ON ACYCLIC KERNELS
AND THE BARYCENTRIC HOMOMORPHISM

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1. Introduction. Throughout this paper (1) we shall assume the reader is familiar with the definitions and basic facts about chain complexes, chain mappings, the homology of a chain complex, and the singular homology of a topological space. We shall use the definitions and notation of [1]. The singular chain complex of a topological space $X$ will be denoted by $C(X)$ and the barycentric homomorphism by $\beta$. If $f: C \to C$ is a chain mapping of the chain complex $C$ into itself and $K$ denotes that subcomplex of $C$ which is the kernel of $f$, then $f$ is said to have an acyclic kernel if and only if $K$ is an acyclic chain complex, i.e., $H_p(K) = 0$ for each $p$.

In [2] Fadell has shown that the kernel of the barycentric homomorphism is acyclic. This condition is not implied by the fact that $\beta_*$ (the homomorphism that $\beta$ induces on $H(C(X))$) is an isomorphism nor even by the fact that $B \sim 1$ ($\beta$ is chain homotopic to the identity homomorphism). It is implied by the fact that there is, as Fadell shows, a chain homotopy connecting $\beta$ and 1 which is stable with respect to the kernel of $\beta$. Precisely, there is a homomorphism $\varphi : C(X) \to C(X)$ such that (1) $\varphi$ has degree 1, (2) $\varphi \partial + \partial \varphi = 1 - \beta$, and (3) $\varphi(K) \subset K$, where $K$ is the kernel. When the condition given in (3) holds we say that $\varphi$ is stable with respect to the kernel of $\beta$. Later we shall consider this notion in a more general setting.

The result that the kernel of $\beta$ is acyclic is used in [2] to obtain an "unessential identifier" for $C(X)$. For any chain complex $C$ and sub-

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complex $K$ it is shown in [3] that $K$ is acyclic if and only if $\pi_*$ is an isomorphism where $\pi : C \rightarrow C/K$ is the natural projection of $C$ onto the factor complex $C/K$. Thus, an unessential identifier for $C$ is a subcomplex which can be factored out without affecting the homology. It is not hard to show that the kernels of the iterates of $\beta$, i.e., $\beta^2 = \beta \circ \beta$, $\beta^3 = \beta \circ \beta \circ \beta$, etc., form a properly increasing sequence of subcomplexes of $C(X)$. We show in § 3 that they are all acyclic. Thus their union can be used to obtain an unessential identifier for $C(X)$ which is larger than that given in [2].

In § 2 we give some examples and facts about acyclic kernels for arbitrary chain complexes.

2. Facts and examples about acyclic kernels. The following theorem gives a condition equivalent to the kernel being acyclic for a chain map $f : C \rightarrow C$ such that $f_*$ is an isomorphism:

**Theorem 2.1.** If $C = \{C_p, \partial\}$ is a chain complex, $f : C \rightarrow C$ is a chain mapping, $K = \{K_p, \partial\}$ is the kernel of $f$, and $i : f(C) \rightarrow C$ is the inclusion homomorphism, then any two of the following conditions imply the third:

**Condition 1.** $i_*$ is an isomorphism.

**Condition 2.** $K$ is acyclic.

**Condition 3.** $f_*$ is an isomorphism.

**Proof.** Let $\tilde{f} : C_p/K_p \rightarrow f(C_p)$ be defined by $\tilde{f}([x]) = f(x)$. Now the theorem immediately follows from the commutativity of the following diagram and the fact, mentioned earlier, that $K$ is acyclic if and only if $\pi_*$ is an isomorphism:

\[
\begin{array}{ccc}
H_p(C/K) & \xrightarrow{\tilde{f}_*} & H_p(f(C)) \\
\uparrow{\pi_*} & & \uparrow{i_*} \\
H_p(C) & & H_p(C)
\end{array}
\]

Some examples illustrate the relationships between the conditions given in Theorem 3.2. In [3] Radó gives an example which illustrates that $K$ may fail to be acyclic even when $f \sim 1$ (and thus $f_* = 1$). Hence, Condition 3 does not imply Condition 2. That Condition 2 does not imply Condition 3 is demonstrated by the following example.

