Subharmonic analogues of MacLane’s classes

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1. Introduction. It is natural to ask whether the classes $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{L}$ introduced by MacLane in [3] have subharmonic (s.h.) analogues, and if so what relations exist between them. We shall see that these analogues do exist and are identical as are $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{L}$. The analogues $\mathcal{A}_s$ and $\mathcal{B}_s$ of $\mathcal{A}$ and $\mathcal{B}$ are defined in the natural way, but we must take care when defining $\mathcal{C}_s$, the analogue of $\mathcal{L}$, in order to cater for the possibility of a s.h. function being locally constant, in which case it would have both a Koebe sequence of level curves and asymptotic values. The criterion we adopt is therefore somewhat different, as is seen in Definition 6.

We start off by defining what we mean by a continuum tending to the boundary of a domain.

Definition 1. If $D$ is a domain we say that $\Gamma$ is a continuum in $D$ tending to the boundary of $D$ if

$$\Gamma = \bigcup_{n=1}^{\infty} \gamma_n,$$

where

(i) $\gamma_n$ is a continuum lying in $D$ for each $n$,

(ii) $\gamma_{n+1} \cap \gamma_n \neq \emptyset$ for any $n$,

(iii) given any compact subset $E$ of $D$, we can find an integer $n_0$ such that $E \cap \gamma_n = \emptyset$ for $n > n_0$.

If in addition $\gamma_n \to \zeta$ as $n \to \infty$ in the sense that $\text{diam} \gamma_n \to 0$ as $n \to \infty$ and $d(\gamma_n, \zeta) \to 0$ as $n \to \infty$, then we say that $\Gamma$ tends to the point $\zeta$. Clearly by (iii) $\zeta$ must lie on the boundary of $D$.

Definition 2. Let $D$ be a domain and let $\{\gamma_n\}$ be a sequence of continua in $D$ satisfying (i) and (iii) of Definition 1 but such that $\gamma_m \cap \gamma_n = \emptyset$ for all $m, n \geq 1$ with $m \neq n$. Then $\{\gamma_n\}$ is said to be a sequence of Koebe continua, and we shall use the term Koebe sequence in this paper to denote such a sequence. (Thus we are dropping the condition normally imposed on Koebe sequences that the $\gamma_n$’s be analytic arcs, and asking merely that they be continua.) If $\gamma$ is an arc of $|z| = 1$ and
\[
\max_{z \in \gamma_n} \min_{z' \in \gamma} |z - z'| + \max_{z' \in \gamma} \min_{z \in \gamma_n} |z - z'| \to 0 \quad \text{as} \ n \to \infty,
\]
we say that \( \gamma_n \to \gamma \) as \( n \to \infty \).

**Definition 3**: \( \mathcal{A}_s \). Suppose that \( u(z) \) is s.h. in the unit disc \( |z| < 1 \) and that \( u(z) \to a \) as \( |z| \to 1 \) along some continuum \( \Gamma \) tending to a point \( \zeta \) of \( |z| = 1 \) (i.e. \( u(z) \to a \) as \( z \in \Gamma \to \zeta \)).

Then we say \( \zeta \in \mathcal{A}'_a \) if the set
\[
\mathcal{A}' = \bigcup_{-\infty < a < +\infty} \mathcal{A}'_a
\]
is dense on \( |z| = 1 \), then we say that \( u \in \mathcal{A}_s \).

**Definition 4**: \( \mathcal{B}_s \). Suppose that \( u(z) \) is s.h. in \( |z| < 1 \) and that \( u(z) \) is bounded above by some constant \( M(\zeta) \) on a continuum \( \Gamma \) tending to a point \( \zeta \) (\( |\zeta| = 1 \)). Then we say that \( \zeta \in \mathcal{B}' \).

If \( B' \cup \mathcal{A}'_c \) is dense on \( |z| = 1 \), we say that \( u \in \mathcal{B}_s \).

**Definition 5**. For our purposes, a _subarc_ of an arc \( \gamma \) will be a closed arc \( \gamma' \) lying in \( \gamma \) and not containing either of the end points of \( \gamma \).

**Definition 6**: \( \mathcal{C}_s \). Suppose that \( u(z) \) is s.h. in \( |z| < 1 \) and that \( \{\gamma_n\} \) is a Koebe sequence tending to an arc \( \gamma \) of \( |z| = 1 \), where \( \gamma \) does not reduce to a point. Suppose that \( u(z) \) is bounded above by \( M \) on the sequence \( \{\gamma_n\} \). If these conditions always imply that for any interior point \( \zeta \) of \( \gamma \), \( u(z) \) is bounded above in some neighborhood \( N_\delta(\zeta) = \{z: |\zeta - z| < \delta \land |z| < 1\} \) of \( \zeta \) in \( |z| < 1 \), we say that \( u \in \mathcal{C}_s \).

