A NOTE ON THE MARCINKIEWICZ INTEGRAL

BY

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In their work on Fourier series Littlewood and Paley, and in his work on the boundary values of analytic functions Lusin, introduced the well-known $g$ and $g^*_\lambda$, and the area $S$ functions, respectively. In this context, Marcinkiewicz [4] considered the expression $\mu(f)(x)$ given by

$$\mu(f)(x) = \left( \int_{[0, 2\pi]} \frac{|F(x + t) + F(x - t) - 2F(x)|^2}{t^3} \, dt \right)^{1/2}, \quad x \in [0, 2\pi],$$

where $F(x) = \int_{[0, x]} f(t) \, dt$. The Marcinkiewicz integral $\mu(f)(x)$ was introduced in order to give an analogue of the Littlewood–Paley $g$ function without going into the interior of the unit disk for its definition; that there are similar results along the lines of the $S$ and $g^*_\lambda$ functions is exemplified by Theorem 5 below. It was Zygmund [7] who, among other interesting results, proved that

$$\|\mu(f)\|_p \leq c_p \|f\|_p, \quad 1 < p < \infty.$$  

Stein [5] defined a generalization of the Marcinkiewicz integral to higher dimensions, and proved similar results by means of the so-called real variables method, in the following setting. Let $\Omega(x)$ be a function which is homogeneous of degree 0 and which, in addition, satisfies the following two conditions:

(i) $\Omega(x)$ is continuous on $\Sigma$, the unit sphere of $\mathbb{R}^n$, and satisfies a Lipschitz condition of order $\alpha$ there, i.e.,

$$|\Omega(x') - \Omega(y')| \leq c|x' - y'|^\alpha, \quad x', y' \in \Sigma.$$

(ii) $\int_{\Sigma} \Omega(x') \, dx' = 0$.

For a locally integrable function $f$ on $\mathbb{R}^n$ and $t > 0$, let $F_t(f, x) = F_t(x)$ be given by

$$F_t(x) = \int_{\{|y| \leq t\}} \frac{\Omega(y)}{|y|^{n-1}} f(x - y) \, dy, \quad x \in \mathbb{R}^n,$$
and define now $\mu(f)(x)$ by
\[
\mu(f)(x) = \left( \int_{[0,\infty)} \frac{|F_t(x)|^2}{t^3} \, dt \right)^{1/2}.
\]

Stein showed that if $f \in L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then
\[
\|\mu(f)\|_p \leq c_p \|f\|_p, \quad 1 < p \leq 2,
\]
and when $p = 1$,
\[
\lambda|\{\mu(f) > \lambda\}| \leq c\|f\|_1, \quad \text{all } \lambda > 0.
\]

Benedek, Calderón and Panzone [1] showed that if $\Omega$ is continuously differentiable in $x \neq 0$, then (1) above holds for $1 < p < \infty$.

In order to state our first result we set
\[
M_p f(x) = \sup_{x \in Q} \left( |Q|^{-1} \int_Q |f(y)|^p \, dy \right)^{1/p},
\]
where $Q$ is a cube containing $x$ with sides parallel to the coordinate axes; this is the generalized Hardy–Littlewood maximal function. We put $M_1 f = M f$.

We also need the generalized sharp maximal function $M_p^# f$ given by
\[
M_p^# f(x) = \sup_{x \in Q} \left( |Q|^{-1} \int_Q |f(y) - f_Q|^p \, dy \right)^{1/p},
\]
where $f_Q$ is the average of $f$ over $Q$. We put $M^# f = M^#$, and we set $\text{BMO}(\mathbb{R}^n) = \{ f : \|f\|_* = \|M^# f\|_\infty < \infty \}$; by the John–Nirenberg inequality the expressions $\|M_p^# f\|_\infty$ all give equivalent BMO norms for a given function $f$.