**Example 2.2.** Let a chain complex $C = \{C_p, \partial\}$ be defined as follows:

- $C_p = 0$ for $p \geq 2$ and for $p < 0$.
- $C_1 = \langle t \rangle$, the free abelian group with one generator, $t$.
- $C_0 = \langle z \rangle$, the free abelian group with one generator, $z$.
- $\partial : C \rightarrow C$ is given by $\partial(t) = 2z$ and $\partial(C_p) = 0$ whenever $p \neq 1$. 

A chain mapping \( f : C \to C \) is given by the relations \( f(t) = 2t \) and \( f(z) = 2z \).

Note that if \( K_p \) is the kernel of \( f|_{C_p} \), then \( K_p = 0 \) for every \( p \) and, hence, \( K = \{K_p, \partial\} \) is acyclic. However, \( H_0(C) \) contains an element \([z] \neq 0 \) while \( f_*([z]) = [f(z)] = [2z] = 0 \). Thus \( f \) is not an isomorphism.

Example 2.3 shows that Condition 1 does not imply Condition 3.

Example 2.3. Let \( C_1 = ((a_1, a_2, a_3, \ldots)) \), the free abelian group generated by an infinite countable set, and \( C_p = 0 \) for \( p \neq 1 \). Let the boundary operator be defined by \( \partial(x) = 0 \) for all \( x \). Define a chain map \( f : C \to C \) so that \( f(a_1) = f(a_2) = a_1 \) and \( f(a_j) = a_{j-1} \) for \( j \geq 3 \). In this case \( H_0(C) = C_p \) for each \( p \). Furthermore, \( i_* \) is an isomorphism but \( f_* \) is not 1-1.

A chain complex \( C \) is said to be free provided that \( C_p \) is a free abelian group for each \( p \). It is well known (\(^2\)) that if \( f : C \to C, C \) is free, and \( f_* \) is an isomorphism, then \( f \) is a chain equivalence. Using this fact and Theorem 2.1 we obtain the following corollary about the chain complex \( C \otimes G \) obtained from tensoring a free chain complex \( C \) with an abelian group \( G \). For \( f : C \to C, f \otimes 1 \) denotes that chain map of \( C \otimes G \) into itself defined by \( f(x \otimes g) = f(x) \otimes g \).

Corollary 2.4. Suppose that \( f : C \to C \) is a chain map of a free chain complex \( C \) into itself, \( f_* \) is an isomorphism, and \( f \) has an acyclic kernel. Then the chain map \( f \otimes 1 : C \otimes G \to C \otimes G \) has an acyclic kernel.

Proof. By Theorem 2.1, \( i_* \) is an isomorphism where \( i : f(C) \to C \). Since both of \( C \) and \( f(C) \) are free chain complexes, \( f \) and \( i \) are both chain equivalences. It follows that \( (f \otimes 1)_* \) and \( (i \otimes 1)_* \) are both isomorphisms. Note that here \( i \otimes 1 : f(C) \otimes G \to C \otimes G \). However, \( f(C) \otimes G = (f \oplus 1)(C \otimes G) \), if we consider \( f(C) \otimes G \) as a subgroup of \( C \otimes G \). Now, applying Theorem 2.1 to the chain complex \( C \otimes G \), we conclude that the kernel of \( f \otimes 1 \) is acyclic.

Next we give an example which deals with the iterates of a chain mapping. Example 2.5 shows that for every integer \( n \geq 1 \) there is a chain complex \( C \) and a chain mapping \( f : C \to C \) such that \( f \sim 1 \) (and thus \( f^k \sim 1 \) for each \( k \)), the kernel of \( f^k \) is acyclic for \( k \neq n \), but the kernel of \( f^n \) is not acyclic.

