REMARKS ON STABILITY AND SATURATED MODELS

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In this note we give a description of consequences of stability for the existence of saturated models. In [5] Shelah gave a characterization of those cardinals in which an unstable theory $T$ has saturated models. Our theorems complete this characterization in the case where stability is assumed.

We use the standard notation as in [3]. If $\kappa$ is an ordinal number, then

$$\mathcal{A} = \{f: f \text{ is a function}, \text{dom}(f) \in \kappa \text{ and } \text{ran}(f) \subseteq \mathcal{A}\}.$$ 

$T$ always denotes a countable complete first-order theory with equality in a language $L$, having an infinite model. Without loss of generality we may assume that $T$ has an elimination of quantifiers (see [1]).

Let $\mathcal{A}$ be a model of $T$ and $C \subseteq A$; $p$ is a type over $C$ if $p$ is a set of formulas of the language of $\text{Th}(\mathcal{A}, c)_{c \in C}$ such that, for every formula $\varphi \in p$,

$$\text{Fr}(\varphi) \subseteq \{x_0\}$$

and, for every finite $q \subseteq p$,

$$\langle \mathcal{A}, c \rangle_{c \in C} \models \exists x_0 \wedge \varphi.$$ 

A type $p$ is called a $\varphi$-type, where $\varphi$ is a formula of $L$, if, for every $\psi \in p$, $\psi$ is one of the forms $\varphi(x_0, c)$ or $\neg \varphi(x_0, c)$. The maximal $\varphi$-type included in $p$ is denoted by $p[\varphi]$. A type $p$ over $C$ is complete if, for every formula $\varphi$ of the language of $\text{Th}(\mathcal{A}, c)_{c \in C}$ with at most $x_0$ free, either $\varphi \in p$ or $\neg \varphi \in p$. Similarly for $\varphi$-types, $S(C)\langle S_{\varphi}(C) \rangle$ denotes the set of all complete types ($\varphi$-types) over $C$. A type $p$ over $C$ is realized in $\mathcal{A}$ if there exists $a \in A$ such that, for every $\varphi \in p$, we have $\langle \mathcal{A}, c \rangle_{c \in C} \models \varphi[a]$. Let $\kappa$ be an infinite cardinal number. A model $\mathcal{A}$ of $T$ is $\kappa$-saturated whenever, for every $C \subseteq A$ and $p \in S(C)$, if $|C| < \kappa$, then $p$ is realized in $\mathcal{A}$. A model $\mathcal{A}$ is saturated if it is $|A|$-saturated. $T$ is $\kappa$-stable if $|S(A)| < \kappa$ for every $\mathcal{A} \in \text{Mod}(T)$ such that $|\mathcal{A}| < \kappa$. $T$ is stable if it is $\kappa$-stable for some $\kappa \geq \omega$. 
$T$ is superstable if it is $\kappa$-stable for every $\kappa \geq 2^\omega$. $T$ is unstable if it is not stable. Let $\mathfrak{A} \in \text{Mod}(T)$ and $C \subseteq A$. $I \subseteq A$ is a set of indiscernibles in $\mathfrak{A}$ over $C$ if, for every formula $\varphi$ of the language of $\text{Th}(\mathfrak{A}, e)_{CcC}$ and for every $i_1, \ldots, i_n, i'_1, \ldots, i'_n \in I$ such that $i_k \neq i'_i$, $i'_k \neq i'_i$ for $k \neq l$ ($n \geq |\text{Fr}(\varphi)|$), the following holds:

$$(\mathfrak{A}, c)_{CcC} \models \varphi[i_1, \ldots, i_n] \leftrightarrow \varphi[i'_1, \ldots, i'_n].$$

**Lemma 1.** Let $T$ be stable but not superstable, and $\kappa$ an infinite cardinal number. Then there exist formulas $\{\varphi_n(x, \bar{y}_n): 0 < n < \omega\}$, a structure $\mathfrak{A} \in \text{Mod}(T)$ with $|A| = \kappa$ and sequences $\{\bar{a}_n: \tau \in ^\omega \kappa\}$, $\bar{a}_\tau \in ^\omega A$, such that

(i) for every $\eta \in ^\omega \kappa$

$$p_\eta = \{\varphi_n(x, \bar{a}_{\eta|n}): 0 < n < \omega\}$$

is consistent,

(ii) for every $\tau \in ^\omega \kappa$ and every $\xi_0 < \xi_1 < \kappa$,

$$\mathfrak{A} \models \neg \exists x(\varphi_n(x, \bar{a}_\tau \cdot \xi_0) \land \varphi_n(x, \bar{a}_\tau \cdot \xi_1)),$$

where $n = \text{lh}(\tau) + 1$.

