \in S_i(x) \) for \( x \in G \) and \( i \in B' \), we may assume that \( t_i(x, h) = h \). Condition (a) of Assumption H₁ is satisfied in this case.

Assumption H₂. Suppose that:

1° the functions \( \kappa, \tilde{h} \in C(G, R_+) \), \( K = (K_1, \ldots, K_m) \in C(G, R^n) \), \( \beta \in C(G, G) \) are given and \( \beta(x) \leq x \) for \( x \in G \),

\( Z_2 \) we have

\[ m(x) = \sum_{i=0}^{\infty} k^{(i)}(x) \tilde{h}(\beta^{(i)}(x)) < +\infty \quad \text{for} \ x \in G, \]

where

\[ k^{(0)}(x) = 1 \quad \text{for} \ x \in G, \quad k^{(i+1)}(x) = k(x)k^{(i)}(\beta(x)) \quad \text{for} \ x \in G, \ i = 0, 1, 2, \ldots, \]

\[ \beta^{(0)}(x) = x \quad \text{for} \ x \in G, \quad \beta^{(i+1)}(x) = \beta(\beta^{(i)}(x)) \quad \text{for} \ x \in G, \ i = 0, 1, 2, \ldots, \]

3° we have

\[ M(x) = \sum_{i=0}^{\infty} k^{(i)}(x) [K(\beta^{(i)}(x))L(G(\beta^{(i)}(x)))] < +\infty \quad \text{for} \ x \in G, \]

4° \( M, \tilde{m} \in C(G, R_+) \), the function

\[ \tilde{M}(x) = \sum_{i=0}^{\infty} k^{(i)}(x) \left[ \sum_{j=1}^{m} K_j(\beta^{(i)}(x))L_{p_j}(G_j(\beta^{(i)}(x))(\prod_{x \in G} x_j)^{-1}) \right] \]

is bounded for \( x \in G \).
We adopt the following notation:

\[ \bar{m}(x) = \sum_{i=0}^{\infty} k^{i}(x) \bar{h}(\beta^{i}(x)), \]

\[ (Vz)(x) = \sum_{i=0}^{\infty} k^{i}(x) \int_{H(\beta^{i}(x))} z(\alpha(s)) ds. \]

Remark 3. If
(a) conditions 1° H₂ - 3° H₂ are satisfied,
(b) \( \bar{h} \in C(G, R_{+}), \bar{h}(x) \leq \bar{h}(x) \) for \( x \in G \),
(c) \( z \) is a non-negative and upper-semicontinuous function,
then \( \bar{m} \) and \( Vz \) are functions defined in \( G \).

2. The main lemma.

Lemma 1. If Assumptions H₁, H₂ are satisfied and \( \bar{h} \in C(G, R_{+}), \bar{h}(x) \leq \bar{h}(x) \) for \( x \in G \), then:

1° There exist solutions \( \bar{z}, \tilde{z} \in C(G, R_{+}) \) of the equations

\[ z(x) = \bar{m}(x) + (Vz)(x), \quad x \in G \]

and

\[ z(x) = \bar{m}(x) + (Vz)(x), \quad x \in G, \]

respectively. The solutions \( \bar{z} \) and \( \tilde{z} \) of (5) and (6), respectively, are unique in the set \( M(G, R_{+}) \) of non-negative upper-semicontinuous functions.

2° The functions \( \bar{z} \) and \( \tilde{z} \) are solutions of the equations

\[ z(x) = K(x) \int_{H(x)} z(\alpha(s)) ds + k(x) z(\beta(x)) + \bar{h}(x), \quad x \in G \]

and

\[ z(x) = K(x) \int_{H(x)} z(\alpha(s)) ds + k(x) z(\beta(x)) + \tilde{h}(x), \quad x \in G, \]

respectively. Moreover, these solutions are unique in the class \( \tilde{M}(G, R_{+}, \bar{z}), \)

where

\[ \tilde{M}(G, R_{+}, \bar{z}) = \{ z : z \in M(G, R_{+}) \text{ and } \inf [c : z(x) \leq c\bar{z}(x)] < +\infty \}. \]

The function \( \tilde{z} \) satisfies the condition

\[ \lim_{r \to \infty} k^{r}(x) \tilde{h}(\beta^{r}(x)) = 0 \]

uniformly with respect to \( x \in G \).

3° The function \( z(x) = 0 \) for \( x \in G \) is the unique solution of the inequality

\[ z(x) \leq K(x) \int_{H(x)} z(\alpha(s)) ds + k(x) z(\beta(x)), \quad x \in G, \]

in the class \( \tilde{M}(G, R_{+}, \bar{z}). \)
4° If \( k, \tilde{h}, \tilde{h}, K, \beta \) are non-decreasing in \( G \) and \( H_j(x) \subset H_j(\tilde{x}) \) for \( x < \tilde{x}, x, \tilde{x} \in G, j = 1, 2, \ldots, m \), then \( \bar{z} \) and \( \tilde{z} \) are non-decreasing in \( G \).

Proof. We shall show that equation (5) has exactly one solution in the set \( M(G, R_+) \). Let \( T \) be the operator defined by the right-hand side of equation (5). We prove that \( T: M(G, R_+) \to M(G, R_+) \). Let \( z \in M(G, R_+) \),

\[
v_{ij}(x) = \int_{H_j(\beta^{(i)}(x))} z(\alpha_j(s)) (ds)_{pj}.
\]

Then there exists a sequence \( \{z_r\} \) such that \( z_r \in C(G, R_+) \) and

\[
z_{r+1}(x) \leq z_r(x), \quad x \in G, \quad r = 1, 2, \ldots,
\]

and \( z(x) = \lim_{r \to \infty} z_r(x), \quad x \in G. \)

Let \( v_{ij}^{(r)}(x) = \int_{H_j(\beta^{(i)}(x))} z_r(\alpha_j(s)) (ds)_{pj}, \quad x \in G, \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad r = 1, 2, \ldots. \) The functions \( v_{ij}^{(r)} \) are continuous in \( G \) (cf. [12], p. 972), and \( v_{ij}^{(r+1)}(x) \leq v_{ij}^{(r)}(x) \). From (11) and by the Lebesgue theorem on integration of non-increasing sequences we have \( v_{ij}(x) = \lim_{r \to \infty} v_{ij}^{(r)}(x), \quad i = 1, \ldots, n, \quad j = 1, \ldots, m, \quad x \in G. \) Since \( v_{ij} \) is the limit of the non-increasing sequence of continuous functions, we see that \( v_{ij} \in M(G, R_+) \). It follows from Dini's theorem and from assumptions \( T \) of \( H_1 \) and \( T \) of \( H_1 \) that series (3) and (4) are uniformly convergent in \( G \). From this fact and by the conditions

\[
k^{(i)}(x) \tilde{h}(\beta^{(i)}(x)) \leq k^{(i)}(x) \tilde{h}(\beta^{(i)}(x)), \quad i = 0, 1, 2, \ldots, \quad x \in G,
\]

\[
k^{(i)}(x) \left[ K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} z(\alpha(s)) ds \right]
\]

\[
\leq \left[ \sup_{x \in G} z(x) \right] k^{(i)}(x) \left[ K(\beta^{(i)}(x)) \cdot L(G(\beta^{(i)}(x))) \right], \quad i = 0, 1, 2, \ldots, \quad x \in G,
\]

we infer the uniform convergence in \( G \) of the following series:

\[
\sum_{i=0}^{\infty} k^{(i)}(x) \tilde{h}(\beta^{(i)}(x)), \quad \sum_{i=0}^{\infty} k^{(i)}(x) \left[ K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} z(\alpha(s)) ds \right].
\]

Hence we get \( \tilde{m} \in C(G, R_+) \), \( \forall z \in M(G, R_+) \) and consequently, \( T: M(G, R_+) \rightarrow M(G, R_+) \).

Now we prove that the operator \( T \) is a contraction. Let

\[
\|z\|_0 = \max_{x \in G} \left[ e^{-\lambda(x_1 + \cdots + x_n)} |z(x)| \right],
\]

where \( z \in M(G, R_+) \), and \( \lambda > \Lambda = \max \left[ 1, \sup_{x \in G} \tilde{M}(x) \right] \). For \( z, w \in M(G, R_+) \),
we get
\[ \| (Tz)(x) - (Tw)(x) \| \leq \sum_{i=0}^{\infty} k^i(x) \left[ K(\beta^{i}(x)) \int_{H(\beta^{i}(s))} \left| z(\alpha(s)) - w(\alpha(s)) \right| ds \right] \]
\[ \leq \| z - w \|_0 \sum_{i=0}^{\infty} k^i(x) \left[ K(\beta^{i}(x)) \int_{H(\beta^{i}(s))} e^{i(s_1 + \ldots + s_n)} ds \right]. \]

We have the following estimates:
\[ \int_{H(\beta^{i}(s))} e^{i(s_1 + \ldots + s_n)} ds \leq \exp \left( \lambda \sum_{p \in \sigma_j} x_p \right) \int_{H(\beta^{i}(s))} \exp \left( \lambda \sum_{p \in \sigma_j} s_p \right) ds \]
\[ = \exp \left( \lambda \sum_{p \in \sigma_j} x_p \right) \prod_{p \in \sigma_j} \left\{ \frac{1}{\lambda} \left( \exp \left[ \lambda \phi_p^{(i)}(\beta^{i}(x)) \right] - 1 \right) \right\} \]
\[ \leq \frac{1}{\lambda} \exp \left( \lambda \sum_{p \in \sigma_j} x_p \right) \prod_{p \in \sigma_j} \left\{ \exp \left[ \lambda x_p \frac{\phi_p^{(i)}(\beta^{i}(x))}{x_p} \right] - 1 \right\} \]
\[ \leq \frac{1}{\lambda} e^{i(x_1 + \ldots + x_n)} L_{p_j}(G_j(\beta^{i}(x))) \left( \prod_{p \in \sigma_j} x_p \right)^{-1}. \]