Example 2.5. Let \( n \) be a fixed integer greater than 1. (An easy modification works for \( n = 1 \).)

Define a chain complex \( C = \{C_p, \partial\} \) as follows:

- \( C_p = 0 \) for \( p > 2 \) and \( p < 0 \).
- \( C_2 = (s_1, s_2, \ldots, s_n) \), the free abelian group with the \( n \) generators \( s_1, s_2, \ldots, s_n \).

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(\(^2\)) For example, see p. 192 of [5].
\[ C_1 = (\langle a_0, a_1, \ldots, a_n, b_1, b_2, \ldots, b_{n-1} \rangle), \] the free abelian group with the \(2n\) generators shown.

\[ C_0 = (\langle x_1, x_2, \ldots, x_n \rangle), \] the free abelian group with the \(n\) generators shown.

\[ \partial : C \to C \] is given by
\[ \begin{align*}
\partial(C_p) &= 0 \quad \text{for } p > 2 \text{ and } p < 1, \\
\partial(s_1) &= a_0, \\
\partial(s_j) &= a_{j-1} - b_{j-1} \quad \text{for } 2 \leq j \leq n, \\
\partial(a_0) &= 0, \\
\partial(a_j) &= x_j \quad \text{for } i \leq j \leq n, \\
\partial(b_j) &= x_j \quad \text{for } 1 \leq j \leq n-1.
\end{align*} \]

Define a homomorphism \(f : C \to C\) as follows:

\[ f(C_p) = 0 \text{ for } p > 2 \text{ and } p < 0. \]

\[ f(C_2) \] is given by \(f(s_1) = 0\) and \(f(s_j) = s_{j-1}\) for \(2 \leq j \leq n\).

\[ f(C_1) \] is given by \(f(a_0) = f(b_1) = 0, f(a_j) = a_{j-1}\) for \(1 \leq j \leq n\) and \(f(b_j) = b_{j-1}\) for \(2 \leq j \leq n-1\).

\[ f(C_0) \] is given by \(f(x_1) = 0\) and \(f(x_j) = x_{j-1}\) for \(2 \leq j \leq n\).

It is easy to show that \(f\) is a chain mapping.

Define a homomorphism \(\varrho : C \to C\) as follows:

\[ \varrho(C_p) = 0 \text{ for } p > 1 \text{ and } p < 0. \]

\[ \varrho(C_0) \] is given by \(\varrho(x_j) = a_{j-1} - a_{j-1}\) for \(1 \leq j \leq n-1\) and \(\varrho(x_n) = a_n - a_{n-1}\).

\[ \varrho(C_1) \] is given by \(\varrho(a_j) = s_{j+1}\) for \(0 \leq j \leq n-1\), \(\varrho(a_n) = 0\) and \(\varrho(b_j) = s_j\) for \(1 \leq j \leq n-1\).

It can be verified that \(\varrho \partial + \partial \varrho = 1 - f\). Hence \(f \sim 1\).

Hereafter, \((y_1, y_2, \ldots, y_m)\) will be used as above to denote the free abelian group with the \(m\) generators \(y_1, y_2, \ldots, y_m\). Also \(K^j = \{ K_p^j, \partial \} \) will denote the kernel of \(f^j\).

A consideration of the definition of \(f\) shows that the kernels are as follows:

\[ K_p^j = 0 \text{ for all } j \text{ whenever } p < 0 \text{ or } p > 2, \]

\[ K_2^j = (\langle s_1, s_2, \ldots, s_j \rangle) \text{ for } 1 \leq j \leq n-1 \text{ and } K_2^j = C_2 \text{ for } j \geq n. \]

\[ K_1^j = (\langle a_0, \ldots, a_{j-1}, b_1, \ldots, b_j \rangle) \text{ for } 1 \leq j \leq n-1, \quad K_1^n = (\langle a_0, \ldots, a_{n-1}, b_1, \ldots, b_{n-1} \rangle). \]

\[ K_1^j = C_1 \text{ for } j \geq n+1. \]

\[ K_0^j = (\langle x_1, \ldots, x_j \rangle) \text{ for } 1 \leq j \leq n-1 \text{ and } K_0^j = C_0 \text{ for } j \geq n. \]