The first result we prove is

**Theorem 1.** Suppose $1 < p < \infty$, and that $\|\mu(f)\|_p \leq k_p \|f\|_p$. Then there is a constant $c_p = c_p(k_p)$ independent of $f$ such that
\[
M^#(\mu(f))(x) \leq c_p M_p f(x), \quad \text{for all } x \in \mathbb{R}^n.
\]

Our next result deals with a commutator of the Marcinkiewicz integral, and for this purpose we take the point of view of the vector-valued singular integral operators of Benedek, Calderón and Panzone. Let, then, $H$ be the Hilbert space
\[
H = \left\{ h : \|h\| = \left( \int_{[0,\infty)} \frac{|h(t)|^2}{t^3} \, dt \right)^{1/2} < \infty \right\}.
\]

For each fixed $x \in \mathbb{R}^n$, we may view $F_t(x)$ as a mapping from $[0, \infty)$ to $H$, and it is clear that $\|F_t(f, x)\| = \mu(f)(x)$. 

For $b \in \text{BMO}(\mathbb{R}^n)$, $C_b(f)(x)$, the commutator of the Marcinkiewicz integral, is then defined as

$$C_b(f)(x) = \|b(x)F_t(f,x) - F_t(bf,x)\|, \quad x \in \mathbb{R}^n.$$ 

We then have

**Theorem 2.** Given $1 < r, s < \infty$, there is a constant $c = c_{r,s}$ independent of $b$ and $f$ such that

$$M^\#(C_b(f))(x) \leq c\|b\|_s(M_r(\mu(f))(x) + M_s f(x)).$$

Theorems 1 and 2 lead to various weighted $L^p(\mathbb{R}^n)$ inequalities; we list some but do not prove them as the proof technique, once we have the pointwise estimates at hand, is by now well known.

**Theorem 3.** Let $1 < p < \infty$, and $w$ a weight in the Muckenhoupt $A_p(\mathbb{R}^n)$ class. Then there is a constant $c = c_p(w)$ independent of $f$ such that

$$\|\mu(f)\|_{L^p_w} \leq c\|f\|_{L^p_w}, \quad 1 < p < \infty.$$ 

**Theorem 4.** Let $1 < p < \infty$, and $w$ a weight in the Muckenhoupt $A_p(\mathbb{R}^n)$ class. Then there is a constant $c = c_p(w)$ independent of $f$ and $b$ such that

$$\|C_b(f)\|_{L^p_w} \leq c\|b\|_{L^\infty} \|f\|_{L^p_w}, \quad 1 < p < \infty.$$ 

Finally, we consider the Marcinkiewicz integrals $\mu_S(f)$ and $\mu_S^*(f)$ corresponding to the $S$ and $g_S^\lambda$ functions; they are defined by

$$\mu_S(f)(x)^2 = \int_{\Gamma(x)} \frac{|F_t(y)|^2}{t^{n+3}} \, dy \, dt,$$

where $\Gamma(x) = \{(y,t) : |x - y| < t\}$, and

$$\mu_S^*(f)(x)^2 = \int_{\mathbb{R}^n_{t+1}} \left( \frac{t}{t + |x - y|} \right)^{n\lambda} \frac{|F_t(y)|^2}{t^{n+3}} \, dy \, dt, \quad \lambda > 1,$$

respectively. As an indication that the theory in this case proceeds along the lines of the $S$ and $g_S^\lambda$, and $\mu$, functions, we show a result that includes the $L^2(\mathbb{R}^n)$ continuity case.

**Theorem 5.** Suppose $w$ is a nonnegative locally integrable function in $\mathbb{R}^n$. Then there is a constant $c$ independent of $f$ and $w$ such that

$$\int_{\mathbb{R}^n} \mu_S^*(f)(x)^2 w(x) \, dx \leq c \int_{\mathbb{R}^n} |f(x)|^2 Mw(x) \, dx.$$ 

We pass now to the proofs.