Our result is

**Theorem 1**. \( \mathcal{A}_s = \mathcal{B}_s = \mathcal{C}_s \).

2. **Topological preliminaries to the proof**. Suppose that \( u(z) \) is s.h. on a compact set \( E \) (i.e. in some open set \( O \supset E \)) and let \( F \) denote the set of points \( \{z: u(z) \geq K, z \in E\} \), where \( K \) is some real number. Then since \( u(z) \) is upper semi-continuous (u.s.c.), \( F \) is a closed set and therefore decomposes into closed maximal connected subsets \( F_a \) called the *components* of \( F \), each point of \( F \) belonging to exactly one component \( F_a \), where \( a \) runs over some index set \( I \).

Now suppose that instead of on a compact set \( E \), \( u(z) \) is s.h. in \( |z| < 1 \). Let \( z_0 \) be a point in \( |z| < 1 \) with \( u(z_0) \geq K \).

**Definition 7**. The _compartment_ of the set \( \{z: u(z) \geq K, |z| < 1\} \) containing \( z_0 \) is defined to be
\[
C(z_0, K, 1) = \bigcup_{r < 1} C(z_0, K, r),
\]
where \( C(z_0, K, r) \) is the component of \( \{z: u(z) \geq K, |z| \leq r\} \) containing \( z_0 \).

Since \( C(z_0, K, r_1) \subset C(z_0, K, r_2) \) for \( r_1 < r_2 < 1 \), it follows that the compartment \( C(z_0, K, 1) \) is connected.
In a similar way we can define compartments of the set \( \{ z : u(z) < K, \ |z| < 1 \} \), but since this set is open there is no difficulty in defining the components, and the definitions of compartment and component give rise to the same sets.

3. **Proof of Theorem 1.** Following the preliminaries, we now set out to prove the theorem. The proof consists of proving the following inclusion relations:

(i) \( \mathcal{A}_s \subset \mathcal{B}_s \), 
(ii) \( \mathcal{B}_s \subset \mathcal{C}_s \), 
(iii) \( \mathcal{C}_s \subset \mathcal{A}_s \),

so that, as opposed to MacLane's proof it is not necessary to show that \( \mathcal{C}_s \subset \mathcal{B}_s \) and \( \mathcal{B}_s \subset \mathcal{A}_s \); these two steps are combined in (iii) above.

3.1. \( \mathcal{A}_s \subset \mathcal{B}_s \).

This is clear, for on any continuum \( \Gamma \) tending to a point \( \zeta \) and on which \( u(z) \to a < \infty \), \( u(z) \) is bounded above so that \( \zeta \in B' \).

3.2. \( \mathcal{B}_s \subset \mathcal{C}_s \).

![Diagram](image)

Fig. 1

Suppose that \( u \in \mathcal{B}_s \) and that \( \gamma_n \to \gamma \) is a Koebe sequence on which \( u(z) < M' < \infty \). Let \( \gamma' \) be a subarc of \( \gamma \).

We observe first that \( \gamma' \cap A'_{\infty} = \emptyset \), since any continuum \( \Gamma \) tending to \( \zeta \in \gamma' \) meets infinitely many \( \gamma_n \) in any neighbourhood of \( \zeta \), so that \( u(z) \) cannot tend to infinity on \( \Gamma \).

Thus \( B' \) is dense on \( \gamma' \); choose distinct points \( a, \beta \in \gamma' \cap B' \) and let \( \Gamma_1, \Gamma_2 \) be continua tending to \( a, \beta \) respectively on which \( u(z) < M'' \) say; let \( M = \max( M', M'' ) \).

Since \( \gamma' \) is a subarc of \( \gamma \), and \( \gamma_n \to \gamma \) as \( n \to \infty \), it follows that \( \Gamma_1 \) and \( \Gamma_2 \) must meet \( \gamma_n \) for \( n \geq n_0 \) say. \( \gamma_n \) is a continuum, therefore compact and lying in \( |z| \leq r_0 < 1 \) say; let \( D_n \) be any domain whose boundary consists of subsets of \( \Gamma_1, \Gamma_2, \gamma_{n_0} \) and \( \gamma_n \) for \( n > n_0 \). Then \( u(z) < M \) on the boundary of \( D_n \) and therefore inside \( D_n \) by the maximum principle.
Let $\gamma''$ be a subarc of $\gamma'$ and let $z_0$ be a point lying in a small neighbourhood of $\gamma''$ with $|z_0| > r_0$. Since $\gamma_a$ lies finally outside any compact subset of $|z| < 1$, it follows that $z_0$ must lie in some domain $D_a$, so that $u(z_0) < M$; since $z_0$ is arbitrary it follows that $u(z) < M$ in a small enough neighbourhood of $\gamma''$. Since $B'$ is dense, $\alpha$ and $\beta$ may be chosen to lie outside $\gamma''$ and since $\gamma'$ is arbitrary, $\gamma''$ is an arbitrary subarc of $\gamma$. Thus $u \in \mathcal{G}_s$.