Since \(K_p^j = 0 \text{ for all } j \text{ whenever } p > 2 \text{ or } p < 0\), \(H_p(K^j) = 0\) for all \(j \text{ whenever } p > 2 \text{ or } p < 0\).
Since $C_2$ contains no cycles, $H_2(K^j) = 0$ for all $j$.
For the case $p = 1$, $K^1_1 \cap Z_1 = (a_0)$,
\[ K_1^j \cap Z_1 = (a_0, a_1 - b_1, a_2 - b_2, \ldots, a_{j-1} - b_{j-1}) \quad \text{for } 2 \leq j \leq n, \]
and
\[ K_1^j \cap Z_1 = (a_0, a_1 - b_1, a_2 - b_2, \ldots, a_{n-1} - b_{n-1}) \quad \text{for } j \geq n. \]

Consideration of the definition of $\partial$ shows that $\partial(K^j_2) = K_1^j \cap Z_1$ for every $j$. This implies that $H_1(K^j) = 0$ for every $j$.

For the case $p = 0$, $C_0 = Z_0$ and thus $K_0^j \cap Z_0 = K_0^j$. Consideration of the definition of $\partial$ shows that $\partial(K_0^j) = K_0^j$ whenever $j \neq n$. Hence $H_0(K^j) = 0$ for all $j \neq n$. However, $x_n \epsilon K_0^n \cap Z_0$ and $x_n \epsilon \partial(K_0^n)$ so $H_0(K^n) \neq 0$.

3. The kernels of the iterates of $\beta$ are acyclic. We start by giving a brief discussion of the notation which we shall use. Let $E_\infty$ denote the set of all square summable real sequences with the usual topology. Let $d_0, d_1, d_2, \ldots$ denote, respectively, the points $(1, 0, 0, 0, \ldots), (0, 1, 0, 0, \ldots), (0, 0, 1, 0, \ldots), \ldots$ in $E_\infty$. If $v_0, v_1, v_2, \ldots, v_p$ are any $p+1$ points in $E_\infty$, then $\langle v_0, v_1, v_2, \ldots, v_p \rangle$ denotes the convex hull of these points. We set $A_p = \langle d_0, d_1, \ldots, d_p \rangle$. For any topological space $X$, $C_p(X)$ is the free abelian group generated by all continuous mappings $T : A_p \rightarrow X$. For any $p+1$ points $v_0, v_1, \ldots, v_p$ in $E_\infty$, $\langle v_0, v_1, v_2, \ldots, v_p \rangle$ will stand for that linear map $L$ of $A_p$ into $E_\infty$ with the property that $L(d_i) = v_i$ for each $i$ and $b(v_0, v_1, \ldots, v_p)$ will denote the barycenter of the $p+1$ points, i.e., the point $(v_0 + v_1 + \ldots + v_p)/p + 1$. We shall assume familiarity with the barycentric homomorphism $\beta : C(X) \rightarrow C(X)$. In any case it suffices, for our purposes, to state that for $T : A_p \rightarrow X$, $\beta(T) = \sum_{\sigma \in P_n} \sgn \sigma (T \circ [\sigma])$, where $P_n$ is the set of all permutations of the set $\{0, 1, 2, \ldots, n\}$ and for $\sigma = (i_0, i_1, \ldots, i_n) \epsilon P_n$, $[\sigma] = [d_{i_0}, b(d_{i_1}, d_{i_2}), b(d_{i_3}, d_{i_4}, d_{i_5}), \ldots, b(d_{i_0}, d_{i_1}, d_{i_2}, \ldots, d_{i_n})]$ and $\sgn \sigma$ is 1 when $\sigma$ is even, and $-1$ when $\sigma$ is odd.