**Proof of Theorem 1.** Given $x \in \mathbb{R}^n$, let $Q = Q(x_0, h)$ be a cube centered at $x_0$ of edgelength $h$ with sides parallel to the axes. If now $Q^* = \cdots$
\( Q(x_0, 4\sqrt{n}h) \), let
\[
f = f_{\chi Q} + f(1 - \chi Q) = f_1 + f_2,
\]
say. Then
\[
\int_{Q} \mu(f_1)(y)^p \, dy \leq \int_{\mathbb{R}^n} \mu(f_1)(y)^p \, dy \leq c_p \int_{\mathbb{R}^n} |f_1(y)|^p \, dy \leq c_p \int_{Q^*} |f(y)|^p \, dy,
\]
and consequently,
\[
(2) \quad |Q|^{-1} \int_{Q} \mu(f_1)(y) \, dy \leq \left( |Q|^{-1} \int_{Q} \mu(f_1)(y)^p \, dy \right)^{1/p} \leq c_p M_p f(x).
\]
Next, given \( w \in Q \), we estimate \( I = |\mu(f_2)(x_0) - \mu(f_2)(w)| \). From the inequality
\[
||F_i(f_2, x_0)|| - ||F_i(f_2, w)|| \leq ||F_i(f_2, x_0) - F_i(f_2, w)||,
\]
it readily follows that \( I \) does not exceed \( I_1 + I_2 + I_3 \), where
\[
I_1 = \left( \int_{[0, \infty)} \frac{1}{t^3} \left( \int_{\{|x_0 - y| < t \leq |w - y|\}} \frac{|\Omega(x_0 - y)|}{|x_0 - y|^{n-1}} |f_2(y)|^2 \, dy \right) \frac{1}{t^3} \right)^{1/2},
\]
\[
I_2 = \left( \int_{[0, \infty)} \frac{1}{t^3} \left( \int_{\{|w - y| < t < |x_0 - y|\}} \frac{|\Omega(w - y)|}{|w - y|^{n-1}} |f_2(y)|^2 \, dy \right) \frac{1}{t^3} \right)^{1/2},
\]
and \( I_3 \) equals
\[
\left( \int_{[0, \infty)} \frac{1}{t^3} \left( \int_{\{|x_0 - y| \leq t, |w - y| \leq t\}} \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-1}} - \frac{\Omega(w - y)}{|w - y|^{n-1}} |f_2(y)|^2 \, dy \right) \frac{1}{t^3} \right)^{1/2}.
\]
Since the estimates for \( I_1 \) and \( I_2 \) follow along similar lines, we only consider \( I_1 \) here. Since \( \Omega \) is bounded and since \( |x_0 - y| \sim |x - y| \), by Minkowski's inequality, \( I_1 \) does not exceed
\[
(3) \quad c \int_{\mathbb{R}^n} \frac{|f_2(y)|}{|x_0 - y|^{n-1}} \left( \int_{\{|x_0 - y|, |w - y|\}} \frac{1}{t^3} \, dt \right)^{1/2}
\]
\[
\leq c \int_{\mathbb{R}^n \setminus Q^*} \frac{|f(y)|}{|x_0 - y|^{n-1}} \left( \frac{1}{|x_0 - y|^2} - \frac{1}{|w - y|^2} \right)^{1/2} \, dy
\]
\[
\leq c h^{1/2} \int_{\mathbb{R}^n \setminus Q^*} \frac{|f(y)|}{|x_0 - y|^{n-1}} \frac{1}{|x_0 - y|^{3/2}} \, dy
\]
\[
\leq c h^{1/2} \int_{\mathbb{R}^n \setminus Q^*} \frac{|f(y)|}{|x - y|^{n+1/2}} \, dy.
\]
It is well known (cf. Torchinsky [6, p. 83]) that the above expression does not exceed \( cMf(x) \leq cM_p f(x) \), \( 1 < p < \infty \). Thus

\[
I_1, I_2 \leq cM_p f(x).
\]