4. The kernel of the proof consists of showing that $\mathcal{G}_s \subset \mathcal{A}_s$; this is quite a lengthy procedure and we need the following result, which is of some interest in itself.

**Theorem 2.** If $u(z)$ is s.h. in $|z| < 1$ and $u(z_0) > M$, then there exists a continuum $\Gamma$, containing $z_0$ and tending to $|z| = 1$, on which $u(z) \geq M$ and $u(z) \to M_1 \geq u(z_0)$ as $|z| \to 1$ along $\Gamma$.

The proof of this result requires five subsidiary lemmas, three of which are due to M. N. M. Talpur ([4], §1) and one of which is an immediate consequence of Lemma 3. The fifth lemma is the s.h. form of Hadamard's convexity theorem.

**Lemma 1.** Let $u(z)$ be s.h. in $|z| < R$. Then if $u(z_0) > M_0$, there is a circle $C$, centre $z_0$ and lying in $|z| < R$, such that $u(z) \geq M_0$ on $C$.

**Note.** Since $u(z)$ is s.h. in $|z| < r$ for all $r < R$, we may choose $C$ to have arbitrarily small radius.

**Lemma 2.** If $u(z)$ is s.h. in $|z| \leq r$, then each component of the set \{z: u(z) \geq K\} contains points of modulus $r$.

From this we deduce immediately

**Lemma 3.** If $u(z)$ is s.h. in $|z| < 1$, then each compartment of the set \{z: u(z) \geq K\} goes to the boundary, i.e. possesses limit points of modulus 1.

**Lemma 4.** Let $u(z)$ be s.h. in $|z| < r$ and let $K$ be finite. Set

$$u_r(z) = \begin{cases} u(z), & z \in C(z_0, K, r), \\ K, & \text{elsewhere in } |z| < r. \end{cases}$$

Then $u_r(z)$ is s.h. in $|z| < r$.

**Lemma 5.** Let $u(z)$ be s.h. in $|z| < 1$ and let $B(r) = \sup |u(z)|$. Then $B(r)$ is an increasing convex function of $\log r$, so that if $0 < r_0 < r < r_1 < 1$ and $|z| = r$ it follows that

$$u(z) \leq \frac{B(r_0) \log r_1/r + B(r_1) \log r/r_0}{\log r_1/r_0}.$$ 

If $B(1) = \lim_{|z| \to 1} \sup |u(z)|$, it follows on letting $r_1 \to 1$ that

$$u(z) \leq \frac{B(r_0) \log 1/r + B(1) \log r/r_0}{\log 1/r_0}.$$
We now prove Theorem 2. If \( u(z) \) is a constant, then the result is trivial; we assume that \( u(z) \) is non-constant in \( |z| < 1 \). Choose \( z_0, |z_0| < 1 \), and suppose that \( u(z_0) > M \). Let \( C(z_0, M, r) \) and \( u_r(z) \) be as previously defined and set

\[
M_0 = \sup_{z \in C(z_0, M, 1)} u(z).
\]

We see first that \( M_0 > u(z_0) \), for \( u(z_0) \leq M_0 \), since \( z_0 \in C(z_0, M, 1) \) and if \( u(z_0) = M_0 \), then \( u_r(z) = M_0 \), so that \( C(z_0, M_0, r) \) contains the whole of \( |z| < r \). Thus \( u_r(z) = u(z) = u(z_0) \) for \( |z| < r \); since this is true for all \( r < 1 \) it follows on letting \( r \to 1 \) that \( u(z) \equiv M_0 \) in \( |z| < 1 \), contrary to our assumption.

Next, we show that given \( r_0 < 1 \) there is an \( M_1 = M_1(r_0) \) such that \( u(z) < M_1 < M_0 \), and so \( u_\varrho(z) < M_1 \), for \( z \in C(z_0, M, 1) \) and \( |z| < r_0 \), and all \( \varrho \) such that \( r_0 < \varrho < 1 \).