Straightforward computation yields the following useful lemma.

**Lemma 3.1.** If $X$ is a topological space, $T$ is a map of $A_p$ into $X$, and $L$ is a linear map of $A_n$ into $A_p$ such that $L(d_j) = L(d_k)$ where $d_j$ and $d_k$ are distinct vertices of $A_n$, then $\beta(T \circ L) = 0$.

The following notation will be used hereafter: If $n \geq 0$ and $f : A_n \rightarrow A_n$ is a map such that $f \circ [\sigma]$ is linear for every $\sigma \epsilon P_n$, then $D(n, f) = \{ \sigma \epsilon P_n \mid f \circ [\sigma] \}$ agrees on two distinct vertices of $A_n$. Note that $D(0, f) = \emptyset$ for any map $f$ from $A_0$ to $A_0$.

**Lemma 3.2.** If $X$ is a topological space, $T$ is a map of $A_n$ into $X$ with $n \geq 0$, and $f : A_n \rightarrow A_n$ is a map such that $f \circ [\sigma]$ is linear for every $\sigma \epsilon P_n$ and $J = P_n - D(n, f)$, then
\[ \beta^2(T \circ f) = \sum_{\sigma \epsilon J} \left( \sum_{\tau \epsilon P_n} (\sgn \sigma)(\sgn \tau)(T \circ f) \circ [\sigma] \circ [\tau] \right). \]
Proof. Write \( \beta^2(T \circ f) = \beta(\beta(T \circ f)) = \beta(A + B) = \beta(A) + \beta(B) \) where

\[
A = \sum_{\sigma \in D(n, f)} \text{sgn}((T \circ f) \circ [\sigma]) \quad \text{and} \quad B = \sum_{\sigma \in D(n, f)} \text{sgn}((T \circ f) \circ [\sigma]).
\]

If \( \sigma \in D(n, f) \), then \( f \circ [\sigma] \) agrees on two distinct vertices of \( \Delta_n \) and thus satisfies the hypothesis of Lemma 3.1. It follows that \( \beta((T \circ f) \circ [\sigma]) = 0 \). Thus \( \beta(B) = 0 \) and

\[
\beta^2(T \circ f) = \beta(A) = \beta \left( \sum_{\sigma \in \Delta} \text{sgn}((T \circ f) \circ [\sigma]) \right)
\]

\[
= \sum_{\sigma \in P_n} \left( \sum_{\tau \in \Delta} (\text{sgn} \tau)(\text{sgn} \sigma)(T \circ f) \circ [\sigma] \circ [\tau] \right).
\]

The order of the summation signs may be changed proving the lemma.

Definition of \( f_n : \Delta_n \rightarrow \Delta_n \). For each integer \( n \geq 0 \), \( f_n \) is to be a map with the following three properties:

P-1. \( f_n \circ [\sigma] \) is linear for each \( \sigma \in P_n \), i.e., \( f_n \) is linear on the first barycentric subdivision of \( \Delta_n \).

P-2. If \( \sigma_0 \) is the identity permutation in \( P_n \), then \( f_n \circ [\sigma_0] \) is the identity map.

P-3. If \( \sigma \) is not the identity permutation in \( P_n \), then \( \sigma \in D(n, f_n) \), i.e., \( f_n \circ [\sigma] \) agrees on two distinct vertices of \( \Delta_n \).

The \( f_n \) are to be defined inductively. Roughly, letting \( \sigma_0 \) be the identity permutation in \( P_n \), \( f_n \) is to expand the "\( \sigma_0 \)-piece" of \( \Delta_n \) onto all of \( \Delta_n \) in a natural linear manner while collapsing the other pieces linearly into the boundary in such a way that for each such piece two distinct vertices are mapped into the same point. This is illustrated as follows for \( n = 2 \).