In order to estimate \( I_3 \) recall that \( x_0, w \in Q, y \notin Q^* \), and note that

\[
\left| \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-1}} - \frac{\Omega(w - y)}{|w - y|^{n-1}} \right| \leq \left| \frac{\Omega(x_0 - y)}{|x_0 - y|^{n-1}} \right| \frac{1}{|x_0 - y|^{n-1}} - \frac{1}{|w - y|^{n-1}} + \frac{1}{|w - y|^{n-1}} |\Omega(x_0 - y) - \Omega(w - y)| = A + B,
\]

say. By the mean value theorem, and since \( |x - y| \sim |x_0 - y| \sim |w - y| \), it readily follows that

\[
A \leq c \frac{|x_0 - w| |x_0 - y|^{n-2}}{|x_0 - y|^{n-1}|w - y|^{n-1}} \leq c \frac{|x_0 - w|}{|x_0 - y|^n} \leq c \frac{h}{|x - y|^n}.
\]

Similarly, since

\[
|\Omega(x_0 - y) - \Omega(w - y)| = \left| \Omega \left( \frac{x_0 - y}{|x_0 - y|} \right) - \Omega \left( \frac{w - y}{|w - y|} \right) \right| \leq c \left| \frac{x_0 - y}{|x_0 - y|} - \frac{w - y}{|w - y|} \right|^\alpha \leq c \frac{|x_0 - w|^\alpha}{|x_0 - y|^\alpha},
\]

we also have

\[
B \leq c \frac{h^\alpha}{|x - y|^{n-1+\alpha}}.
\]

Thus, again by Minkowski’s inequality, it follows that \( I_3 \) does not exceed (as a multiple of)

\[
h \int_{\mathbb{R}^n} \frac{|f_2(y)|}{|x - y|^n} \left( \int_{|x_0 - y| \leq t, |w - y| \leq t} \frac{1}{t^3} dt \right)^{1/2} dy + h^\alpha \int_{\mathbb{R}^n} \frac{|f_2(y)|}{|x - y|^{n-1+\alpha}} \left( \int_{|x_0 - y| \leq t, |w - y| \leq t} \frac{1}{t^3} dt \right)^{1/2} dy.
\]

Since \( |x_0 - y| \sim |x - y| \sim |w - y| \), the innermost integrals involving \( t \) above are of order \( |x - y|^{-1} \). Consequently,

\[
I_3 \leq ch \int_{\mathbb{R}^n \setminus Q^*} \frac{|f(y)|}{|x - y|^{n+1}} dy + ch^\alpha \int_{\mathbb{R}^n \setminus Q^*} \frac{|f(y)|}{|x - y|^{n+\alpha}} dy.
\]

Again by an argument similar to the one used in bounding \( I_1 \) above, the right-hand side of (5) does not exceed \( cMf(x) \leq cM_p f(x) \). Hence, combining the above estimates, it follows that

\[
|\mu(f_2)(x_0) - \mu(f_2)(w)| \leq cM_p f(x), \quad \text{for all } w \in Q,
\]
and consequently,

\begin{equation}
|Q|^{-1} \int_Q |\mu(f_2)(w) - \mu(f_2)(x_0)| \, dw \leq c M_p f(x).
\end{equation}

Finally, since

\begin{equation}
|\mu(f_1 + f_2)(w) - \mu(f_2)(x_0)| \\
\leq |\mu(f_1)(w) + |\mu(f_2)(w) - \mu(f_2)(x_0)|,
\end{equation}

for all \( w \in Q \),

by (2) and (6) above it follows that

\begin{equation}
|Q|^{-1} \int_Q |\mu(f)(w) - \mu(f_2)(x_0)| \, dw \leq c M_p f(x),
\end{equation}

and the proof is complete.