The last inequality is a consequence of the obvious inequality
\[ e^{it} - 1 \leq \gamma e^t \quad \text{for} \quad \gamma \in [0, 1], t \geq 0. \]

Finally, we obtain
\[ \| (Tz)(x) - (Tw)(x) \| \]
\[ \leq \frac{1}{\lambda} \| z - w \|_0 \sum_{i=0}^{\infty} k^i(x) \left[ \sum_{j=1}^{m} K_j(\beta^{i}(x)) L_{p_j}(G_j(\beta^{i}(x))) \right] \times \]
\[ \left( \prod_{p \in \sigma_j} x_p \right)^{-1} e^{i(x_1 + \ldots + x_n)} \]
\[ \leq \frac{1}{\lambda} A \| z - w \|_0 e^{i(x_1 + \ldots + x_n)}, \]

and consequently
\[ \| Tz - Tw \|_0 \leq \frac{A}{\lambda} \| z - w \|_0. \]

Since \( A < \lambda \), then by the Banach fixed point theorem we infer that equation (5) has a unique solution \( \tilde{z} \) being an upper-semicontinuous function.

We prove that \( \tilde{z} \in C(G, R_+) \). The solution \( \tilde{z} \) of equation (5) is the limit of the sequence \( \{ z_r \} \) which is defined in the following way:
\[ z_0 \in M(G, R_+), \quad z_0 \text{ - arbitrarily fixed}, \]
\[ z_{r+1}(x) = \tilde{m}(x) + (Vz_r)(x), \quad x \in G, \ r = 0, 1, 2, \ldots \]
For \( z \in M(G, R_+) \) we define \((V^0 z)(x) = z(x), (V^{i+1} z)(x) = (V(V^i z))(x), x \in G, i = 0, 1, \ldots \) We easily see that

\[
(12) \quad z_{r+1}(x) = \sum_{i=0}^{r} (V^i \bar{m})(x) + (V^{r+1} z_0)(x).
\]

\([V^r z_0]\) is the sequence of successive approximations for the equation \( z(x) = (V_2 z)(x) \). Since this equation has a solution \( z(x) = 0, x \in G \), which is unique in the set \( M(G, R_+) \), it follows that

\[
\lim_{r \to \infty} (V^r z_0)(x) = 0 \quad \text{uniformly with respect to } x \in G.
\]

Since the functions \( V^i \bar{m} \) are continuous in \( G \) and the sequence \( \{z_r\} \) is uniformly convergent, it follows from (12) that

\[
\bar{z}(x) = \sum_{i=0}^{\infty} (V^i \bar{m})(x)
\]

is a continuous function in \( G \). This completes the proof of assertion 1° of Lemma 1.

Now we shall prove assertion 2°. At first we prove that equality (9) holds true. It is easy to check that functions \( k^{(r)} \) and \( \beta^{(r)} \) satisfy the conditions

\[
(13) \quad k^{(r)}(x) k^{(i)}(\beta^{(r)}(x)) = k^{(r+i)}(x), \quad \beta^{(i)}(\beta^{(r)}(x)) = \beta^{(r+i)}(x), \quad x \in G, r, i = 0, 1, \ldots
\]

Formulas (13) and (6) imply

\[
k^{(r)}(x) \bar{z}(\beta^{(r)}(x)) = \sum_{i=0}^{\infty} k^{(r+i)}(x) \bar{h}(\beta^{(r+i)}(x)) + \sum_{i=0}^{\infty} k^{(r+i)}(x) \left[ K(\beta^{(r+i)}(x)) \int_{H^{(\beta^{(r+i)}(x))}} \bar{z}(\alpha(s)) \, ds \right].
\]

This last equality and (3), (4) imply (9). The uniform convergence of \( \{k^{(r)}(x) \bar{z}(\beta^{(r)}(x))\} \) follows from the uniform convergence of series (3) and (4).

We observe that any solution of equation (5) is a solution of (7). Indeed, if \( \bar{z} \) is a solution of equation (5), we have

\[
\bar{z}(x) - K(x) \int_{H^{(\alpha)}} \bar{z}(\alpha(s)) \, ds - k(x) \bar{z}(\beta(x))
\]

\[
= \sum_{i=0}^{\infty} k^{(i)}(x) \bar{h}(\beta^{(0)}(x)) + \sum_{i=0}^{\infty} k^{(i)}(x) \left[ K(\beta^{(0)}(x)) \int_{H^{(\beta^{(0)}(x))}} \bar{z}(\alpha(s)) \, ds \right] - K(x) \int_{H^{(\alpha)}} \bar{z}(\alpha(s)) \, ds - k(x) \left\{ \sum_{i=0}^{\infty} k^{(i)}(\beta(x)) \bar{h}(\beta^{(0)}(\beta(x))) + \sum_{i=0}^{\infty} k^{(i)}(\beta(x)) \left[ K(\beta^{(0)}(\beta(x))) \int_{H^{(\beta^{(0)}(\beta(x)))}} \bar{z}(\alpha(s)) \, ds \right] \right\} \equiv \bar{h}(x),
\]

which means that \( \bar{z} \) is a solution of (7).
Now we prove that $\tilde{z}$ is a unique solution of (7) in the set $\tilde{M}(G, R_+, \tilde{z})$. In fact, if $\tilde{z} \in \tilde{M}(G, R_+, \tilde{z})$ is a solution of (7), then for $r = 1, 2, \ldots$ and any $x \in G$ the equality

\begin{equation}
\tilde{z}(x) = \sum_{i=0}^{r-1} k^{(i)}(x) \left[ K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} \tilde{z}(s) \, ds \right] + \\
+ \sum_{i=0}^{r-1} k^{(i)}(x) \tilde{h}(\beta^{(i)}(x)) + k^{(r)}(x) \tilde{z}(\beta^{(r)}(x))
\end{equation}

holds.

Since $\tilde{z} \in \tilde{M}(G, R_+, \tilde{z})$, we have for some $c \in R_+ : 0 \leq \tilde{z}(x) \leq c\tilde{z}(x)$ for $x \in G$. Now, according to (9), we obtain

\begin{equation}
\lim_{r \to \infty} k^{(r)}(x) \tilde{z}(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G.
\end{equation}

If we let $r \to \infty$ in relation (14), we obtain

\[ \tilde{z}(x) = \tilde{m}(x) + (V \tilde{z})(x), \quad x \in G, \]

i.e. $\tilde{z}$ is the solution of equation (5). This equation has only the solution $\tilde{z}$; hence it results that $\tilde{z} = \tilde{z}$. Thus the proof of $\mathcal{Z}$ is completed.

Now we are going to prove $\mathcal{Z}'$. Let us suppose that $z^* \in \tilde{M}(G, R_+, \tilde{z})$ and $z^*$ is a solution of inequality (10). We obtain easily for $r = 1, 2, \ldots$ and $x \in G$

\begin{equation}
z^*(x) \leq \sum_{i=0}^{r-1} k^{(i)}(x) \left[ K(\beta^{(i)}(x)) \int_{H(\beta^{(i)}(x))} z^*(s) \, ds \right] + k^{(r)}(x) z^*(\beta^{(r)}(x)).
\end{equation}

Since $z^* \in \tilde{M}(G, R_+, \tilde{z})$, we have for some $c \in R_+ : 0 \leq z^*(x) \leq c\tilde{z}(x)$ for $x \in G$. By (9) the last inequalities implies

\begin{equation}
\lim_{r \to \infty} k^{(r)}(x) z^*(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G.
\end{equation}

Letting $r$ in (16) tend to $\infty$ we get

\begin{equation}
z^*(x) \leq (V z^*)(x), \quad x \in G.
\end{equation}

Let $\{z_r\}$ be the sequence defined in the following way:

\begin{equation}
z_0(x) = \tilde{z}(x), \quad x \in G, \quad z_{r+1}(x) = (V z_r)(x), \quad x \in G, \quad r = 0, 1, 2, \ldots
\end{equation}

From assertions $1^o$ and $2^o$ of this Lemma and from (18) it follows that

\begin{equation}
0 \leq z_{r+1}(x) \leq z_r(x), \quad x \in G, \quad r = 0, 1, 2, \ldots
\end{equation}

and

\begin{equation}
\lim_{r \to \infty} z_r(x) = 0 \quad \text{uniformly in } G.
\end{equation}

Further, by (17), we obtain

\[ z^*(x) \leq c z_r(x), \quad x \in G, \quad r = 0, 1, \ldots \]
The last formula together with (20) gives \( z^*(x) = 0 \) for \( x \in G \), which completes the proof of assertion 3°.

The simple proof of assertion 4° is omitted.