![Diagram](image)

Precisely, we use the fact that \( \Delta_1 \subset \Delta_2 \subset \Delta_3 \ldots \), set \( f_n(d_0) = d_0 \) and define \( f_n \) to be linear on the first barycentric subdivision of \( \Delta_n \) with \( f_n|_{\Delta_{n-1}} = f_{n-1} \) and \( f_n(d) = d_n \) for each \( d \) which is a vertex of the first barycentric subdivision of \( \Delta_n \) and is not in \( \Delta_{n-1} \).

Properties P-1 and P-2 are easily verified from the definitions. We give an inductive proof that P-3 is satisfied.

Proof of P-3 is trivially true when \( n = 0 \). Assume that \( P_{n-1} - D(n-1, f_{n-1}) \) contains only the identity permutation. Now let \( \sigma \in P_n \) with \( \sigma \neq \text{the identity} \). Either the \( \sigma \)-piece of \( \Delta_n \) has a face in \( \Delta_{n-1} \) or it does not; these two cases are considered as follows.
In the first case, since \( \sigma \) is not the identity, there is a \( \tau \in P_{n-1} \) such that \( \tau \) is not the identity and \([\sigma]|_{A_{n-1}} = [\tau] \). But then, by the inductive assumption, \( \tau \in D(n-1, f_{n-1}) \) and \( f_n \circ [\sigma]|_{A_{n-1}} = f_{n-1} \circ [\tau] \) agrees on two distinct vertices of \( A_{n-1} \). Thus \( f_n \circ [\sigma] \) agrees on two distinct vertices of \( n \) and \( \sigma \in D(n, f_n) \).

In the second case, \( \sigma = (i_0, i_1, \ldots, i_n) \) where \( i_j = n \) for some \( j < n \). Hence \([\sigma](d_j) = b(d_{i_0}d_{i_1} \ldots d_{i_j}) \notin A_{n-1} \) and \([\sigma](d_n) = b(d_{i_0}d_{i_1} \ldots d_{i_n}) \notin A_{n-1} \). By definition \((f_n \circ [\sigma])(d_j) = (f_n \circ [\sigma])(d_n) = d_n \) and it is proved that \( \sigma \in D(n, f_n) \).

**Theorem 3.3.** If \( C(X) = \{C_n(X), \partial\} \) is the chain complex of singular chains of a topological space \( X \) and \( \beta \) is the barycentric homomorphism, then \( \beta(C(X)) = \beta^j(C(X)) \) for every positive integer \( j \).

**Proof.** Note that \( \beta(C(X)) \supseteq \beta^2(C(X)) \supseteq \beta^3(C(X)) \supseteq \ldots \) Therefore it remains to show that \( \beta(C(X)) \subseteq \beta^2(C(X)) \subseteq \beta^3(C(X)) \subseteq \ldots \) or, equivalently, simply that \( \beta(C(X)) \subseteq \beta^2(C(X)) \).

Let \( T \in C_n(X) \) be an arbitrary generator of \( C(X) \) and let \( f_n : A_n \rightarrow A_n \) be the map defined above.

By Lemma 3.2,

\[
\beta^2(T \circ f_n) = \sum_{\sigma \in P_{n-1}} \sum_{\tau \in P_n} (\text{sgn } \sigma)(\text{sgn } \tau) T \circ f_n \circ [\sigma] \circ [\tau].
\]

Applying P-3,

\[
\beta^2(T \circ f_n) = \sum_{\tau \in P_n} \text{sgn } \tau (T \circ f_n) \circ [\sigma_0] \circ [\tau],
\]

where \( \sigma_0 \) is the identity permutation in \( P_n \). Applying P-2, \( f_n \circ [\sigma_0] \) is the identity map and

\[
\beta^2(T \circ f_n) = \sum_{\tau \in P_n} (\text{sgn } \tau) T \circ [\tau] = \beta(T).
\]

This shows that \( \beta(T) \in \beta^2(C(X)) \) for any generator \( T \) of \( C(X) \) and thus that \( \beta(C(X)) \subseteq \beta^2(C(X)) \).