To see this we use Lemma 5. By a suitable conformal map we may assume that \( z_0 = 0 \). Choose \( \varepsilon > 0 \) so that \( M'' = u(0) + \varepsilon < M_0 \). Then for \( |z| \leq \delta_0 \) say we have \( u(z) \leq M'' \) for otherwise the u.s.c. of \( u(z) \) at \( z = 0 \) would be contradicted. Thus \( u_\varrho(z) \leq M'' \) for \( |z| \leq \delta_0 \) and \( 0 < \varrho < 1 \). Thus for \( |z| = r_0 \), and \( \delta_0 < r_0 < r_1 \leq \varrho < 1 \) we have

\[
u_\varrho(z) \leq \frac{M'' \log r_1/r_0 + M_0 \log r_0/\delta_0}{\log r_1/\delta_0}
\]

by Lemma 5. By letting \( \varrho \to 1 \), we see that the same inequality holds on \( C(z_0, M, 1) \) with \( u(z) \) instead of \( u_\varrho(z) \). Now let \( r_1 \to 1 \); we obtain

(1) \[
u(z) \leq \frac{M'' \log r_1/r_0 + M_0 \log r_0/\delta_0}{\log r_1/\delta_0} = M_1'(r_0) \text{ (say)}
\]

for \( z \in C(z_0, M, 1) \). Since \( r_0 < 1 \), it follows that \( M_1'(r_0) < M_0 \).

Now choose \( z_1 \) and \( \varrho_1 \) with \( |z_1| = r_1 > r_0 \) such that \( u_\varrho_1(z_1) > M_1 \). Then \( z_1 \in C(z_0, M, \varrho_1) \), which also contains \( z_0 \) and is a continuum \( \gamma \) say. It follows from (1) and our choice of \( z_1 \) that \( C(z_1, M_1, 1) \) lies in \( |z| \geq r_0 \).

Next, we repeat the above argument to obtain that given \( r_2 > r_1 \), there is an \( M_2 = M_2(r_2) \) such that \( u(z) < M_2 < M_0' \) for \( z \in C(z_1, M_1, \varrho_1) \) and \( |z| < r_2 \), where \( M_0' = \sup_{z \in C(z_1, M_1, 1)} u(z) \). To do this we use, instead of the function \( u_\varrho(z) \) which is defined relative to \( C(z_0, M, 1) \), the function

\[
v_\varrho(z) = \begin{cases} u(z), & z \in C(z_1, M_1, \varrho), \\ M_1, & \text{elsewhere in } |z| < 1, \end{cases}
\]

which is easily seen to be s.h. in \( |z| < \varrho, r_2 < \varrho < 1 \).

It is clear that \( v_\varrho(z) = v_\varrho(z) \) for \( z \in C(z_1, M_1, \varrho) \), and the existence of \( M_2 < M_0' \) follows since otherwise \( v_\varrho(z) \), therefore \( u_\varrho(z) \) and so \( u(z) \), would be constant in \( |z| < \varrho \) for all \( \varrho < 1 \) and thus in \( |z| < 1 \).
Now choose $z_2$ and $\varrho_2$ such that $u_{\varrho_2}(z_2) > M_2$, and denote the continuum $C(z_1, M_1, \varrho_2)$ by $\gamma_2$. Then $\gamma_1 \cap \gamma_2$ contains the point $z_1$, and on $\gamma_2$, $u(z) \geq M_1$. $\gamma_2$ lies outside $|z| < r_1$ by construction. To carry on the construction, we choose $z_n$ and $\varrho_n$ so that $u_{\varrho_n}(z_n) > M_n$ and set $\gamma_n = C(z_{n-1}, M_{n-1}, \varrho_n)$. Then $\gamma_n$ by construction will lie in $|z| \geq r_{n-1}$; if we choose $r_n \to 1$, then $\bigcup_{n=1}^{\infty} \gamma_n$ is a continuum tending to the boundary.

We obtain in this way an increasing sequence $\{M_n\}$ and a decreasing sequence $\{M_n^k\}$, with $M_0^k = \sup_{z \in C} u(z)\sup_{z \in C} M_k, 1)$. If $M_0^k = +\infty$, then we may choose $M_n$ as large as we please; since $\gamma_n \to |z| = 1$ and $u(z) > M_{n-1}$ on $\gamma_n$, it follows that $u(z) \to \infty$ as $\Gamma$ tends to the boundary.

![Fig. 2](image_url)

If $\lim M_0^k = L < \infty$, then we must choose $M_n$ so that $M_n \to L$ as $n \to \infty$. (It follows from the convexity argument that this is possible.) If the $M_n$ are chosen in this way, then $u(z) \to L$ as $\Gamma \to |z| = 1$; since in both cases $\{M_n\}$ is increasing it is clear that the other conclusions of the theorem also hold.

This completes the proof of Theorem 2.

We next state another standard result which is Brelot's form of the Milloux–Schmidt inequality ([4], § 1).