**Proof of Theorem 2.** The proof follows along the lines of that of Theorem 1. Given a point \( x \in \mathbb{R}^n \), suppose \( Q = Q(x_0, h) \) is a cube containing it, and put \( Q^* = Q(x_0, 4\sqrt{n}h) \). If \( b_Q \) denotes the average of \( b \) over the cube \( Q \), note that

\[
b(y)F_t(f, y) - F_t(bf, y) = (b(y) - b_Q)f_t(f, y) + b_QF_t(f, y) - F_t(bf, y) = (b(y) - b_Q)f_t(f, y) - F_t((b - b_Q)f, y) = A + B,
\]

say. First we estimate the average of \( \|A\| \leq |b(y) - b_Q|\|F_t(f)\| \) over \( Q \). By Hölder’s inequality with indices \( 1 < r, r' < \infty \), and the John–Nirenberg inequality, it does not exceed

\begin{equation}
\left( |Q|^{-1} \int_Q |b(y) - b_Q|^{r'} \, dy \right)^{1/r'} \left( |Q|^{-1} \int_Q \|F_t(f)(y)\|^r \, dy \right)^{1/r} \leq c \|b\| \inf_{y \in Q} M_r(\mu(f))(y).
\end{equation}

To bound the average of \( \|B\| \) over \( Q \) note that

\[
\|B\| \leq \|F_t((b - b_Q)\chi_{Q^*}f, y)\| + \|F_t((b - b_Q)\chi_{\mathbb{R}^n \setminus Q^*}f, y)\| = \|B_1\| + \|B_2\|,
\]

say. Let \( 1 < q, u < \infty \) be such that \( q u = s \). Then by Hölder’s inequality and the boundedness of the Marcinkiewicz integral in \( L^q(\mathbb{R}^n) \) it follows that

\begin{equation}
|Q|^{-1} \int_Q \|B_1\| \, dy \leq \left( |Q|^{-1} \int_Q \|B_1\|^q \, dy \right)^{1/q} \leq c \left( |Q|^{-1} \int_{Q^*} |b(y) - b_Q|^q |f(y)|^r \, dy \right)^{1/q} \leq c \left( |Q^*|^{-1} \int_{Q^*} |b(y) - b_Q|^{qu} \, dy \right)^{1/ru} \left( |Q^*|^{-1} \int_{Q^*} |f(y)|^{qu} \, dy \right)^{1/qu}.
\end{equation}
\[ \leq c \|b\|_* \inf_{y \in Q} M_s f(y). \]

Finally, to bound \( \|B_2\| \), as in the proof of Theorem 1, it suffices to estimate
\[ \|F_i((b - b_Q)\chi_{\mathbb{R}^* \setminus Q^*}, f, y) - F_i((b - b_Q)\chi_{\mathbb{R}^* \setminus Q^*}, x_0)\| . \]
By the argument in the proof of the theorem, with \( f_2(y) \) assuming the value \((b(y) - b_Q)\chi_{\mathbb{R}^* \setminus Q^*}(y)f(y)\) now, we estimate the above expression by a sum of three terms, corresponding to \( I_1, I_2 \) and \( I_3 \), respectively. By estimate (3), then, the terms corresponding to \( I_1 \) and \( I_2 \) are dominated by
\[ ch^{1/2} \int_{\mathbb{R}^* \setminus Q^*} \frac{|b(y) - b_Q| |f(y)|}{|x - y|^{n+1/2}} \, dy, \]
which, by Hölder’s inequality with indices \( 1 < s, s' < \infty \), is in turn bounded by
\[ c \left( h^{s'/2} \int_{\mathbb{R}^* \setminus Q^*} \frac{|b(y) - b_Q|^{s'}}{|x - y|^{s'(n+1/2)}} \, dy \right)^{1/s} \left( h^{s/2} \int_{\mathbb{R}^* \setminus Q^*} \frac{|f(y)|^s}{|x - y|^{s(n+1/2)}} \, dy \right)^{1/s}. \]
The second term above does not exceed \( cM_s f(x) \). Similarly, by a result of Fefferman and Stein [3], the first term does not exceed \( cM_{s'} b(x) \leq c\|b\|_* \).