**Theorem 3.4.** If \( C(X) \) is the chain complex of singular chains of a topological space \( X \), \( \beta \) is the barycentric homomorphism, \( j \) is a positive integer, and \( K^j \) denotes the kernel of chain mapping \( \beta^j \), then \( K^j \) is acyclic.

**Proof.** Consider the following diagrams, where \( i, i' \), and \( i'' \) are the inclusion chain maps:

\[
\begin{array}{c}
\xymatrix{ H(\beta^j(C(X))) & H(C(X)) \\
  & H(\beta(C(X))) \\
\downarrow i_* & \downarrow i_* \ar@{..>}[u] \\
\downarrow i'_* & \downarrow i''_* }
\end{array}
\]
By Theorem 3.3, $i'$ is an isomorphism of $\beta'(C(X))$ onto $\beta(C(X))$ and thus $i'_*\beta$ is an isomorphism. Also, $\beta$ is an isomorphism and Fadell proved in [2] that the kernel of $\beta$ is acyclic. Hence, by Theorem 2.1, $i'_*\beta$ is an isomorphism. Then the commutativity of the diagram yields the conclusion that $i'_*$ is an isomorphism. Applying Theorem 2.1 again gives the result that $K'\beta$, the kernel of $\beta$, is acyclic and completes the proof of the theorem.

4. On chain homotopies which are stable with respect to the kernel.
If $f : C \to C$ and there is a chain homotopy $\varrho$ connecting $f$ and $1$ which has the property that $\varrho(K) = K$, where $K$ is the kernel of $f$, then it is easy to show that $K$ is acyclic. In fact, we stated earlier that Fadell used such a homotopy to show that the kernel of $\beta$ is acyclic. It might be difficult to apply this technique to $\beta^2$ since the usual chain homotopies connecting $\beta^2$ and $1$ are not stable with respect to the kernel. For example, if $\varrho$ is the chain homotopy connecting $\beta$ and $1$ given in [4] and $\varrho' = \varrho + \varrho\beta$, then $\varrho'$ is a homotopy connecting $\beta^2$ and $1$ which is not stable with respect to the kernel. Stable homotopy operators for $\beta^2$ do exist, however, as the following theorem implies:

**Theorem 4.1.** If $C$ is a free chain complex and $f$ is a chain mapping from $C$ to $C$ such that $f \sim 1$ and the kernel of $f$ is acyclic then there is a chain homotopy connecting $f$ and $1$ which is stable with respect to $K$, the kernel of $f$.

The following known result (3) is useful in proving Theorem 4.1. We state it as a lemma.

**Lemma 4.2.** Given the hypothesis of Theorem 4.1, there is a chain mapping $g : f(C) \to C$ such that $f \circ g = 1$.

**Proof of Theorem 4.1.** $Z^K_p$ and $B^K_p$ will denote the $p$-cycles and $p$-bounds, respectively, of $K$. Since $B^K_{p-1}$ is a free abelian group for every $p$, there is a homomorphism $r$ of degree $1$ from $\Sigma B^K_p$ to $\Sigma K_p$ and, for every $p$, a split exact sequence $0 \to Z^K_p \xrightarrow{i} K_p \xrightarrow{\delta} B^K_{p-1} \to 0$, where $j$ is the inclusion homomorphism. Furthermore, since $K$ is acyclic, $H_p(K) = 0$ and $Z^K_p = B^K_p$ for every integer $p$. Thus we can write the sequence $0 \to Z^K_p \xrightarrow{j} K_p \xrightarrow{r} Z^K_{p-1} \to 0$ is split exact. Then $K_p = Z^K_p \oplus r(Z^K_{p-1})$ is the direct sum of two of its subgroups.