**Lemma 2.** If Assumptions H_1 and H_2 are satisfied and the sequence \( \{w_r\} \) is defined by the formulas

\[
(21) \quad w_0(x) = \bar{z}(x), \quad w_{r+1}(x) = K(x) \int_{H(x)} w_r(x(s)) \, ds + k(x) w_r(\beta(x)), \quad x \in G, \ r = 0, 1, \ldots
\]

then

\[
(22) \quad 0 \leq w_{r+1}(x) \leq w_r(x) \leq w_0(x), \quad x \in G, \ r = 0, 1, 2, \ldots,
\]

\[
(23) \quad \lim_{r \to \infty} w_r(x) = 0 \quad \text{uniformly with respect to} \ x \in G.
\]

**Proof.** Relations (22) follow by induction. The convergence of the sequence \( \{w_r\} \) is implied by (22). Since \( w_r \in C(G, R_+) \), it follows that the limit \( \bar{w} \) of the sequence \( \{w_r\} \) is an upper-semicontinuous function. From (21) it follows that \( \bar{w} \) satisfies inequality (10). According to assertion 3° of Lemma 1 we have \( \bar{w}(x) = 0 \) for \( x \in G \). The uniform convergence of the sequence \( \{w_r\} \) follows from Dini's theorem.

3. **The existence and uniqueness of solutions of equation** (2). We introduce the following

**Assumption H_3.** Suppose that:

1° There exist functions \( l_i \in C(G, R_+) \), \( i = 1, \ldots, m \), \( k \in C(G, R_+) \) such that

\[
\|F(x, u, v) - F(x, \bar{u}, \bar{v})\| \leq \sum_{i=1}^{m} l_i(x) \|u_i - \bar{u}_i\| + k(x) \|v - \bar{v}\|,
\]

where \( u = (u_1, \ldots, u_m) \), \( \bar{u} = (\bar{u}_1, \ldots, \bar{u}_m) \), \( x \in G \), \( u, \bar{u} \in B^m \), \( v, \bar{v} \in B \).

2° There exist functions \( l_i \in C(G, R_+) \), \( i = 1, \ldots, m \), such that

\[
\|f_i(x, s, z) - f_i(x, s, \bar{z})\| \leq l_i(x) \|z - \bar{z}\|, \quad x, s \in G, \ z, \bar{z} \in B.
\]

3° There exists a function \( u_0 \in C(G, B) \) such that Assumption H_2 is fulfilled for \( h, K \) defined by relations

\[
\tilde{h}(x) = \left\| F(x, \int_{H(x)} f(x, s, u_0(x(s))) \, ds, u_0(\beta(x)) - u_0(x) \right\|,
\]

\[
K(x) = (l_1(x)\tilde{l}_1(x), \ldots, l_m(x)\tilde{l}_m(x)),
\]

and for \( k \) defined by condition 1° of Assumption H_3.
**Lemma 3.** If Assumptions $H_1$ and $H_3$ are satisfied and the sequence \( \{u_r\} \) is defined by the relations

\[
(24) \quad u_{r+1}(x) = F \left( x, \int_{H(x)} f \left( x, s, u_r(\alpha(s)) \right) ds, u_r(\beta(x)) \right), \quad x \in G, \ r = 0, 1, 2, \ldots,
\]

where $u_0$ is given by condition 3' of Assumption $H_3$, then

\[
(25) \quad \|u_r(x) - u_0(x)\| \leq \tilde{z}(x), \quad x \in G, \ r = 0, 1, 2, \ldots,
\]

\[
(26) \quad \|u_{r+p}(x) - u_r(x)\| \leq w_r(x), \quad x \in G, \ r = 0, 1, 2, \ldots,
\]

where $\tilde{z}$ is defined in Lemma 1, and the sequence $\{w_r\}$ is defined by relations (21).

**Proof.** We prove that (25) is fulfilled. For $r = 0$ this inequality is evidently satisfied. If we assume that $\|u_r(x) - u_0(x)\| \leq \tilde{z}(x)$ for $x \in G$, then

\[
\|u_{r+1}(x) - u_0(x)\| \leq \left| F \left( x, \int_{H(x)} f \left( x, s, u_r(\alpha(s)) \right) ds, u_r(\beta(x)) \right) - \right.
\]

\[
\left. - F \left( x, \int_{H(x)} f \left( x, s, u_0(\alpha(s)) \right) ds, u_0(\beta(x)) \right) \right| + \tilde{h}(x)
\]

\[
\leq K(x) \int_{H(x)} \tilde{z}(\alpha(s)) ds + k(x) \tilde{z}(\beta(x)) + \tilde{h}(x) = \tilde{z}(x), \quad x \in G.
\]

Now we obtain (25) by induction.

Next we prove (26). From (21) and (25) it follows that (26) is satisfied for $r = 0, p = 0, 1, 2, \ldots, x \in G$. If we assume that (26) holds for arbitrarily fixed $r$ and any $p = 0, 1, 2, \ldots, x \in G$, then

\[
\|u_{r+1+p}(x) - u_{r+1}(x)\| \leq \left| F \left( x, \int_{H(x)} f \left( x, s, u_{r+p}(\alpha(s)) \right) ds, u_{r+p}(\beta(x)) \right) - \right.
\]

\[
\left. - F \left( x, \int_{H(x)} f \left( x, s, u_r(\alpha(s)) \right) ds, u_r(\beta(x)) \right) \right| \leq K(x) \int_{H(x)} w_r(\alpha(s)) ds + k(x) w_r(\beta(x)) = w_{r+1}(x).
\]

Now (26) follows by induction with respect to $r$.

We have the following

**Theorem 1.** If Assumptions $H_2$ and $H_3$ are satisfied, then there exists a solution $\bar{u} \in C(G, B)$ of equation (2) such that

\[
(27) \quad \|\bar{u}(x) - u_0(x)\| \leq \tilde{z}(x), \quad x \in G,
\]

\[
(28) \quad \|\bar{u}(x) - u_r(x)\| \leq w_r(x), \quad x \in G, \ r = 0, 1, 2, \ldots,
\]

where $u_r$ and $w_r$ are defined by formulas (24) and (21), respectively. The solution $\bar{u}$ of (2) is unique in the class

\[
X(G, B) \overset{\text{def}}{=} \bigcup_{c > 0} \{u: u \in C(G, B), \|u(x) - u_0(x)\| \leq c\tilde{z}(x), x \in G\}.
\]
Proof. It follows from (23) and (26) that the sequence \( \{u_r\} \) is uniformly convergent in \( G \) to a certain function \( \bar{u} \in C(G, B) \). Obviously \( \bar{u} \) is a solution of (2). The estimates (27) and (28) are implied by (25) and (26), respectively. To prove that the solution \( \bar{u} \) of (2) is unique in the class considered, let us suppose that there exists another solution \( \tilde{u} \) of equation (2) and \( \tilde{u} \in X(G, B) \). It is easy to check that the function \( z(x) = \|\bar{u}(x) - \tilde{u}(x)\| \) is an element of the set \( \tilde{M}(G, R_+, \tilde{\varepsilon}) \) and

\[
z(x) \leq K(x) \int_{\tilde{H}(x)} z(\alpha(s)) ds + k(x) z(\beta(x)), \quad x \in G.
\]

By assertion 3' of Lemma 1 we get \( z(x) = 0 \) for \( x \in G \), and hence \( \bar{u}(x) = \tilde{u}(x) \) for \( x \in G \). Thus the proof of Theorem 1 is complete.

4. Continuous dependence of solutions on the right-hand side of equation (2).

Let us consider another equation:

\[
v(x) = \tilde{F} \left( x, \int_{\tilde{H}(x)} \tilde{f}(x, s, v(\alpha(s))) ds, v(\beta(x)) \right), \quad x \in G,
\]

where the functions \( \tilde{F}, \tilde{f} = (\tilde{f}_1, \ldots, \tilde{f}_m), \tilde{\alpha} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_m), \tilde{\beta} \) and the sets \( \tilde{H}(x) = (\tilde{H}_1(x), \ldots, \tilde{H}_m(x)) \) have the same properties as \( F, f, \alpha, \beta, H(x) \), which are formulated Assumptions \( H_1, H_2 \). Suppose that \( \bar{u} \) and \( \tilde{v} \) are solutions of equations (2) and (29), respectively. Let \( \tilde{r} \in C(G, R_+) \) be a function such that \( \|\bar{u}(x) - \tilde{v}(x)\| \leq \tilde{r}(x) \) for \( x \in G \). Let

\[
q(x) \equiv \left\| F \left( x, \int_{\tilde{H}(x)} f(x, s, \tilde{v}(\alpha(s))) ds, \tilde{v}(\beta(x)) \right) - \right.
\]
\[\left. \quad \tilde{F} \left( x, \int_{\tilde{H}(x)} \tilde{f}(x, s, \tilde{v}(\alpha(s))) ds, \tilde{v}(\beta(x)) \right) \right\|,
\]

\[
\tilde{h}_1(x) = \max [F(x), q(x), \tilde{r}(x)], \quad x \in G.
\]

Now we have the following

**Theorem 2.** If the functions \( F, f, \alpha, \beta \) and \( \tilde{F}, \tilde{f}, \tilde{\alpha}, \tilde{\beta} \) and \( \tilde{h}_1 \) satisfy Assumption \( H_2 \) and the sets \( H(x), \tilde{H}(x), x \in G \), satisfy Assumption \( H_1 \), then there exists a solution \( \tilde{w} \in C(G, R_+) \) of the equation

\[
z(x) = K(x) \int_{\tilde{H}(x)} z(\alpha(s)) ds + k(x) z(\beta(x)) + q(x), \quad x \in G,
\]

such that

\[
\|\bar{u}(x) - \tilde{v}(x)\| \leq \tilde{w}(x) \quad \text{for} \ x \in G.
\]

**Proof.** Let \( \tilde{z} \) be a solution of the equation

\[
z(x) = K(x) \int_{\tilde{H}(x)} z(\alpha(s)) ds + k(x) z(\beta(x)) + \tilde{h}_1(x), \quad x \in G.
\]
Put
\[ w_0(x) = \bar{z}(x), \quad x \in G, \]
\[ w_{r+1}(x) = K(x) \int_{H(x)} w_{r}(\alpha(s)) \, ds + k(x) w_{r}(\beta(x)) + q(x), \quad x \in G, \quad r = 0, 1, 2, \ldots \]

By induction we get \( 0 \leq w_{r+1}(x) \leq w_r(x) \leq \bar{z}(x), \quad x \in G, \quad r = 0, 1, 2, \ldots \) From these inequalities we see that the sequence \( \{w_r\} \) is convergent to the solution \( \bar{w} \) of equation (30) and \( 0 \leq \bar{w}(x) \leq \bar{z}(x) \) for \( x \in G \). However, in view of Lemma 1, there exists only one solution of this equation in the class of upper-semicontinuous functions satisfying the condition \( 0 \leq \bar{w}(x) \leq \bar{z}(x) \) for \( x \in G \).