**Lemma 6.** Let $u(z)$ be s.h. in $|z| \leq R$ and suppose that $u(z) \leq 1$ there.

Let

$$\inf_{|z| = r} u(z) \leq 0$$

for all $r$, $0 \leq r \leq R$. Then

$$u(z) \leq \frac{4}{\pi} \tan^{-1} \sqrt{\frac{r}{R}} \quad (r = |z|).$$

This lemma gives almost immediately

**Lemma 7.** Let $u(z)$ be s.h. in $|z| < 1$ and let $|z_0| < 1$. Let $\{\gamma_n\}$ be a sequence of continua in $|z| < 1$ on which $u(z) < M$, and such that $\text{diam}(\gamma_n)$
Suppose that any neighbourhood $N$ of $z_0$ meets infinitely many of the $\gamma_n$. Then $u(z_0) \leq M$.

Proof. Let $z_n \in \gamma_n$ be a sequence of points tending to $z_0$, suppose that for $n > n_0$, $|z_n - z_0| < \frac{1}{3}(1 - |z_0|)$. Let $R = \min\left(\delta, \frac{1}{3}(1 - |z_0|)\right)$. Let $B(r) = \sup_{|z| < 1 - \frac{1}{3}(1 - |z_0|)} u(z) < +\infty$.

We apply Lemma 6 to the function $\frac{1}{B(r)} \{u(z) - M\}$ on circles of radius $R$ centred on the points $z_n, n > n_0$, and let $r_n$ denote $|z_n - z_0|$. Since for $n$ large enough $z_0$ will lie inside these circles, we obtain that

$$u(z_0) \leq M + \frac{4}{\pi} \tan^{-1} \sqrt{\frac{r_n}{R}} \{B(r) - M\}.$$

Letting $z_n \to z_0$, i.e. $r_n \to 0$, we see that $u(z_0) \leq M$.

5. Our proof of Theorem 1 is now completed by introducing the following classification of boundary points, and investigating their properties.

**Definition 7.** Let $\zeta_0$ be a point of $|z| = 1$, and let $M$ be given. If for every small $\delta > 0$, the neighbourhood $N_{\delta}(\zeta_0) = \{|z| < 1\} \cap \{|\zeta_0 - z| < \delta\}$ contains points belonging to a component of $u(z) < M$ which has at least one limit point of modulus 1, then we say that $\zeta_0$ is $M$-barred. Here and subsequently ‘component’ in this context means a component relative to the neighbourhood.

If $\zeta_0$ is $M$-barred for some $M$, we say that $\zeta_0$ is barred.

Otherwise, we say that $\zeta_0$ is free.

We now establish results on free and barred points. We first need another lemma due to Talpur ([4], § 1) namely

**Lemma 8.** Suppose that $u(z)$ is s.h. in a neighbourhood $N$ of a continuum $\gamma$ and that $u(z) \geq K$ for $z \in \gamma$. Let $z_1, z_2$ be two points of $\gamma$. Then given $\epsilon > 0$, we can find a polygonal path joining $z_1$ to $z_2$ in $N$ such that $u(z) > K - \epsilon$ on this path.

Talpur proves this result with $K - 1$ instead of $K - \epsilon$. However, our result follows immediately by applying this result to the function $v(z) = \epsilon u(z) + (1 - \epsilon)K$.

The proof includes the following result, which is a deduction from a theorem of Hayman ([1], Theorem 4, p. 193) and which we shall use explicitly:

**Lemma 9.** Suppose that $u(z)$ is s.h. in $|z| < 1$ and that $z_1, z_2$ are two
points in $|z| \leq r_0 < 1$ at which $u(z) \geq K$. Then we can find a $\delta = \delta(\varepsilon, r_0)$ such that if $|z_1 - z_2| < \delta(\varepsilon, r_0)$, then $z_1$ can be joined to $z_2$ by a zigzag path $[z_1, \xi] \cup [\xi, z_2]$ on which $u(z) \geq K - \varepsilon$, and such that

\[(*) \quad |\xi| < r_0 \quad \text{and} \quad |z_1 - \xi| + |z_2 - \xi| \leq 2|z_1 - z_2|.
\]

We show with the aid of these results

**Lemma 10.** Let $\zeta$ be a free point of $|z| = 1$. Then one of the three following possibilities must hold:

(i) $\zeta \in A'_\infty$.

(ii) $u \notin B'_\infty$.

(iii) There exist points in $A'$ arbitrarily near $\zeta$.

**Proof.** We note first that $u(z)$ is unbounded on the radius at $\zeta$, for if $u(z) < M$ on this radius $R$, then $R$ would belong to a component of $u < M$; thus since $R$ ends at $\zeta$, $\zeta$ would be $M$-barred contrary to hypothesis.

