BAND COPULAS AS SPECTRAL MEASURES FOR TWO-DIMENSIONAL STABLE RANDOM VECTORS

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Abstract

In this paper, we study basic properties of symmetric stable random vectors for which the spectral measure is a copula, i.e., a distribution having uniformly distributed marginals.

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1. Introduction

We say that a symmetric random variable is stable if there exists a positive constant $A$ and index of stability $\alpha \in (0, 2]$ such that

$$Ee^{itX} = \exp\{-A|t|^\alpha\}, \quad t \in \mathbb{R}.$$  

A random vector $X = (X_1, \ldots, X_n)$ is symmetric $\alpha$-stable if for every $\xi = (\xi_1, \ldots, \xi_n)$ the random variable $<\xi, X> = \sum_{k=1}^{n} \xi_k X_k$ is symmetric $\alpha$-stable. The following, well known theorem was proven by Feldheim in 1937 and presented in P. Levy [5] in 1937 (first edition).
Theorem 1.1. A random vector $X = (X_1, \ldots, X_n)$ is symmetric $\alpha$-stable if and only if there exists a finite measure $\nu$ on the unit sphere $S_{n-1} \subset \mathbb{R}^n$ such that

$$\mathbb{E}e^{i\langle \xi, X \rangle} = \exp\left\{-\int_{S_{n-1}} \int |\langle \xi, x \rangle|^\alpha \nu(dx)\right\}.$$ 

The measure $\nu$ on $S_{n-1}$ is uniquely determined and it is called the canonical spectral measure for the symmetric $\alpha$-stable random vector $X$.

Remark 1. Usually the measure $\nu$ given in the previous theorem is simply called the spectral measure for the symmetric $\alpha$-stable vector $X$. However we will also consider other representations for the characteristic functions of $X$, so in this paper a canonical spectral measure will always mean the measure concentrated on the unit sphere. The existence of many representations of the characteristic functions for the given symmetric $\alpha$-stable vector $X$ follows from the following theorem:

Theorem 1.2. For every symmetric finite measure $\nu$ on $\mathbb{R}^n$ such that:

$$\int_{\mathbb{R}^n} \int \|x\|^\alpha \nu(dx) < \infty$$

the following function:

$$\varphi(\xi) \overset{\text{def}}{=} \exp\left\{-\int_{\mathbb{R}^n} \int |\langle \xi, x \rangle|^\alpha \nu(dx)\right\}$$

is a characteristic function of a symmetric $\alpha$-stable vector $X = (X_1, \ldots, X_n)$. The measure $\nu$ given by equality (1) we will call a spectral measure for the random vector $X$. This measure is not uniquely determined.

Proof. We shall prove that for every fixed $\xi \in \mathbb{R}^n$ the function $\varphi(t\xi)$, as function of $t \in \mathbb{R}$, is a characteristic function of an $\text{S}_\alpha$ random variable, i.e., there exists $A > 0$ such that $\varphi(\xi t) = e^{-A|t|^\alpha}$. Indeed:

$$\varphi(\xi t) = \exp\left\{-\int_{\mathbb{R}^n} \int |\langle \xi t, x \rangle|^\alpha \nu(dx)\right\}$$

$$= \exp\left\{-|t|^\alpha \int_{\mathbb{R}^n} \int |\langle \xi, x \rangle|^\alpha \nu(dx)\right\}.$$
It is enough to take

\[ A = A(\xi) = \int \cdots \int_{\mathbb{R}^n} \langle \xi, x \rangle |^\alpha \nu(dx). \]

**Remark 2.** If the characteristic function of a symmetric \( \alpha \)-stable random vector \( X \) is given by the formula (1) with the spectral measure \( \nu \), then the canonical spectral measure \( \nu_0 \) for this vector we obtain substituting \( u = x/\|x\| \), and \( r = \|x\| \) and integrating with respect to \( r \). Notice that if \( \nu \) has an atom at zero, then this atom has no influence on the formula (1), thus we can always assume that \( \nu(\{0\}) = 0 \).

**Remark 3.** Assume that \( n = 2 \) and assume that the canonical measure \( \nu \) in formula (1) is absolutely continuous with the density function \( f(x,y) \). Then we can write:

\[
\int \cdots \int_{\mathbb{R}^n} \langle \xi, x \rangle |^\alpha \nu(dx) = \\
= \int_0^{2\pi} |\xi_1 \cos t + \xi_2 \sin t|^\alpha \int_0^\infty r^{\alpha+1} f(r \cos t, r \sin t)drdt.
\]

This means that the canonical spectral measure \( \nu_0 \) for this random vector has the density given by:

\[
g(u) = \int_0^\infty f(ru)r^{\alpha+1}dr, \quad u \in S_1 \subset \mathbb{R}^2.
\]