Now, we show that
\[ \| \bar{u}(x) - \bar{v}(x) \| \leq w_r(x) \quad \text{for} \quad x \in G, \quad r = 0, 1, 2, \ldots \]

Since \( \bar{r}(x) \leq \bar{h}_1(x) \leq \bar{z}(x) = w_0(x), \quad x \in G \), it follows that (32) is satisfied for \( r = 0 \) and \( x \in G \). If we assume that \( \| \bar{u}(x) - \bar{v}(x) \| \leq w_r(x) \) for \( x \in G \) and for some \( r \), then
\[ \| \bar{u}(x) - \bar{v}(x) \| \leq \left\| F \left( x, \int_{H(x)} f \left( x, s, \bar{u}(x(s)) \right) ds, \bar{u}(\beta(x)) \right) - 
\quad - F \left( x, \int_{H(x)} f \left( x, s, \bar{v}(x(s)) \right) ds, \bar{v}(\beta(x)) \right) \right\| + 
\quad + \left\| F \left( x, \int_{H(x)} f \left( x, s, \bar{v}(x(s)) \right) ds, \bar{v}(\beta(x)) \right) - 
\quad - F \left( x, \int_{H(x)} f \left( x, s, \bar{u}(x(s)) \right) ds, \bar{u}(\beta(x)) \right) \right\| 
\leq K(x) \int_{H(x)} \| \bar{u}(\alpha(s)) - \bar{v}(\alpha(s)) \| \, ds + k(x) \| \bar{u}(\beta(x)) - \bar{v}(\beta(x)) \| + q(x) \]
\[ \leq K(x) \int_{H(x)} w_r(\alpha(s)) \, ds + k(x) w_r(\beta(x)) + q(x) = w_{r+1}(x), \quad x \in G. \]

Now (32) follows by induction. Letting \( r \to \infty \) in (32), we get estimation (31).

5. Some effective conditions. We give here simple sufficient conditions for assumptions 2$^0$–4$^0$ in H$_2$ to be satisfied.

Lemma 4. Assume that
1$^o$ \( k(x) \leq \bar{k}, \quad K(x) = (K_1(x), \ldots, K_m(x)) \leq (\bar{K}_1, \ldots, \bar{K}_m), \quad \bar{k}, \bar{K}_1 \in R_+, \)
2$^o$ \( \varphi_1(x) = (\varphi_{i_1}^{(i)}(x), \ldots, \varphi_{i_p}^{(i)}(x)) \leq (\bar{z}_{i_1}^{(0)} x_{i_1}, \ldots, \bar{z}_{i_p}^{(0)} x_{i_p}), \) where \( \bar{z}_{i_j}^{(0)} \in R_+, t_j \in \sigma_i, \bar{z}_{i_j}^{(0)} \leq 1, \)
3$^o$ \( \beta(x) = (\bar{\beta}_1(x), \ldots, \bar{\beta}_n(x)) \leq (\bar{\beta}_1 x_1, \ldots, \bar{\beta}_n x_n), \) where \( \bar{\beta}_i \in R_+, \bar{\beta}_i \leq 1, \)
4$^o$ \( \sum_{i=0}^{\infty} \bar{h}_i (\bar{\beta}_1 x_1, \ldots, \bar{\beta}_n x_n) < +\infty, \)
5$^o$ \( \bar{k} \prod_{j=0}^{m} \bar{\beta}_j < 1 \) for \( j = 1, 2, \ldots, m. \)
Under these assumptions conditions 2°–4° of Assumption H₂ are satisfied.

Proof. By induction we easily obtain the estimates $k^{(i)}(x) \leq \bar{k}^{i}$, $x \in G$, $i = 0, 1, 2, \ldots$ and $\beta^{(i)}(x) \leq (\bar{\beta}^{i}_1 x_1, \ldots, \bar{\beta}^{i}_n x_n)$, $x \in G$, $i = 0, 1, 2, \ldots$. From these inequalities we get the following estimation for series (3):

$$
\sum_{i=0}^{\infty} k^{(i)}(x) h(\beta^{(i)}(x)) \leq \sum_{i=0}^{\infty} \bar{k}^{i} h(\bar{\beta}^{i}_1 x_1, \ldots, \bar{\beta}^{i}_n x_n), \quad x \in G.
$$

Now 4° implies condition 2° of Assumption H₂. Since

$$
\sum_{i=0}^{\infty} k^{(i)}(x) \left[ K(\beta^{(i)}(x)) L(G(\beta^{(i)}(x))) \right] = \sum_{j=1}^{m} \left[ \sum_{i=0}^{\infty} k^{(i)}(x) K_j(\beta^{(i)}(x)) L_{\nu_j}(G(\beta^{(i)}(x))) \right],
$$

then for a fixed index $j$ we have

$$
\sum_{i=0}^{\infty} k^{(i)}(x) K_j(\beta^{(i)}(x)) L_{\nu_j}(G(\beta^{(i)}(x))) \leq \sum_{i=0}^{\infty} \bar{k}^{i} K_j \prod_{s \neq j} \varphi^{(i)}_s(\beta^{(i)}(x))
$$

$$
\leq \bar{K}_j \sum_{i=0}^{\infty} \bar{k}^{i} \prod_{s \neq j} \bar{\beta}^{i}_s \cdot x_s \leq \bar{K}_j \left( \prod_{s \neq j} \bar{k} \right) \left( \prod_{s \neq j} x_s \right) \sum_{i=0}^{\infty} \left( \bar{k} \prod_{s \neq j} \bar{\beta}_s \right)^i.
$$

Hence and from assumption 5° of this Lemma it follows that condition 3° of Assumption H₂ is satisfied. From Dini’s theorem and from the last inequalities it follows that condition 4° of Assumption H₂ is satisfied, too.

Remark 4. Suppose that conditions 1°–4° of Lemma 4 are satisfied. A sufficient condition for the existence of a solution of equations of the type (2) given in Lemma 11 in [11] is of the form

$$(33) \quad \bar{k} \max_{1 \leq s \leq n} \bar{\beta}_s < 1.$$

We see easily that condition 5° of Lemma 4 is more general than condition (33).

By a similar argument we can prove the following lemmas:

Lemma 5. If

1° $k(x) \leq \bar{k}$, $K_j(x) \leq \bar{K}_1 x_1 + \ldots + \bar{K}_n x_n$, $j = 1, 2, \ldots, m$, $\bar{k}, \bar{K}_i \in R_+$,

2° assumptions 2°–4° of Lemma 4 are satisfied, then conditions 2°–4° of Assumption H₂ are fulfilled.

Lemma 6. If

1° $G = [0, a]$, $a = (a_1, \ldots, a_n)$, $a_i > 0$, $i = 1, \ldots, n$,

2° $k(x) \leq \bar{k}_1 x_1 + \ldots + \bar{k}_n x_n$, $K(x) = (K_1(x), \ldots, K_n(x)) \leq (\bar{K}_1, \ldots, \bar{K}_n)$, $\bar{k}_i, \bar{K}_i \in R_+$,

3° assumptions 2°, 3° of Lemma 4 are satisfied,
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4° \( \bar{K}_1 \bar{β}_1 a_1 + ... + \bar{K}_n \bar{β}_n a_n < 1 \),
then conditions 2°–4° of Assumption H₂ are fulfilled.

**Lemma 7.** If
1° \( k(x) \leq \bar{k}, \ K_j(x) \leq \bar{K}_1 x_1 + ... + \bar{K}_n x_n, \ j = 1, ..., m, \ \bar{k}, \ \bar{K}_i \in R_+ \),
2° \( φ_j(x) \leq (\bar{α}_{j1} x_1^2, ..., \bar{α}_{jp_j x_j^2}), t_i \in \bar{σ}_j, \ \bar{α}_{ij} \in R_+ \),
3° assumptions 3°, 4° of Lemma 4 are satisfied,
4° \( \prod \limits_{\bar{α}_{ij}} \bar{β}_i j^2 < 1, \ i = 1, 2, ..., n, \ j = 1, 2, ..., m \),
then conditions 2°–4° of Assumption H₂ are satisfied.