Now if $M(\xi) = \lim_{\xi \to \xi \text{ radially}} u(z)$, then $M(\xi) \to \infty$ as $\xi \to \zeta$, for if $\lim_{\xi \to \xi \text{ radially}} M(\xi) < M_1 < \infty$, then every neighbourhood $N_\delta(\zeta)$ contains points $\xi$, where $M(\xi) < M_1 + 1$, and so an end of a radius on which $u(z) < M_1 + 2$. Thus $\zeta$ would be barred, which is again contrary to hypothesis.

Next, suppose that $u \notin B'_{\infty}$; then given $N'_{M}$, we can find a neighbourhood $N_\delta(M)$, of $\zeta$ such that all components of $u < M$ for fixed $M$ which contain points in $N_\delta(M)$ have compact closures in $N_\delta(M)$, or else (iii) holds.

To see this, suppose that we cannot find such neighbourhoods $N_\delta(M)$, $N_\delta(M)$, $N_\delta(M)$. Then we recall that since $\zeta$ is a free point these components have no limit points on $|z| = 1$ for small enough $\delta$; suppose that we can find a sequence of points $z_n \to \zeta$ such that each $z_n$ lies in a different component $K_n$ of $u < M$, in some fixed neighbourhood $N$ for all $n$.

If in $N$ these components lie finally outside any compact subset of $|z| < 1$, then we can select from them a sequence of Koebe continua on which $u < M$, tending to an arc of $|z| = 1$ which contains $\zeta$ (possibly as an endpoint). Choose $\xi_1$ and $\xi_2$ on this arc; then since $u \notin B_{\infty}$, $u$ is bounded in a neighbourhood of the arc $[\xi_1, \xi_2]$ and so has finite asymptotic values. Since $\xi_1$ can be chosen arbitrarily near to $\zeta$, it follows that (iii) holds.

If the components do not lie finally outside any compact subset of $|z| < 1$, then a subsequence $\{K_n\}$ must meet the circle $|z| = r_1$ for some $r_1 < 1$, at points $\{z'_n\}$ say. These points $\{z'_n\}$ have a limit point $z_0$ with $|z_0| = r_1$. We may apply Lemma 7 to deduce that $u(z_0) \leq M$, and from the u.s.c. of the function it follows that if $\delta'$ is small enough, $u(z) < M + 1$ in $|z - z_0| < \delta'$; thus the component of $u < M + 1$ containing $z_0$ also contains infinitely many $K_n$, and so infinitely many $z_n$. Since $z_n \to \zeta$, it follows that $\zeta$ is a limit point of this component of $u < M + 1$. 

Unfortunately, this is not enough to show that \( \zeta \) is \((M + 1)\)-barred, for the components are defined relative to \( N \) and as \( N \) decreases the component of \( u < M + 1 \) may well split up into infinitely many components in a smaller neighbourhood, as indicated below (Fig. 3).

![Fig. 3](image)

To overcome this difficulty, the argument must now be repeated in some smaller neighbourhood \( N' \); we obtain in a similar way that there exists a point of \( \{|z| = 1\} \cap \delta N' \) which is a limit point of a component of \( u < [(M + 1) + (\frac{1}{2})^k] \) say; repeating the argument as many times as we please, we obtain a sequence of limit points \( \{\zeta_n\} \) of some component of \( u < M + \sum_0^\infty 2^{-n} = M + 2 \), with \( \zeta_n \rightarrow \zeta \) \((n \rightarrow \infty)\) and \(|\zeta_n| = 1\) for all \( n \).

It follows that \( \zeta \) is \((M + 2)\)-barred, contrary to hypothesis, so that if neither (i) nor (iii) hold, then given \( N_\delta^M \) we can find \( N_\delta^M \), with the required properties.

Now choose a sequence of neighbourhoods \( N_\delta^M \) \((k \geqslant 0)\) which are nested and whose diameter tends to zero. Then by the above we can find \( \delta' < \delta \) such that any component meeting \( N_\delta^M \) has compact closure in \( N_\delta^M \); we choose \( \delta_0 \) so that no component of \( u < M + 1 \) in \(|z - \zeta| < \delta_0\) has a limit point on \(|z| = 1\).

Choose \( z_1 \in N_{\delta_1}^{M+1} \) and \( z_2 \in N_{\delta_2}^{M+2} \) such that
(a) \( z_1 \) and \( z_2 \) lie on the radius \( R \) and \(|z_1| < |z_2| = r_0 < 1\).
(b) \( u(z_1) \geqslant M, \ u(z_2) \geqslant M + 1\).