In order to estimate the term corresponding to \( I_3 \), we make use of the estimate (5). Thus, this term does not exceed
\[ ch \int_{\mathbb{R}^* \setminus Q^*} \frac{|b(y) - b_Q| |f(y)|}{|x - y|^{n+1}} \, dy + ch^\alpha \int_{\mathbb{R}^* \setminus Q^*} \frac{|b(y) - b_Q| |f(y)|}{|x - y|^{n+\alpha}} \, dy. \]
Hence, as above, this term is also bounded by \( c\|b\|_* M_s f(x) \). It is now a simple matter, left to the reader, to complete the proof. \( \blacksquare \)

To close this paper, and show how these results extend to the more general Marcinkiewicz integrals introduced above, we prove Theorem 5.

**Proof of Theorem 5** (cf. Chanillo–Wheeden [2]). Clearly
\[ \int_{\mathbb{R}^n} \mu_n^*(f)(x)^2 w(x) \, dx \]
\[ = \int_{\mathbb{R}^{n+1}_+} |F_i(y)|^2 \left( \frac{1}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - y|} \right)^{n\lambda} \, dx \right) \, dy \, dt. \]
If \( A_k \) denotes the set
\[ \left\{(y, t) \in \mathbb{R}^{n+1}_+: 2^{k-1} < \frac{1}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - y|} \right)^{n\lambda} \, dx \leq 2^k \right\}, \]
\[ k = 0, \pm 1, \ldots, \]
the above expression is bounded by

\[ \sum_{k} 2^k \int_{\mathbb{R}^{n+1}_+} |F_t(y)|^2 \chi_{A_k}(y, t) \frac{dy \, dt}{t^3}. \]

Now, note that if \((y, t) \in A_k\), then also

\[ \frac{1}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - y|} \right)^{n \lambda} dx \leq c M w(y). \]

Now, if \(|y - z| < t\), then \(t + |x - y| \sim t + |x - z|\), and consequently, if \(|y - z| < t\) and \((y, t) \in A_k\), then

\[ 2^{k-1} \leq c \frac{1}{t^n} \int_{\mathbb{R}^n} w(x) \left( \frac{t}{t + |x - z|} \right)^{n \lambda} dx \leq c M w(z). \]

In particular, if \((y, t) \in A_k\) and \(|y - z| < t\), then \(z \in E_k = \{ z \in \mathbb{R}^n : M w(z) \geq c 2^k \}\). Thus, for \((y, t) \in A_k\) and \(z \in \mathbb{R}^n\) such that \(|y - z| \leq t\), we have \(f(z) = f(z) \chi_{E_k}(z)\), and consequently, \(F_t(f, y) = F_t(f \chi_{E_k}, y)\). Hence it follows that

\[ \int_{\mathbb{R}^{n+1}_+} |F_t(y)|^2 \chi_{A_k}(y, t) \frac{dy \, dt}{t^3} = \int_{\mathbb{R}^{n+1}_+} |F_t(f \chi_{E_k}, y)|^2 \chi_{A_k}(y, t) \frac{dy \, dt}{t^3} \leq \int_{\mathbb{R}^n} \mu(f \chi_{E_k}, y)^2 dy \leq c \|f \chi_{E_k}\|_2^2, \]

for all \(k\). Thus,

\[ \int_{\mathbb{R}^n} \mu^*_\lambda(f)(x)^2 w(x) \, dx \leq c \sum_k 2^k \|f \chi_{E_k}\|_2^2 \]

\[ = c \int_{\mathbb{R}^n} |f(x)|^2 \sum_k 2^k \chi_{E_k}(x) \, dx. \]

By the definition of the sets \(E_k\), the above sum does not exceed \(c M w(x)\). Thus, the right-hand side above is bounded by \(c \int_{\mathbb{R}^n} |f(x)|^2 M w(x) \, dx\), and the proof is complete. \(\blacksquare\)

Since \(\mu_S(f)(x) \leq c \mu^*_\lambda(f)(x)\), the same result holds for the Marcinkiewicz integral related to the Lusin function.

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