**Lemma 8.** If
1° \( G = [0, a], \ a = (a_1, ..., a_n), \ 0 < a_i \leq 1, \ i = 1, ..., n, \prod \limits_{\bar{α}_{ij}} a_i^2 < 1, \ j = 1, ..., m, \)
2° \( β(x) \leq (x_1^2, ..., x_n^2) \),
3° assumptions 1°, 2° of Lemma 4 are satisfied,
4° \( \sum \limits_{i=0}^∞ \bar{K}_i \bar{h}(x_1^2, ..., x_n^2) < ∞ \),
then conditions 2°–4° of Assumption H₂ are fulfilled.

**Lemma 9.** If
1° assumptions 1°, 2°, 3° of Lemma 4 are satisfied,
2° \( \bar{h}(x) \leq hx^p, \ h = \text{const}, \ x^p = x_1^p \cdot x_2^p \cdot ... \cdot x_n^p \),
3° \( \bar{k}(\prod \limits_{\bar{α}_{ij}} \bar{β}_i j^v < 1, \ where \ v = \min [1, p] \),
then conditions 2°–4° of Assumption H₂ are fulfilled.

**PART II**

In this part of the paper we give sufficient conditions for the existence of at least one continuous solution of equation (2) considered in a finite dimensional Banach space \( B \). Now we do not assume the Lipschitz condition for the function \( F \) with respect to \( u_i \) for \( i \in A' \) and for the functions \( f_i \) with respect to \( z \) for \( i \in A' \) (see Assumption H₃ in Part I).

**1. Assumptions.** We introduce the following

**Assumption H₄.** Suppose that:
1° There exist functions \( h_i \in C(G, R_+), \ i = 0, 1, ..., m, g, \ \bar{h}_i, \ g_i \in C(G, R_+), \)
\( i = 1, 2, ..., m, \) such that
\[ \|F(x, u, v)\| \leq \sum \limits_{i=1}^m h_i(x)\|u_i\| + h_0(x)\|v\| + g(x), \quad x \in G, \]
where \( u = (u_1, \ldots, u_m) \in B^m, \ v \in B, \ B \) is a finite dimensional Banach space and
\[
\| f_i(x, s, z) \| \leq \bar{h}_i(x) \| z \| + g_i(x), \quad i = 1, 2, \ldots, m, \ x, s \in G.
\]

2° There exist functions \( l_i \in C(G, R_+), \ i = k_0 + 1, \ldots, m, \ k \in C(G, R_+) \) and \( \bar{l}_i \in C(G, R_+), \ i = k_0 + 1, \ldots, m, \) such that
\[
\| F(x, u_A, u_B, v) - F(x, u_A, \bar{u}_B, \bar{v}) \| \leq \sum_{i=k_0+1}^{m} l_i(x) \| u_i - \bar{u}_i \| + k(x) \| v - \bar{v} \|,
\]
where \( x \in G, \ u_A = (u_1, \ldots, u_{k_0}), \ u_B = (u_{k_0+1}, \ldots, u_m), \ u_i, \bar{u}_i, v, \bar{v} \in B, \) and
\[
\| f_i(x, s, z) - f_i(x, s, \bar{z}) \| \leq \bar{l}_i(x) \| z - \bar{z} \| \quad \text{for} \ i = k_0 + 1, \ldots, m,
\]
\( x, s \in G, \ z, \bar{z} \in B. \)

3° Assumption H_2 is fulfilled for \( \bar{h}, K \) defined by the relations
\[
K(x) = (h_1(x) \bar{h}_1(x), \ldots, h_m(x) \bar{h}_m(x)),
\]
\[
\bar{h}(x) = \sum_{i=1}^{m} [h_i(x) \bar{g}_i(x) \sup_{x \in G} L_{p_i}(H_i(x))] + g(x)
\]
and for \( k \) defined in condition 2° of Assumption H_4.

Assumption H_5. We assume that the functions \( D_j, d_i, \bar{d}_i, \tilde{d}_i, \bar{d}_i, \tilde{d}_i, \bar{d}_s, \) \( \bar{d}_s \in C(R_+, R_+), \ i = 1, \ldots, m, \ j = 0, 1, \ldots, m + 1, \ s = k_0 + 1, \ldots, m \) are subadditive, non-decreasing and such that \( D_j(0) = 0, d_i(0) = \bar{d}_i(0) = \tilde{d}_i(0) = \bar{d}_s(0) = \tilde{d}_s(0) = 0, \) and, moreover:

1° \( \| F(x, u, v) - F(\bar{x}, \bar{u}, \bar{v}) \| \leq D_0(\| x - \bar{x} \|) + \sum_{i=1}^{m} D_i(\| u_i - \bar{u}_i \|) + D_{m+1}(\| v - \bar{v} \|) \)
for \( x, \bar{x} \in G, \ \| u_i \|, \ \| \bar{u}_i \| \leq R_i, \ \| v \|, \ \| \bar{v} \| \leq \bar{r}, \) where \( \bar{r} \triangleq \sup_{x \in G} \bar{z}(x), \ \bar{r}_i \triangleq \sup_{x \in G} L_{p_i}(H_i(x)) \sup_{x \in G} \bar{h}_i(x) \sup_{x \in G} \bar{z}(x) + \sup_{x \in G} \bar{g}_i(x), \) and \( \bar{z} \) defined in Lemma 1.

2° \( \| f_i(x, s, z) - f_i(\bar{x}, \bar{s}, \bar{z}) \| \leq d_i(\| x - \bar{x} \|) + \tilde{d}_i(\| s - \bar{s} \|) + \tilde{d}_i(\| z - \bar{z} \|) \), \( i = 1, \ldots, m \)
for \( x, \bar{x}, s, \bar{s} \in G, \ \| z \|, \ \| \bar{z} \| \leq \bar{r}. \)

3° \( L_{p_i}(H_i(x + h) - H_i(x)) \leq \bar{d}_i(|h|) \) for \( i \in A', \ x, x + h \in G, \)
\( L_{p_i}[H_i(x) - (\alpha_i(x, h) + H_i(x + h))] \leq \bar{d}_i(|h|) \) for \( i \in B', \ x, x + h \in G. \)

4° \( |t_i(x, h)| \leq \bar{d}_i(|h|) \) for \( i \in B', \ |h| \in [0, r_0] \), where \( r_0 \) is the diameter of the set \( G. \)

Let \( \bar{d}_i(t) = \bar{d}_i(\bar{d}_i(t)), \ i \in B', \ t \in [0, r_0]. \)

2. A certain functional equation.

**Lemma 10.** If

1° Assumption H_5 and conditions 1°, 3° from Assumption H_4 are satisfied,
2° the Lipschitz condition
\[ \| F(x, u, v) - F(x, u, \bar{v}) \| \leq k(x) \| v - \bar{v} \|, \quad x \in G, \; v, \bar{v} \in B \]
holds,

3° \( W \overset{\text{def}}{=} \{ y : y \in C(G, B), \; \| y(x) \| \leq \bar{z}(x) \text{ for } x \in G \} \),

then for any \( y \in W \) there exists the unique \( u(\cdot, y) \in W \) being a solution of the equation
\[ u(x) = F\left(x, \int_{H(x)} f(x, s, y(x(s))) ds, u(\beta(x))\right), \quad x \in G. \]

Proof. Put
\[ u_0(x) = 0, \quad u_{r+1}(x) = F\left(x, \int_{H(x)} f(x, s, y(x(s))) ds, u_r(\beta(x))\right), \quad x \in G, \; r = 0, 1, \ldots \]

We prove that
\[ \| u_r(x) \| \leq \bar{z}(x), \quad x \in G, \; r = 0, 1, 2, \ldots \]

For \( r = 0 \) this inequality is evidently satisfied. If we assume that \( \| u_r(x) \| \leq \bar{z}(x) \) for \( x \in G \), then
\[ \| u_{r+1}(x) \| \leq \left\| F\left(x, \int_{H(x)} f(x, s, y(x(s))) ds, u_r(\beta(x))\right) - 
\quad - F\left(x, \int_{H(x)} f(x, s, y(x(s))) ds, 0\right) \right\| + \left\| F\left(x, \int_{H(x)} f(x, s, y(x(s))) ds, 0\right) \right\|
\leq k(x) \bar{z}(\beta(x)) + K(x) \int_{H(x)} \bar{z}(x(s)) ds + \bar{z}(x) = \bar{z}(x) \]

for \( x \in G \). Now we obtain (38) by induction.