The proof follows by 6.10 of [6], Compactness Theorem and Downward Skolem-Löwenheim Theorem.

**Lemma 2.** Let $T$ be stable. If $T$ has a saturated model of power $\kappa$ for some $\kappa$ such that $\omega < \kappa < 2^\omega$, then $T$ is superstable.

**Proof.** Suppose $T$ is not superstable. Let $\mathfrak{B}$ be a saturated model of $T$ of power $\kappa$ and let $\mathfrak{A}$ be as in Lemma 1. By [2] we may assume $\mathfrak{A} \prec \mathfrak{B}$. Then, for every $\eta \in ^\omega \kappa$, $p_\eta$ is realized in $\mathfrak{B}$ by a $c_\eta \in B$, and if $\eta_0, \eta_1 \in ^\omega \kappa$, $\eta_0 \neq \eta_1$, then $c_{\eta_0} \neq c_{\eta_1}$. Hence

$$|B| \geq |\{c_\eta: \eta \in ^\omega \kappa\}| = 2^\omega > \kappa$$

and we get a contradiction.

The following theorem was stated in [6]. As far as we know the proof was never published.

**Theorem 1.** Let $\lambda$ be an infinite cardinal number. If $T$ is $\lambda$-stable, then it has a saturated model of power $\lambda$.

**Proof.** We consider three cases.

Case 1. $\lambda = \text{cf}(\lambda)$. The proof can be found in [1].

Case 2. $\omega < \text{cf}(\lambda) < \lambda$.

Let $\mathfrak{B}^*$ denote an elementary extension of $\mathfrak{B}$ of minimal cardinality and such that, for every $p \in S(B)$, $p$ is realized in $\mathfrak{B}^*$. Define, for every ordinal number $\alpha$ and structure $\mathfrak{B}$, a sequence of structures $\mathfrak{B}^{(\alpha)}$ in the following way:

$$\mathfrak{B}^{(0)} = \mathfrak{B}, \quad \mathfrak{B}^{(\alpha + 1)} = (\mathfrak{B}^{(\alpha)})^*,$$

$$\mathfrak{B}^{(\alpha)} = \bigcup \{\mathfrak{B}^{(\xi)}: \xi < \alpha\} \quad \text{for limit } \alpha.$$
Let $\mathfrak{U}$ be a model of $T$ of cardinality at most $\lambda$. We define a sequence of structures in the following way:

$$\mathfrak{U}_0 = \mathfrak{U}, \quad \mathfrak{U}_{\xi+1} = \mathfrak{U}_{\xi}^{(\xi+\omega]^+)} \text{ for } \xi < \lambda,$$

$$\mathfrak{U}_\sigma = \bigcup \{\mathfrak{U}_\xi : \xi < \sigma\} \text{ for limit } \sigma \leq \lambda, \quad \mathfrak{U}_{\lambda+1} = \mathfrak{U}_{\lambda}^{(\lambda^+)}.$$

It is easy to check that

(a1) $\{\mathfrak{U}_\xi : \xi \leq \lambda+1\}$ forms an elementary continuous chain of models of $T$.

(a2) If $\xi \leq \lambda$, then $|A_\xi| \leq \lambda$ (by $\lambda$-stability of $T$ and the cardinality assumption on $\mathfrak{U}$).

(a3) If $\xi < \lambda$, then $\mathfrak{U}_{\xi+1}$ is $|\xi + \omega|^+$-saturated.

(a4) $\mathfrak{U}_{\lambda+1}$ is $\lambda^+$-saturated.

CLAIM. $\mathfrak{U}_\lambda$ is a saturated model of power $\lambda$.