![Fig. 4](image)

Then by Lemma 9 with \( \varepsilon = \frac{1}{4} \), we can find a value \( \rho > 0 \) such that if \(|\zeta_1|, |\zeta_2| < r_0 < 1\) and \( u(\zeta_1), u(\zeta_2) \geqslant K \), then if \(|\zeta_1 - \zeta_2| < \rho, \zeta_1 \) and \( \zeta_2 \) can be joined by a zig-zag on which \( u(z) \geqslant K - \frac{1}{4} \).
We now split the radial segment \([z_1, z_2]\) into two classes of points, those where \(u(z) \geq M\) and those where \(u(z) < M\). The latter set is the intersection of \(R\) with the set \(u < M\) and so consists of open intervals of various lengths; at the end points of these intervals \(u(z) \geq M\). The component of \(u < M\) separating them is a simply connected domain whose closure lies in \(N_{\delta_1}^M\), so that its boundary is a continuum, which we can surround by an open neighbourhood lying in \(N_{\delta_1}^M\), and so we may apply Lemma 8 with \(\epsilon = \frac{1}{2}\) to deduce the existence of a polygonal path \(\pi\) lying in \(N_{\delta_1}^M\) and joining \(\zeta_1\) to \(\zeta_2\), on which \(u(z) \geq M - \frac{1}{2}\).

We now construct our asymptotic path; we consider first the ‘large’ intervals of \(R\) where \(u(z) < M\), namely those whose length is at least \(\frac{1}{2}\). These are only finite in number, and we connect their endpoints by a polygonal path as above on which \(u(z) \geq M - \frac{1}{2}\).

On a complementary interval \([a, b]\) of \(R\), we can find a sequence of points \(a = \xi_0, \xi_1, \ldots, \xi_n = b\) such that \(\frac{1}{2}\theta < |\xi_{i+1} - \xi_i| < \theta \) and \(u(\xi_i) \geq M\). We then join \(\xi_i\) to \(\xi_{i+1}\) by a zig-zag which \(u(z) \geq M - \frac{1}{2}\). Since \(n\) is finite, it follows that \(a, b\) can be joined by a polygonal path.

Combining these two sets of polygonal paths, we obtain a polygonal path \(\pi_1\) joining \(z_1\) to \(z_2\) on which \(u(z) \geq M - \frac{1}{2}\) and which lies in a \((\delta_1 + \theta_1)\) neighbourhood of \(\zeta\).

We now choose \(z_3\) to lie in \(N_{\delta_3}^{(M+1)} \cap R\) with \(u(z_3) \geq M + 2\) and \(|z_3| = r_1\), \(r_0 < r_1 < 1\). Proceeding as above, we construct a polygonal path \(\pi_2\) joining \(z_2\) to \(z_3\), lying in a \((\delta_2 + \theta_2)\) neighbourhood of \(\zeta\) and on which \(u(z) \geq M + \frac{1}{2}\).

Continuing in this way, choosing \(\delta_n \to 0\) and \(\theta_n \to 0\), we obtain a sectionally polygonal path \(\pi = \bigcup_{r^{-1}} \pi_n\) ending at \(\zeta\), such that \(u(z) \to \infty\) as \(z \to \zeta\) on \(R\). Thus \(\zeta \in A_\infty\).

**Lemma 11.** If \(\gamma\) is an arc of \(|z| = 1\) containing no free points, then there is a value \(M\) and a subarc \(\gamma'\) of \(\gamma\) such that the set of \(M\)-barred points is dense on \(\gamma'\).

**Proof.** The set of all barred points is the union of the sets \(E_n\) of \(n\)-barred points for integral \(n\), and so is a countable union. If no such arc \(\gamma'\) exists, then \(\gamma\) is the union of a countable number of nowhere dense sets and so is of the first category in itself, which is impossible. Thus one of the sets \(E_n\) must be dense on some arc \(\gamma'\), and the lemma is proved with \(M = n\).

**Lemma 12.** If the set of \(M\)-barred points is dense on an arc \(\gamma\), and \(u(z)\) is unbounded near every point of \(\gamma\), then there exists a continuum \(\Gamma\) in \(|z| < 1\) tending to a point of \(\gamma\) and on which \(u(z) \to M' > M\).

**Proof.** Let \(\zeta_1, \zeta_2\) be distinct \(M\)-barred points on \(\gamma\), and let \(D_1\) and \(D_2\) be components of \(u < M\) which have limit points of modulus 1 in disjoint
neighbourhoods of $\zeta_1$, $\zeta_2$ respectively. Let $K_1$, $K_2$ be components containing limit points of modulus 1 of the restrictions of $D_1$, $D_2$ respectively to the neighbourhoods $N_1$, $N_2$ of $\zeta_1$, and $\zeta_2$.

Let $r_0 < 1$ be chosen so that $|z| = r_0$ meets both $K_1$ and $K_2$ and let $B(r_0) = \sup_{|z| = r_0} u(z)$.