Next we prove that \( u_r \) are continuous in \( G \). Since \( u_0 \) is continuous in \( G \), it is sufficient to prove that the continuity of \( u_r \) implies the continuity of \( u_{r+1} \). Let
\[ \bar{R}_1 = \max \left[ \sup_{j} \bar{h}_j(x) \sup_{x \in G} \| y(x) \| + \sup_{x \in G} \bar{g}_j(x) \right]. \]

By Assumption H₅ we have
\[ \| \dot{u}_{r+1}(x+h) - u_{r+1}(x) \|
\leq \sum_{j=1}^{m} D_j \left( \left\| \int_{H_j(x+h)} f_j(x+h, s, y(x_j(s))) (ds)_{p_j} - \int_{H_j(x)} f(x, s, y(x_j(s))) (ds)_{p_j} \right\| \right) + \sum_{j=1}^{m} \left( \| u_r(\beta(x+h)) - u_r(\beta(x)) \| \right). \]
Now for \( j \in A' \) we get, writing \( ds \) instead of \( (ds)_n \)

\[
(40) \quad \left\| \int_{H_j(x+h)} f_j(x+h, s, y(\alpha_j(s))) \, ds - \int_{H_{j(x)}} f_j(x, s, y(\alpha_j(s))) \, ds \right\|
\leq \int_{H_j(x+h) \land H_{j(x)}} \left\| f_j(x+h, s, y(\alpha_j(s))) - f_j(x, s, y(\alpha_j(s))) \right\| \, ds + \int_{H_j(x+h) - H_{j(x)}} \tilde{R}_1 \, ds
\leq L_n(G) \delta_j(|h|) + \tilde{R}_1 \delta_j(|h|).
\]

If for \( j \in B' \) we define the sets \( H_j^0(x, h), \ H_j^1(x, h), \ H_j^2(x, h), \ H_j^3(x, h) \) by

\[
(41) \quad \begin{align*}
H_j^0(x, h) &= H_j(x) - (-t_j(x, h) + H_j(x+h)), \\
H_j^1(x, h) &= (-t_j(x, h) + H_j(x+h)) \cap H_j(x), \\
H_j^2(x, h) &= (-t_j(x, h) + H_j(x+h)) - H_j(x), \\
H_j^3(x, h) &= H_j(x) - (-t_j(x, h) + H_j(x+h)),
\end{align*}
\]

then

\[
(42) \quad \left\| \int_{H_j(x+h)} f_j(x+h, s, y(\alpha_j(s))) \, (ds)_{p_j} - \int_{H_{j(x)}} f_j(x, s, y(\alpha_j(s))) \, (ds)_{p_j} \right\|
\leq \int_{H_j^1(x,h)} \left\| f_j(x+h, s+t_j(x, h), y(\alpha_j(s+t_j(x, h)))) - f_j(x, s, y(\alpha_j(s))) \right\| \, (ds)_{p_j} + \\
+ \int_{H_j^2(x,h)} \left\| f_j(x+h, s+t_j(x, h), y(\alpha_j(s+t_j(x, h)))) \right\| \, (ds)_{p_j} + \\
+ \int_{H_j^3(x,h)} \left\| f_j(x, s, y(\alpha_j(s))) \right\| \, (ds)_{p_j}
\leq L_{p_j}(H_j(x)) \left[ d_j(|h|) + \delta_j(\bar{\omega}_j(|h|)) \right]
+ \sup_{s \in G} \left\| y(\alpha_j(s+t_j(x, h)) - y(\alpha_j(s)) \right\| + \int_{H_j^0(x,h)} \tilde{R}_1 \, (ds)_{p_j}
\leq L_{p_j}(H_j(x)) \left[ d_j(|h|) + \delta_j(\bar{\omega}_j(|h|)) + \delta_j(\tilde{\delta}_j(\bar{\omega}_j(|h|))) + \tilde{R}_1 \delta_j(|h|) \right],
\]

where \( \tilde{\alpha} \) is a modulus of continuity for the function \( y \). It follows from

(39), (40), (42) and from the continuity of \( y, u_r, \alpha_j, \beta \) that \( u_{r+1} \) is continuous.

We put

\[
z_0(x) = \tilde{\alpha}(x) \quad \text{for} \ x \in G, \quad z_r(x) = k(x) z_{r-1}(\beta(x)), \quad r = 1, 2, \ldots, x \in G.
\]

By induction we get

\[
z_r(x) = k^{(r)}(x) \tilde{\alpha}(\beta^{(r)}(x)), \quad r = 1, 2, \ldots, x \in G.
\]

In virtue of condition (9) of Lemma 1 it follows from (43) that

\[
\lim_{r \to \infty} z_r(x) = 0
\]
and the convergence is uniform with respect to $x \in G$. Further, we get easily
\begin{equation}
\|u_{r+p}(x) - u_r(x)\| \leq z_r(x), \quad x \in G, \quad r, p = 0, 1, 2, \ldots
\end{equation}
Indeed, from (37) and (38) it follows that (45) is satisfied for $r = 0, p = 0, 1, 2, \ldots, x \in G$. If we assume that (45) holds for a fixed $r$ and $p = 0, 1, 2, \ldots, x \in G$, then
\begin{align*}
\|u_{r+1+p}(x) - u_{r+1}(x)\| &\leq k(x) \|u_{r+p}(\beta(x)) - u_r(\beta(x))\| \\
&\leq \bar{k}(x) z_r(\beta(x)) = z_{r+1}(x).
\end{align*}
Now we obtain (45) by induction.

By (37), (44), (45) we infer that the sequence $\{u_r\}$ is uniformly convergent in $G$ to the solution $\bar{u}$ of equation (36). Since the sequence $\{u_r\}$ is uniformly convergent in $G$ and $u_r \in C(G, B)$, we conclude by (38) that $\bar{u} \in W$.

To prove that the solution $\bar{u}$ of (36) is unique in $W$, let us suppose that there exists another solution $\vec{u} \neq \bar{u}$ and $\vec{u} \in W$. Now, from (35) we have
\begin{equation}
\|\vec{u}(x) - \bar{u}(x)\| \leq k(x) \|\vec{u}(\beta(x)) - \bar{u}(\beta(x))\|, \quad x \in G,
\end{equation}
and by induction we get
\begin{equation}
\|\vec{u}(x) - \bar{u}(x)\| \leq k^{(r)}(x) \|\vec{u}(\beta^{(r)}(x)) - \bar{u}(\beta^{(r)}(x))\|, \quad r = 0, 1, 2, \ldots
\end{equation}
Since
\begin{align*}
k^{(r)}(x) \|\vec{u}(\beta^{(r)}(x))\| &\leq k^{(r)}(x) \bar{z}(\beta^{(r)}(x)), \quad r = 0, 1, 2, \ldots, x \in G, \\
k^{(r)}(x) \|\vec{u}(\beta^{(r)}(x))\| &\leq k^{(r)}(x) \bar{z}(\beta^{(r)}(x)), \quad r = 0, 1, 2, \ldots, x \in G,
\end{align*}
and
\[ \lim_{r \to \infty} k^{(r)}(x) \bar{z}(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G, \]
we infer by (46) that $\vec{u} = \bar{u}$. This contradiction proves the uniqueness of $\bar{u}$ in $W$.

3. Further assumptions. We introduce

**Assumption $H_\delta$.** Suppose that
\begin{align*}
1^o \quad &m_0(x, \delta_1, \delta_2, \ldots, \delta_m) = \sum_{i=0}^{\infty} k^{(i)}(x) \sum_{j=1}^{m} D_j \delta_j L_{p_j}(G_j(\beta^{(i)}(x)))) < +\infty \quad \text{for } x \in G, \quad \delta_j \in R_+ \\
2^o \quad &\text{the function } m_0 \text{ is continuous with respect to } (x, \delta_1, \ldots, \delta_m) \in G \times R_+^m.
\end{align*}

**Remark 5.** It is obvious that Assumption $H_\delta$ is fulfilled if, for instance, $k(x) \leq \bar{k} < 1$ for $x \in G$. If condition $1^o$ of $H_\delta$ is satisfied and the functions $k, \beta, \alpha_j$ are non-decreasing in $G$, then condition $2^o$ of $H_\delta$ is fulfilled.
Remark 6. If the functions \( \varphi_i \) and \( \beta \) satisfy conditions 2' and 3' of Lemma 4, respectively, and
\[
k(x) \leq \bar{k} = \text{const,} \quad D_j(t) \leq Dt, \quad D = \text{const,} \quad j = 1, \ldots, m, \\
\bar{k} \prod_{j \in \bar{A}_j} \bar{\beta}_j < 1, \quad j = 1, 2, \ldots, m,
\]
then Assumption H_6 is fulfilled (see the proof of Lemma 4).

We adopt the following notation:
\[
\bar{K} = (K_{k_0+1}, \ldots, K_m), \quad \bar{L}(G(x)) = (L_{p_{k_0+1}}(G_{k_0+1}(x)), \ldots, L_{p_m}(G_m(x))),
\]
\[
\bar{K}(x) \cdot \bar{L}(G(x)) = \sum_{j=k_0+1}^m K_j(x) \cdot L_{p_j}(G_j(x)),
\]
\[
\int_{H(s)} d(z(s), \bar{\omega}(t)) ds = \left( \int_{H_{k_0+1}} d(z_{k_0+1}(s) \bar{\omega}_{k_0+1}(t)) (ds)_{p_{k_0+1}}, \ldots, \int_{H_m} d(z_m(s), \bar{\omega}_m(t)) (ds)_{p_m} \right),
\]
\[
\bar{K}(x) \int_{H(s)} d(z(s), \bar{\omega}(t)) ds = \sum_{j=k_0+1}^m K_j(x) \int_{H_j(s)} d(z_j(s), \bar{\omega}_j(t)) (ds)_{p_j},
\]

where \( \bar{\omega}_j \) are real-valued functions of one variable.