Define a homomorphism $\tilde{r}$ of degree $1$ from $\Sigma K_p$ to $\Sigma K_p$ by considering $k = z + r(z') \epsilon K_p$ with $z \epsilon Z^K_p$ and $r(z') \epsilon r(Z^K_{p-1})$ and setting $\tilde{r}(z + r(z')) = r(z) \epsilon K_{p+1}$.

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(3) See exercise on p. 158-9 of [1].
Note that \( \tilde{r} | \Sigma Z_p^K = r \) and \( \tilde{r} \circ r = 0 \) and recall that \( \partial \circ r = 1 \). It follows that for \( z \in Z_p^K \) and \( z' \in Z_p^{K-1} \), \( \tilde{r}(\partial(z)) = r(0) = 0 \), \( \tilde{r}(\partial(r(z'))) = r(z') \), \( \partial(\tilde{r}(z)) = \partial(r(z)) = z \), and \( \partial(\tilde{r}(r(z'))) = \partial(0) = 0 \).

Combining these results yields \( (\tilde{r} \partial + \partial \tilde{r})(z + r(z')) = r(z') + z \). Thus \( \tilde{r} \partial + \partial \tilde{r} = 1 \).

From the hypothesis that \( f \sim 1 \) there is a homomorphism \( \tilde{\varrho} \) of degree 1 from \( \Sigma C_p \) to \( \Sigma C_p \) such that \( \tilde{\varrho} \partial + \partial \tilde{\varrho} = 1 - f \).

By Lemma 4.2, there is a chain mapping \( g \) from \( f(C) \) to \( C \) such that the sequence \( 0 \rightarrow K_p \overset{j}{\rightarrow} C_p \overset{f}{\rightarrow} f(C_p) \rightarrow 0 \) is split exact, where \( j \) is the inclusion homomorphism. Thus, for each \( p \), \( C_p = K_p \oplus g(f(C_p)) \) is the direct sum of two of its subgroups.

Now define a homomorphism \( \varrho \) of degree 1 from \( \Sigma C_p \) to \( \Sigma C_p \) by considering \( x = k + y \in C_p \) with \( k \in K_p \) and \( y \in g(f(C_p)) \) and setting \( \varrho(x) = \tilde{r}(k) + \tilde{\varrho}(y) \), where \( \tilde{r} \) and \( \tilde{\varrho} \) are as given above. It is easy to show that \( \varrho \partial + \partial \varrho = 1 - f \). Also, \( \varrho(K_p) = \tilde{r}(K_p) \subseteq K_p+1 \), so \( \varrho \) is stable with respect to the kernel of \( f \) and the theorem is proved.

5. Remarks and questions. Corollary 2.4 shows that acyclicity of the kernel of \( f \otimes 1 \) follows from that of \( f \) in the case where \( C \) is free and \( f_* \) is an isomorphism. The proof made no use of any relationship between \( \text{Ker}(f \otimes 1) \) and \( (\text{Ker} f) \otimes G \). If \( \text{Ker}(f \otimes 1) = (\text{Ker} f) \otimes G \), then one could use the stable homotopy operator given by Theorem 4.1 to obtain an alternate proof for the corollary. Is it in fact true that the equality just mentioned must hold given the hypothesis of Corollary 2.4? (P 651)

Results in [2], [3], and [4] are concerned with finding unessential identifiers for the complex \( C(X) \) and for a somewhat larger complex \( R(X) \). In [2] Fadell obtains a largest known unessential identifier for \( R(X) \). A still larger one could be obtained using our result on the acyclicity of the kernels of the iterates of \( \beta \). Is there an even larger one (P 652)? Or, we can ask related questions concerning \( C(X) \). First let \( K^i(X) \) denote the kernel of \( \beta^i \). Let \( K(X) = \bigcup_{i=1}^{\infty} K^i(X) \). It is easy to see that \( K(X) \) is an unessential identifier for \( C(X) \). Is there a larger one? Is there a largest unessential identifier for \( C(X) \)? (P 653)

REFERENCES


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