### 2. Copulaes

In general, by the term copula we understand a two-dimensional (or \( n \)-dimensional) distribution with given marginals. The inversion method restricts the problem of constructing such distributions into constructing distributions on \([0,1]^2\) (or \([-1,1]^2\)) having marginals uniform on the interval \([0,1]\) (or \([-1,1]\) respectively). Many types of copulas are well known in the literature. Recently there appeared a book written by Nelsen [6] which is entirely devoted to the theory of copulae and a two dimensional distribution on \([0,1]^2\). In this paper, we will use copulae from a very wide class constructed independently by T.S. Ferguson in [2] and J. Bojarski in [1]. The construction is follows:
Construction:
Let $Z$ be a random variable with a density function $f(z)$, concentrated on an interval $[-2, 2]$ such that $f(z) = f(-z)$. We define a two-dimensional density function $g(x, y)$ concentrated on $[-1, 1]^2$ by the formula

$$
g(x, y) = \begin{cases} 
    f(x - y) + f(x + y - 2) & \text{for } x + y \geq 0, \\
    f(x - y) + f(x + y + 2) & \text{for } x + y \leq 0.
\end{cases}
$$

The density $g(x, y)$ has marginals uniform on the interval $[-1, 1]$, thus it defines a two-dimensional copulae.

3. Copulae as a spectral measure for an $\mathcal{S}\mathcal{S}$ random vector

Let $x^{<p>} = |x|^p \text{sign}(x)$. This notation is very useful in describing properties and moments of random variables with an infinite variance. In our considerations, we will use the following formulas:

$$
\int (ax + b)^{<\alpha>} dx = \frac{1}{a(\alpha + 1)} (ax + b)^{<\alpha + 1>} + C,
$$

and

$$
\int |ax + b|^\alpha dx = \frac{1}{a(\alpha + 1)} (ax + b)^{<\alpha + 1>} + C.
$$

Theorem 3.1. Assume that $Z$ is a random variable with the density function $f(z)$ concentrated on $[-2, 2]$. If the spectral measure $\nu$ of an $\mathcal{S}\mathcal{S}$ random vector $(X, Y)$ has the density $g(x, y)$ given by formula (2), then the characteristic function of $(X, Y)$ at the point $(a, b)$ is given by $\exp\{-c(a, b)^\alpha\}$ where

$$
c(a, b)^\alpha = \frac{2(1 + \alpha)^{-1}}{(a^2 - b^2)} \mathbb{E} \left[ b(b - a(1 - |Z|))^{<\alpha + 1>} + a(b(1 - |Z|) - a)^{<\alpha + 1>} \right].
$$
The James correlation coefficient for the random vector \((X, Y)\) is given by:

\[
\rho_{\alpha}(X, Y) \overset{\text{def}}{=} \int \cdots \int x^{<\alpha - 1>} y^{\nu}(dx, dy)
\]

\[
= \frac{2}{\alpha(\alpha + 1)} \mathbb{E} \left[ (\alpha + 1)(1 - |Z|) - (1 - |Z|)^{<\alpha + 1>} \right].
\]

**Proof.** The proof is only a matter of laborious calculations, and the integral formulas given at the beginning of this section simplify these calculations slightly. The formula for \(\rho_{\alpha}\) holds for every \(\alpha \in (0, 2]\) as long as the right hand side makes sense. \(\blacksquare\)

**Examples.** In the following three examples we want to illustrate the dependence between the distribution \(f(x)\), the distribution of spectral measure \(g(x, y)\), the shape of level curves of the characteristic function of the corresponding \(S\alpha S\) vector for different \(\alpha\)'s. For each example we give also

\[
h(\alpha) = \frac{\rho_{\alpha}(X, Y)}{\rho_{\alpha}(X, X)}
\]

describing the dependence between \(\alpha\) and the James correlation function. In the definition of the function \(h(\alpha)\) we shall explain something more. Since \(g(x, y)\) is a copula density function, then it has identical marginals, and from the Hölder inequality we obtain

\[
\left| \int \cdots \int x^{<\alpha - 1>} y^{\nu}(dx, dy) \right|
\]

\[
\leq \left( \int \cdots \int |x|^\alpha \nu(dx, dy) \right)^{\frac{\alpha - 1}{\alpha}} \left( \int \cdots \int |y|^\alpha \nu(dx, dy) \right)^{\frac{1}{\alpha}}
\]

\[
= \int \cdots \int |x|^\alpha \nu(dx, dy).
\]

This means that

\[
|h(\alpha)| \leq 1,
\]
thus the function $h(\alpha)$ can play the same role for the $S\alpha S$ random vector as the correlation coefficient for the second order random vector. In our examples the function $h(\alpha)$ makes sense on the whole interval $(0, 2]$.

**Example 1.** Notice that the shape of level curves for the characteristic function suggests positive dependence coefficients, while in fact we have here $h(\alpha) = 0$ for every $\alpha \in (0, 2]$.
Example 3.

References


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