We introduce

ASSUMPTION H_7. Suppose that

1\(^o\) \( |\beta(x+h) - \beta(x)| \leq \omega(|h|) \), for \( x, x+h \in G \), where \( \omega \in C(R_+, R_+) \) is subadditive and non-decreasing and
\[
\omega(0) = 0, \quad \bar{\omega} = (\bar{\omega}_{k_0+1}, \ldots, \bar{\omega}_m) \in C(R_+, R_2^{k_0}).
\]

\( \bar{\omega}_i \) are subadditive and non-decreasing, and \( \bar{\omega}_i(0) = 0 \),

2\(^o\) there is given a function \( p \) such that

(a) \( p \in C(G \times [0, r_0], R_+) \), where \( r_0 \in R_+ \) is defined in 4\(^o\) of H_5,
(b) \( p \) is non-decreasing and subadditive with respect to the last variable,
(c) \( p(x, 0) = 0 \) for \( x \in G \),

3\(^o\) \( \bar{m}(x, t) = \sum_{i=0}^{\infty} k^{(i)}(x) p(\beta^{(i)}(x), \omega^{(i)}(t)) < +\infty \) for \( (x, t) \in G \times [0, r_0] \),

where \( \omega^{(i)}(t) = t, \omega^{(i+1)}(t) = \omega(\omega^{(i)}(t)), i = 0, 1, 2, \ldots, t \in [0, r_0] \),

4\(^o\) \( \bar{M}(x) = \sum_{i=0}^{\infty} k^{(i)}(x) [\bar{K}(\beta^{(i)}(x)) \cdot \bar{L}(G(\beta^{(i)}(x)))] < +\infty, x \in G \),

5\(^o\) \( \bar{m} \in C(G \times [0, r_0], R_+), \bar{M} \in C(G, R_+) \),
the function
\[ M(x) = \sum_{i=0}^{\infty} k_i(x) \left[ \sum_{j=k_0+1}^{m} K_j(\beta_i(x)) L_{\rho_j}(G_j(\beta_i(x))) \cdot \left( \prod_{s \in \sigma_j} x_s \right)^{-1} \right] \]
is bounded in \( G \).

We have the following

**Lemma 11.** If Assumption \( H_2 \) and condition 2° of Assumption \( H_1 \) are satisfied, then:

1° There exists a solution \( \bar{d} \in C(G \times [0, r_0], R_+) \) of the equation

\[
d(x, t) = \sum_{i=0}^{\infty} k_i(x) p(\beta_i(x), \omega(t)) + \\
+ \sum_{i=0}^{\infty} k_i(x) \left[ \bar{K}(\beta_i(x)) \cdot \int_{H(\beta_i(x))} d(\alpha(s), \omega(t)) ds \right],
\]

\((x, t) \in G \times [0, r_0].\)

The solution \( \bar{d} \) of (47) is unique in the class \( M(G \times [0, r_0], R_+) \) of non-negative upper-semicontinuous functions defined on \( G \times [0, r_0] \). The function \( \bar{d} \) is non-decreasing and subadditive with respect to the last variable and \( \bar{d}(x, 0) = 0 \) for \( x \in G \).

2° The function \( \bar{d} \) is a solution of the equation

\[
d(x, t) = \bar{K}(x) \int_{H(x)} d(\alpha(s), \omega(t)) ds + k(x) d(\beta(x), \omega(t)) + p(x, t),
\]

\((x, t) \in G \times [0, r_0].\)

Moreover, this solution is unique in the class \( \bar{M}(G \times [0, r_0], R_+, \bar{d}) \), where \( \bar{M}(G \times [0, r_0], R_+, \bar{d}) = \{ z : z \in M(G \times [0, r_0], R_+) \}, \inf \{c : z(x, t) \leq c\bar{d}(x, t) \}< + \infty \} \).

The proof of this Lemma is similar to the proof of assertions 1°, 2° of Lemma 1.

4. **Properties of the operator \( U \).** Let \( \tilde{W} = \{ y : y \in C(G, B), \| y(x) \| \leq \tilde{z}(x), \| y(x + h) - y(x) \| \leq \tilde{d}(x, |h|) \} \), where the functions \( \tilde{z} \) and \( \tilde{d} \) are defined by Lemma 1 and Lemma 11, respectively. We consider the operator \( U \) defined by the formula \( U y = u(\cdot, y) \), where \( u(\cdot, y) \) is the solution of functional equation (36).

We have

**Lemma 12.** If Assumptions \( H_5, H_6 \), conditions 1° and 3° from \( H_4 \) and the Lipschitz condition (35) are satisfied, then the operator \( U \) is continuous in the set \( \tilde{W} \).
Proof. Let \( y_1, y_2 \in W, \ u_1 = u(\cdot, y_1), \ u_2(\cdot, y_2), \ v(x) = \|u_1(x) - u_2(x)\| \). Then we have for \( x \in G \)

\[
v(x) = \left\| F(x, \int_{H(x)} f(x, s, y_1(\alpha(s))) \, ds, u_1(\beta(x))) - F(x, \int_{H(x)} f(x, s, y_2(\alpha(s))) \, ds, u_2(\beta(x))) \right\|
\]

\[
\leq \sum_{j=1}^{m} D_j \left( \left\| \int_{H_j(x)} \left( f_j(x, s, y_1(\alpha_j(s))) - f_j(x, s, y_2(\alpha_j(s))) \right) \, ds \right\| \right) +
\]

\[
+ k(x) \left\| u_1(\beta(x)) - u_2(\beta(x)) \right\|
\]

\[
\leq \sum_{j=1}^{m} D_j \left( \left\| y_1(\alpha_j(s)) - y_2(\alpha_j(s)) \right\| \right) + k(x) v(\beta(x)).
\]

Let \( \delta_j = \bar{d}_j \left( \sup_{s \in G} \|y_1(s) - y_2(s)\| \right) \). Then we have the inequality

\[
v(x) \leq \sum_{j=1}^{m} D_j \left( \delta_j L_{\rho_j}(G_j(x)) \right) + k(x) v(\beta(x)), \quad x \in G,
\]

and we get by induction

\[
(49) \quad v(x) \leq \sum_{i=0}^{r-1} k^{(i)}(x) \left[ \sum_{j=1}^{m} D_j \left( \delta_j L_{\rho_j}(G_j(\beta^{(i)}(x))) \right) \right] + k^{(r)}(x) v(\beta^{(r)}(x)), \quad x \in G, \ r = 1, 2, ...
\]

Since

\[
0 \leq k^{(r)}(x) v(\beta^{(r)}(x)) \leq 2k^{(r)}(x) \bar{z}(\beta^{(r)}(x)), \quad r = 0, 1, 2, ..., x \in G
\]

and

\[
\lim_{r \to \infty} k^{(r)}(x) \bar{z}(\beta^{(r)}(x)) = 0 \quad \text{uniformly with respect to } x \in G,
\]

we get, making \( r \to \infty \) in (49), that

\[
v(x) \leq \sum_{i=0}^{\infty} k^{(i)}(x) \left[ \sum_{j=1}^{m} D_j \left( \delta_j L_{\rho_j}(G_j(\beta^{(i)}(x))) \right) \right] = m_0(x, \delta_1, ..., \delta_m).
\]

In view of the continuity of the function \( m_0 \) we conclude the assertion of Lemma 12.
Lemma 13. Suppose that:

1° Assumptions $H_4$, $H_5$, $H_6$ are satisfied and Assumption $H_7$ is fulfilled for $p$, $\tilde{K}$ defined by the relations

\begin{equation}
\begin{aligned}
p(x, t) &= D_0(t) + \sum_{j=1}^{k_0} D_j(L_n(G) d_j(t) + P_j \tilde{d}_j(t)) + \\
&+ \sum_{j=k_0+1}^{m} l_j(x) [L_{p_j}(H_j(x))(d_j(t) + \tilde{d}_j(\tilde{w}_j(t)))] + P_j \tilde{d}_j(t),
\end{aligned}
\end{equation}

where $P_j = \sup_{x \in G} \tilde{h}_j(x) \sup_{x \in G} \bar{Z}(x) + \sup_{x \in G} \bar{g}_j(x)$,

\begin{equation}
\tilde{K}(x) = (l_{k_0+1}(x) \bar{l}_{k_0+1}(x), \ldots, l_m(x) \bar{l}_m(x)), \quad x \in G.
\end{equation}

2° For $x$, $x+h \in G$ we have

\begin{equation}
\lim_{r \to \infty} k^{(r)}(x) \bar{Z}(\beta^{(r)}(x+h)) = 0 \quad \text{uniformly with respect to } x, x+h \in G.
\end{equation}

Under these assumptions the operator $U$ maps $\hat{W}$ into itself.

Proof. In virtue of Lemma 10 it follows that for each $y \in \hat{W}$ the function $Uy$ satisfies the condition $\|Uy(x)\| \leq \bar{Z}(x)$ for $x \in G$. To prove that $Uy \in \hat{W}$ for $y \in \hat{W}$, it is sufficient to show that $\|Uy(x+h) - Uy(x)\| \leq \bar{a}(x, |h|)$ for $x$, $x+h \in G$.