Of course, $|A_\lambda| = \lambda$. Suppose $B \subseteq A_\lambda$, $|B| = \kappa < \lambda$, $p \in S(B)$, $q \in S(A_\lambda)$, and $q \supseteq p$. By 2.5, 2.9 and 2.13 of [6], for every formula $\varphi(x, y)$ of $L$, $\text{Rank}_\varphi(q|\varphi) < \infty$ and there is a finite $q_\varphi \subseteq q|\varphi$ such that $\text{Rank}_\varphi(q|\varphi) = \text{Rank}_\varphi(q_\varphi)$ (for definitions see [4] and [6]). Let

$$\bar{q} = \bigcup \{q_\varphi : \varphi \text{ is a formula of } L\}.$$

By the countability of $T$ and the assumption $\text{cf}(\lambda) > \omega$, we may suppose that $\bar{q}$ is a type over some countable $C \subseteq A_\lambda$. By 3.4 of [6], there exists $q_{2\lambda} \in S(A_{2\lambda+1})$ such that $q_{2\lambda} \supseteq q$ and, for every formula $\varphi$ of $L$,

$$\text{Rank}_\varphi(q_{2\lambda}|\varphi) = \text{Rank}_\varphi(q|\varphi).$$

Now we define by induction sequences $\{q_\xi : \xi < 2\lambda\}$ and $\{c_\xi : \xi < 2\lambda\}$ such that

(b1) $\xi \leq \eta < 2\lambda \rightarrow q_\xi \subseteq q_\eta$, $\xi < \lambda \rightarrow q_\xi \subseteq q_\xi$ and $\xi < 2\lambda \rightarrow q_\xi \subseteq q_{2\lambda}$;

(b2) $\eta < 2\lambda \rightarrow q_\eta \in S(C \cup \{c_\xi : \xi < \eta\})$;

(b3) $\xi < \lambda \rightarrow c_\xi \in A_{\xi+1}$ and $\lambda < \xi < 2\lambda \rightarrow c_\xi \in A_{\lambda+1}$;

(b4) $\xi < 2\lambda \rightarrow c_\xi$ realizes $q_\xi$, and $\lambda < \xi < 2\lambda \rightarrow c_\xi$ realizes $q$ (hence also $p$).

Let $q_0 = \bar{q}$. By the construction of $\bar{q}$ and (a3), there exists $a \in A_\lambda$ which realizes $\bar{q}$. Take $c_0 = a$. Suppose that $\{q_\xi : \xi < \eta < \lambda\}$ and $\{c_\xi : \xi < \eta < \lambda\}$ have been defined. Then take

$$q_\eta = \{\varphi \in q : \varphi \text{ contains only constants from } C \cup \{c_\xi : \xi < \eta\}\}.$$

Of course, (b1) and (b2) hold. By (a3), there exists $b \in A_{\eta+1}$ which realizes $q_\eta$. Put $c_\eta = b$. Then (b3) and (b4) hold. Let $q_1 = \bigcup \{q_\xi : \xi < \lambda\}$. By (a4), there exists $c_\xi \in A_{\lambda+1}$ which realizes $q$ (hence also $q_1$). We complete the construction of these sequences, similarly as above, taking care of (b1)-(b4) ($q_\xi \cup q \subseteq q_{2\lambda}$, so it is a type and, by (a4), can be realized in $\mathfrak{U}_{\lambda+1}$).
In view of the choice of $\tilde{q}$ and $q_2$, we conclude, by (b1) and 2.5 of [6], that, for every formula $\varphi$ of $L$ and every $\xi \leq 2\lambda$,

$$\text{Rank}_p(q_0|\varphi) = \text{Rank}_p(q_\xi|\varphi).$$

So, by 5.7 of [6], $I = \{c_\xi : \xi < 2\lambda\}$ is a set of indiscernibles in $\mathfrak{A}_{k+1}$ over $C$. By 6.13 of [6], there exists $I_0 \subseteq I$ such that $|I_0| = \kappa = |B|$ and $I \setminus I_0$ is a set of indiscernibles in $\mathfrak{A}_{k+1}$ over $B$. Then

$$J_1 = \{c_\xi : \lambda \leq \xi < 2\lambda\} \cap (I \setminus I_0) \neq 0, \quad J_2 = (I \setminus I_0) \cap A_\lambda \neq 0.$$

By (b4), there is $d_1 \in J_1$ which realizes $q$, and hence also $p$. Since $p$ is a type over $B$, there exists $d_2 \in J_2$ which realizes $p$.

**Case 3.** $\omega = \text{cf}(\lambda) < \lambda$.

In this case $\lambda^\omega > \lambda$, and hence, by 6.10 of [6], $T$ is superstable. We proceed similarly as in Case 2, but using the degree of type (see Section 6 of [6]) instead of rank.