Let $D$ be the domain bounded by $\partial K_1$, $\partial K_2$, $|z| = r_0$ and $\gamma$, and choose $z_0 \in D$ such that

$$\mu(z_0) > M_1 = \sup\{M, B(r_0)\}.$$ 

Then using Theorem 2 we construct a continuum $\Gamma$ tending to $|z| = 1$ such that $u(z) \to M > M_1$ on $\Gamma$, and $u(z) > M_1$ on $\Gamma$. Then $\Gamma'$ must end on $\gamma$, for otherwise it would have to cross the boundary of $D$ and we would have $u(z) \leq M$ at such a point.

Further, $\Gamma'$ must tend to a point, for if $\zeta_1$, $\zeta_2'$ are distinct limit points of $\Gamma$ with modulus 1, we choose $\zeta$ between them so that $\zeta$ is $M$-barred. $\Gamma'$ must then meet a component of $u < M$ which has a limit point near $\zeta$, thus contradicting the fact that $u > M_1$ on $\Gamma$.

Proof that $\mathcal{C}_s \subset \mathcal{A}_s$. Suppose that $u \notin \mathcal{A}_s$. Then there is an arc $\gamma$ free of $A'$, so that in particular $u(z)$ is unbounded near every point of $\gamma$ by Littlewood's theorem ([4], p. 169). If $\gamma'$ is a subarc of $\gamma$, and some $\zeta \in \gamma'$ is free, then $u \notin \mathcal{C}_s$ by Lemma 10 since $\gamma'$ is free of $A'$.

If not, then by Lemma 11 we can find $M$ and $\gamma'' \subset \gamma'$ such that the set of $M$-barred points is dense on $\gamma''$. We then apply Lemma 12 to deduce the existence of asymptotic values at points of $\gamma''$, which contradicts the assumption that $\gamma'$ is free of $A'$.

Thus $\mathcal{C}_s \subset \mathcal{A}_s$, and the proof of Theorem 1 is complete.

We can show slightly more, namely that instead of asymptotic values on continua we can have asymptotic values on sectionally polygonal paths. This is

**Theorem 3.** Let $u(z)$ be s.h. in $|z| < 1$. Then if $u(z) \to a$ as $z \to \zeta$ ($|\zeta| = 1$) on a continuum $\Gamma$, there exists a sectionally polygonal Jordan arc $\pi$ in $|z| < 1$ ending at $\zeta$ on which $u(z) \to a$.

Proof. Let $\Gamma = \bigcup_{n=1}^{\infty} \gamma_n$, where $\gamma_n \cap \gamma_{n+1} \neq \emptyset$, and choose $\varepsilon_n \to 0$.

Suppose that $a < \infty$. Let

$$\mu^*_n = \begin{cases} \sup u(z), \\ \inf \varepsilon \in \gamma_n, \end{cases}$$

Then $\mu^*_n \to a$, $\mu^*_n \to a$ and by u.s.c., $\gamma_n$ lies in a domain $D_n$, where $u(z) < \mu^*_n + \varepsilon_n$. Let $z_n \in \gamma_{n-1} \cap \gamma_n$ ($n \geq 2$). Then by Lemma 8, $z_n$, $z_{n+1}$ can be joined by a polygonal path $\pi_n$ in $D_n$ on which

$$u(z) \geq \mu^*_n - \varepsilon_n,$$

so that $\mu^*_n - \varepsilon_n \leq u(z) \leq \mu^*_n + \varepsilon_n$ on $\pi_n$. 

.
Then \( \pi' = \bigcup_{n=1}^{\infty} \pi_n \) is the required sectionally polygonal path; \( \pi \) will end at a point provided the neighbourhoods \( D_n \) are chosen small enough, since \( \Gamma \) itself ends at a point.

By removing a countable number of loops, the path may be made into a sectionally polygonal Jordan arc.

Further, in view of [2], we have

**Theorem 4.** Let \( u(z) \) be subharmonic in \( |z| < 1 \) and let \( B(r) = \sup_{|z|=r} u(z) \).

Suppose that

\[
\int_0^1 \log B(r) \, dr < \infty.
\]

Then \( u \in A_s \).

**Proof.** The proof uses Lemma 1 of [2] and proceeds as ([2], Theorem 1) with the auxiliary function \( v(\zeta) = \log |\Phi(\zeta)| \) instead of \( \Phi(\zeta) \). Since the maximum principle holds for subharmonic functions we obtain that \( u(z) \) is locally bounded near \( \gamma^* \), and so lies in \( C_s \). The result then follows from ([2], Theorem 1)

**Theorem 5.** Given \( \epsilon > 0 \), there exists \( u(z) \) subharmonic in \( |z| < 1 \) such that

\[
B(r) < \exp \frac{\epsilon}{(1-r) \log \frac{1}{1-r}}
\]

but \( u \notin A_s \).

**Proof.** The subharmonic function \( u(z) \) constructed in the proof of [2], Theorem 2, provides such an example.

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**References**


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