Let us suppose that $y \in \hat{W}$ and $u(x) = (Uy)(x)$. We show that

\begin{equation}
\|u(x+h) - u(x)\| \leq \bar{a}(x, |h|), \quad x, x+h \in G.
\end{equation}

For $j \in A'$ we get, writing $ds$ instead of $(ds)_n$ for simplicity,

\begin{align*}
\| &\int_{H_j(x+h)} f_j(x+h, s, y(\alpha_j(s))) ds - \int_{H_j(x)} f_j(x, s, y(\alpha_j(s))) ds \| \\
&\leq \int_{H_j(x+h) \setminus H_j(x)} \| f_j(x+h, s, y(\alpha_j(s))) - f_j(x, s, y(\alpha_j(s))) \| ds + \\
&+ \int_{H_j(x+h) \setminus H_j(x)} [\bar{h}_j(x+h) \| y(\alpha_j(s)) \| + \bar{g}_j(x+h)] ds + \\
&+ \int_{H_j(x+h) \setminus H_j(x)} [\bar{h}_j(x) \| y(\alpha_j(s)) \| + \bar{g}_j(x)] ds \\
&\leq L_n(G) d_j(|h|) + P_j \tilde{d}_j(|h|).
\end{align*}

In the case $j \in B'$ we define the sets $H_j^k(x, h)$, $k = 0, 1, 2, 3$, by (41) and note that integration over the set $H_j(x+h)$ is equivalent to integration over $-t_j(x, h) + H_j(x+h)$ if one replaces the variable $s$ by $s + t_j(x, h)$. 
In this way we arrive at

\[ \left\| \int_{H_j(x+h)} f_j(x+h, s, y(\alpha_j(s))) (ds)_{p_j} - \int_{H_j(x)} f_j(x, s, y(\alpha_j(s))) (ds)_{p_j} \right\| \]

\[ = \left\| \int_{H_j(x,h)} f_j(x+h, s+t_j(x,h), y(\alpha_j(s+t_j(x,h)))) (ds)_{p_j} - \int_{H_j(x,h)} f_j(x, s, y(\alpha_j(s))) (ds)_{p_j} + \int_{H_j(x,h)} f_j(x+h, s+t_j(x,h), y(\alpha_j(s+t_j(x,h)))) (ds)_{p_j} - \int_{H_j(x,h)} f_j(x, s, y(\alpha_j(s))) (ds)_{p_j} \right\| \]

\[ \leq \int_{H_j(x,h)} \left( f_j(x+h, s+t_j(x,h), y(\alpha_j(s+t_j(x,h)))) - f_j(x, s, y(\alpha_j(s))) \right) (ds)_{p_j} + \int_{H_j(x,h)} P_j (ds)_{p_j} \]

\[ \leq \int_{H_j(x,h)} \left[ d_j(|h|) + \bar{d}_j(\bar{\alpha}_j(|h|)) + \bar{l}_j(x) \left\| y(\alpha_j(s+t_j(x,h))) - y(\alpha_j(s)) \right\| \right] (ds)_{p_j} + \int_{H_j(x,h)} P_j (ds)_{p_j} \]

\[ \leq L_{p_j}(H_j(x)) [d_j(|h|) + \bar{d}_j(\bar{\alpha}_j(|h|))] + \bar{l}_j(x) \int_{H_j(x)} \left\| y(\alpha_j(s+t_j(x,h))) - y(\alpha_j(s)) \right\| (ds)_{p_j} + \bar{P}_j \bar{d}_j(|h|). \]

It follows from Assumption H.5 and from the above estimates that

\[ \|u(x+h) - u(x)\| = \left\| F(x+h, \int_{H(x+h)} f(x+h, s, y(\alpha(s))) ds, u(\beta(x+h))) - F(x, \int_{H(x)} f(x, s, y(\alpha(s))) ds, u(\beta(x))) \right\| \]

\[ \leq D_0(|h|) + \sum_{j=1}^{k_0} D_j \left[ \left\| \int_{H_j(x+h)} f_j(x+h, s, y(\alpha_j(s))) (ds)_{p_j} - \int_{H_j(x)} f_j(x, s, y(\alpha_j(s))) (ds)_{p_j} \right\| + \sum_{j=k_0+1}^{m} l_j(x) \left\| \int_{H_j(x+h)} f_j(x+h, s, y(\alpha_j(s))) (ds)_{p_j} - \int_{H_j(x)} f_j(x, s, y(\alpha_j(s))) (ds)_{p_j} \right\| + k(x) \|u(\beta(x+h)) - u(\beta(x))\| \]
\[
\leq D_0 (|h|) + \sum_{j=1}^{k_0} D_j \left[ L_n (G) d_j (|h|) + P_j \tilde{d}_j (|h|) \right] + \\
+ \sum_{j=k_0+1}^{m} l_j (x) \left\{ L_{p_j} (H_j (x)) \left[ d_j (|h|) + \tilde{d}_j (\bar{\omega}_j (|h|)) \right] + P_j \tilde{d}_j (|h|) \right\} + \\
+ \sum_{j=k_0+1}^{m} l_j (x) \cdot I_j (x) \int_{H_j (x)} \left\| y \left( \alpha_j (s + t_j (x, h)) \right) - y \left( \alpha_j (s) \right) \right\| (ds)_{p_j} + \\
+ k (x) \left\| u (\beta (x + h)) - u (\beta (x)) \right\|.
\]

Since \( \| y (x + h) - y (x) \| \leq \tilde{d} (x, |h|) \), we have
\[
\left\| y \left( \alpha_j (s + t_j (x, h)) \right) - y \left( \alpha_j (s) \right) \right\| \leq \tilde{d} (\alpha_j (s), \bar{\omega}_j (|h|))
\]
and consequently
\[
\| u (x + h) - u (x) \| \leq p (x, |h|) + \overline{K} (x) \int_{H_j (x)} \tilde{d} (x, \bar{\omega} (|h|)) ds + \\
+ k (x) \left\| u (\beta (x + h)) - u (\beta (x)) \right\|.
\]

The last inequality implies the following:

(54) \[
\| u (x + h) - u (x) \| \leq \sum_{i=0}^{r-1} k^{(i)} (x) p (\beta^{(i)} (x), |\beta^{(i)} (x) + h| - \beta^{(i)} (x))) + \\
+ \sum_{i=0}^{r-1} k^{(i)} (x) \left[ \overline{K} (\beta^{(i)} (x)) \int_{H (\beta^{(i)} (x))} \tilde{d} (x, \bar{\omega} (|\beta^{(i)} (x) + h| - \beta^{(i)} (x)))) ds \right] + \\
+ k^{(r)} (x) \left\| u (\beta^{(r)} (x + h)) - u (\beta^{(r)} (x)) \right\|, \quad x, x + h \in G, \quad r = 1, 2, \ldots
\]

It follows from the inequalities
\[
k^{(r)} (x) \left\| u (\beta^{(r)} (x + h)) - u (\beta^{(r)} (x)) \right\| \leq k^{(r)} (x) \left\| u (\beta^{(r)} (x)) \right\| + k^{(r)} (x) \left\| u (\beta^{(r)} (x + h)) \right\| \\
\leq k^{(r)} (x) \tilde{z} (\beta^{(r)} (x)) + k^{(r)} (x) \tilde{z} (\beta^{(r)} (x + h)), \quad x, x + h \in G, \quad r = 0, 1, 2, \ldots,
\]
and from conditions (9) and (52) that

(55) \[
\lim_{r \to \infty} k^{(r)} (x) \left\| u (\beta^{(r)} (x + h)) - u (\beta^{(r)} (x)) \right\| = 0 \quad \text{uniformly in } G.
\]

By induction we easily obtain

(56) \[
|\beta^{(i)} (x + h) - \beta^{(i)} (x)| \leq \omega^{(i)} (|h|), \quad x, x + h \in G, \quad i = 0, 1, 2, \ldots
\]
Now, from (55), (56) and by the definition of $\mathcal{I}$, we have, letting $r \to \infty$ in (54),
\[
\|u(x + h) - u(x)\| \leq \sum_{i=0}^{\infty} k_i(x) p\left(\beta_i(x), \omega_i(\|h\|)\right) +
\]
\[
+ \sum_{i=0}^{\infty} k_i(x) \left[ K_0(x) \int_{\mathcal{H}(\beta_i(x), \omega_i(\|h\|))} \mathcal{I}(x(s), \omega_i(\|h\|)) ds \right] = \mathcal{I}(x, \|h\|),
\]
which completes the proof of (53).

Remark 7. If the functions $k, \bar{k}, K, \beta$ are non-decreasing in $G$ and $H_j(x) < H_j(\bar{x})$ for $x < \bar{x}, x, \bar{x} \in G, j = 1, 2, \ldots, m$, then assumption 2' of Lemma 13 is satisfied. This fact follows from assertion 4' of Lemma 1 and from (9).

Now, we have the following

THEOREM 3. Suppose that:

1° Assumptions $H_4, H_5, H_6$ are satisfied,

2° Assumption $H_7$ is fulfilled for $p, \bar{K}$ defined by relations (50), (51),

3° condition (52) of Lemma 13 holds.

Under these assumptions equation (2) has at least one solution $\bar{u} \in \bar{W}$.

Proof. It follows from Lemmas 10, 12, 13 that the continuous operator $U$ maps the compact and convex set $\bar{W} \subset C(G, \mathcal{B})$ into itself. By the Schauder fixed point theorem there exists at least one solution $\bar{u} \in \bar{W}$ of equation (2).

LEMMA 14. If

1° $k(x) \leq \bar{k} = \text{const}$, $K(x) = (K_{k_0+1}(x), \ldots, K_m(x)) \leq (\bar{K}_{k_0+1}, \ldots, \bar{K}_m) = \text{const},$

2° the functions $\phi_i$ and $\beta$ satisfy conditions 2' and 3', respectively, of Lemma 4,

3° there exist constants $\omega_0$ and $D$ such that $D_i(t), \bar{d}_j(t), \bar{d}_r(t), \bar{d}_\tau(t) \leq D t, i = 1, \ldots, k_0, j = 1, \ldots, m, r = k_0+1, \ldots, m, \text{ and } \omega(t) \leq \omega_0 \cdot t$,

4° $\bar{k} \prod_{i=1}^{k_0+1} \bar{\beta}_i < 1$ for $j = k_0+1, \ldots, m$,

5° $\bar{k} \cdot \omega_0 < 1$,

then conditions 3°–6° of Assumption $H_7$ are fulfilled.

The proof of this lemma is similar to the proof of Lemma 4. Using this Lemma we can easily formulate a theorem which is more effective than Theorem 3.
References


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