A PROPERTY OF QUASI-COMPLEMENTS

BY

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A closed linear subspace $Y$ of the normed linear space $X$ is said to be quasi-complemented in $X$ if there exists a closed linear subspace $Z$ such that $Y \cap Z = \{0\}$ and $Y + Z$ is dense in $X$. In this case $Z$ is called a quasi-complement of $Y$. If $Z$ is closed, $Y \cap Z = \{0\}$ and $X = Y + Z$, then $Z$ is called a complement of $Y$. Not every closed subspace of a normed linear space has a complement. However, every closed subspace of a separable normed linear space has a quasi-complement by a result of Mackey [2]. Hence, this is one property of quasi-complements which does not hold for complements.

The purpose of this note is to give another nice property of quasi-complements which is not necessarily shared by complements and which has not been mentioned in the literature. If $Y$ and $Z$ are closed subspaces of a Banach space $X$ such that $Y \cap Z = \{0\}$, then, even if $Y$ is complemented in $X$, there does not necessarily exist a complement $Z_1$ of $Y$ such that $Z \subset Z_1$. For example, in $X = c_0$, let $Y$ be the set of all sequences of the form $(x_1, 0, x_2, 0, \ldots)$ and let $Z$ be the set of all sequences of the form $(y_1, 2^{-1}y_1, y_2, 2^{-2}y_2, \ldots)$. Then $Y$ and $Z$ are closed linear subspaces of $c_0$, $Y$ is complemented in $c_0$, and $Y \cap Z = \{0\}$. However, there is no complement of $Y$ in $c_0$ which contains $Z$. In other words, complements of a subspace do not necessarily occur homogeneously throughout $X$. We now show, however, that quasi-complements do occur more homogeneously than complements.

THEOREM 1. Let $X$ be a separable normed linear space and let $Y$ be a closed linear subspace of $X$. If $Z$ is a closed linear subspace of $X$ such that $Y \cap Z = \{0\}$, then $Y$ has a quasi-complement $Z_1$ such that $Z \subset Z_1$.

Proof. The normed linear space $X/Z$ is separable. Let $Q : X \to X/Z$ denote the quotient map $g$. By Mackey's theorem, there is a quasi-complement $Y_1$ of $Q(Y)$ in $X/Z$. Thus $Y_1 \cap Q(Y) = \{0\}$ and $Y_1 + Q(Y)$ is dense in $X/Z$. Now, $Q^{-1}(Y_1)$ is closed in $X$ and $Z \subset Q^{-1}(Y_1)$. Let $Z_1 = Q^{-1}(Y_1)$. If $x \in Y \cap Z_1$, then $Q(x) \in Y_1 \cap Q(Y) = \{0\}$, so that $x \in Z$. Since $Y \cap Z = \{0\}$, we have $x = 0$. Consequently, $Y \cap Z_1 = \{0\}$.
It remains only to show that $Y + Z_1$ is dense in $X$. Let $x \in X$ and let $U$ be an open set containing $x$. Then $Q(U)$ is an open set in $X/Z$ which contains $Q(x)$. Since $Y_1 + Q(Y)$ (and hence $Y_1 + Q(Y)$) is dense in $X/Z$, there exists a $z_1 \in Z_1$ and a $y \in Y$ such that $Q(z_1) + Q(y) \in Q(U)$. It follows that $z_1 + y \in Z + U$, so that there exists a $z \in Z$ with $y + (z + z_1) \in U$. Noting that $z + z_1 \in Z_1$, the proof is complete.

Remark. Weber [3] has generalized Mackey's result by showing that every closed linear subspace of a separable metrizable locally convex space has a quasi-complement. It is then clear that Theorem 1 holds when $X$ is such a space.

A Banach space is said to be weakly compactly generated (WCG) if there is a weakly compact set $A$ in $X$ such that $X$ is the closed linear span of $A$. In particular, every reflexive Banach space is (WCG). Lindenstrauss [1] has shown that if $X$ is a (WCG) Banach space and $Y$ is a closed linear subspace of $X$ which is also (WCG), then $Y$ is quasi-complemented in $X$. Since continuous linear images of (WCG) spaces are (WCG), Lindenstrauss' theorem, together with the same argument as in Theorem 1, yields the following

**Theorem 2.** Let $X$ be a (WCG) Banach space and let $Y$ be a closed linear subspace of $X$ which is also (WCG). If $Z$ is a closed linear subspace of $X$ such that $Y \cap Z = \{0\}$, then $Y$ has a quasi-complement $Z_1$ such that $Z \subset Z_1$.

**References**


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