**Remark.** By exactly the same method we can prove the following theorem (stated without proof in [6]):

Let $T$ be stable and let $\lambda$ be an infinite cardinal number such that $\text{cf}(\lambda) > \omega$. Suppose that $\{\mathfrak{A}_\xi : \xi < \lambda\}$ is an elementary increasing chain of models of $T$, in which every $\mathfrak{A}_\xi$ is $\kappa$-saturated. Then $\mathfrak{A} = \bigcup \{\mathfrak{A}_\xi : \xi < \lambda\}$ is $\kappa$-saturated. Moreover, if $T$ is superstable, we can eliminate the assumption $\text{cf}(\lambda) > \omega$.

We note the well-known fact: A countable Boolean algebra has $\omega$ or $2^\omega$ ultrafilters.

**Corollary 1.** If $T$ has a saturated model of power $\omega_1$, then either $T$ is $\omega$-stable or $2^\omega = \omega_1$.

**Theorem 2.** Let $T$ be stable and let $\kappa > \omega$. Then $T$ is $\kappa$-stable iff $T$ has a saturated model of power $\kappa$.

**Proof.** The necessity follows by Theorem 1.

**Sufficiency. Case 1.** $T$ is $\omega$-stable.

In this case $T$ is $\kappa$-stable by 2.7 of [1].

**Case 2.** $T$ is superstable but not $\omega$-stable.

If $\kappa \geq 2^\omega$, then, of course, $T$ is $\kappa$-stable. Let $\omega < \kappa < 2^\omega$ and let $\mathfrak{B}$ be a saturated (and hence, by [2], universal) model of $T$ of power $\kappa$. Let $\mathfrak{A} < \mathfrak{B}$ and $|S(A)| > \omega = |A|$. By the above-mentioned fact, $|S(A)| = 2^\omega$, and hence $\kappa = |B| \geq 2^\omega$. We get a contradiction.

**Case 3.** $T$ is stable but not superstable.

By Lemma 2, $\kappa^\omega = \kappa$ and, by 2.13 of [6], $T$ is $\kappa$-stable.

As a corollary we obtain a natural characterization of theories having saturated model for every uncountable power.

**Corollary 2.** (i) $T$ has a saturated model for every uncountable power iff $T$ is $\kappa$-stable for every $\kappa > \omega$.
(ii) $T$ has a saturated model for every power $\kappa \geq 2^n$ iff $T$ is superstable.

The proof follows by Theorems 1 and 2 and by the following result of Shelah [5]: Let $T$ be unstable and let $\kappa > \omega$. Then $T$ has a saturated model of power $\kappa$ iff

$$\kappa = \sum_{i < n} \kappa^i.$$

The next proposition shows that we cannot generalize this characterization for $\kappa \geq \omega$.

**Proposition.** (i) There is a superstable theory $T_1$ which is not $\omega$-stable but has a countable saturated model.

(ii) There is a superstable theory $T_2$ which has not a countable saturated model.

**Proof.** We shall describe the examples, but shall not prove the required properties.

(i) Let $\mathcal{A} = \langle \omega \setminus \{0\}, R_p \rangle_{p \in P}$, where $P$ is the set of all prime natural numbers, and $R_p$ are binary relations defined in the following way:

$$n R_p m \text{ iff } \forall r \leq p \ (r \in P \rightarrow (p \text{ divides } n \leftrightarrow p \text{ divides } m)).$$

$T_1 = \text{Th}(\mathcal{A})$ satisfies (i).

(ii) Let $\mathcal{B}_1 = \langle \omega^2, R_i \rangle_{i \in \omega}$, where $R_i$ are unary relations defined in the following way: $R_i(f)$ iff $f(i) = 0$. Thus $T_2 = \text{Th}(\mathcal{B}_1)$ satisfies (ii).

**Remark.** If in the definition of $\kappa$-stable theories we admit also finite cardinals (i.e., if we say $T$ is $\kappa$-stable whenever, for every $\mathcal{A} \in \text{Mod}(T)$ and every $C \subseteq A$ such that $|C| \leq \kappa$, we have $|S(C)| \leq \kappa + \omega$), we get a somewhat artificial characterization:

$T$ has a saturated model for every infinite power iff $T$ is $\kappa$-stable for all $\kappa \neq \omega$.

**REFERENCES**


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