

EDGE MAXIMAL \( C_{2k+1} \)-EDGE DISJOINT FREE GRAPHS

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Abstract

For two positive integers \( r \) and \( s \), \( G(n;r,s) \) denotes the class of graphs on \( n \) vertices containing no \( r \) of \( s \)-edge disjoint cycles and \( f(n;r,s) = \max\{\varepsilon(G) : G \in G(n;r,s)\} \). In this paper, for integers \( r \geq 2 \) and \( k \geq 1 \), we determine \( f(n;r,2k+1) \) and characterize the edge maximal members in \( G(n;r,2k+1) \).

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1. Introduction

The graphs considered in this paper are finite, undirected and have no loops or multiple edges. Most of the notations that follow can be found in [5]. For a given graph \( G \), we denote the vertex set of a graph \( G \) by \( V(G) \) and the edge set by \( E(G) \). The cardinalities of these sets are denoted by \( \nu(G) \) and \( \varepsilon(G) \), respectively. The cycle on \( n \) vertices is denoted by \( C_n \).
Let $G_1$ and $G_2$ be graphs. The union of $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Two graphs $G_1$ and $G_2$ are vertex disjoint if and only if $V(G_1) \cap V(G_2) = \emptyset$; $G_1$ and $G_2$ are edge disjoint if $E(G_1) \cap E(G_2) = \emptyset$. If $G_1$ and $G_2$ are vertex disjoint, we denote their union by $G_1 + G_2$. The intersection $G_1 \cap G_2$ of graphs $G_1$ and $G_2$ is defined similarly, but in this case we need to assume that $V(G_1) \cap V(G_2) \neq \emptyset$. The join $G_1 \vee H$ of two vertex disjoint graphs $G_1$ and $H$ is the graph obtained from $G_1 + H$ by joining each vertex of $G_1$ to each vertex of $H$. For two vertex disjoint subgraphs $H_1$ and $H_2$ of $G$, we let $E_G(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}$ and $E_G(H_1, H_2) = |E_G(H_1, H_2)|$.

In this paper we consider the Turán-type extremal problem with the odd edge disjoint cycles being the forbidden subgraph. Since a bipartite graph contains no odd cycles, the non-bipartite graphs have been considered by some authors. First, we recall some notations and terminologies. For a positive integer $n$ and a set of graphs $\mathcal{F}$, let $G(n; \mathcal{F})$ denote the class of non-bipartite $\mathcal{F}$-free graphs on $n$ vertices, and

$$f(n; \mathcal{F}) = \max\{\mathcal{E}(G) : G \in G(n; \mathcal{F})\}.$$ 

For simplicity, in the case when $\mathcal{F}$ consists only of one member $C_s$, where $s$ is an odd integer, we write $G(n; s) = G(n; \mathcal{F})$ and $f(n; s) = f(n; \mathcal{F})$.

An important problem in extremal graph theory is that of determining the values of the function $f(n; \mathcal{F})$. Further, characterize the extremal graphs $G(n; \mathcal{F})$ where $f(n; \mathcal{F})$ is attained. For a given $r$, the edge maximal graphs of $G(n; r)$ have been studied by a number of authors [1, 2, 3, 7, 8, 9, 10, 12]. In 1998, Jia [11] proved the following result:

**Theorem 1 (Jia).** Let $G \in G(n; 5)$, $n \geq 10$. Then $\mathcal{E}(G) \leq \lfloor (n-2)^2/4 \rfloor + 3$. Furthermore, equality holds if and only if $G \in G^*(n)$ where $G^*(n)$ is the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph $K_{\lfloor (n-2)/2 \rfloor, \lceil (n-2)/2 \rceil}$. Figure 1(a) displays a member of $G^*(n)$.

![Figure 1](image-url)
Jia, also conjectured that \( f(n; 2k + 1) \leq \lfloor (n - 2)^2/4 \rfloor + 3 \) for all \( n \geq 4k + 2 \). In 2007, Bataineh, confirmed positively the conjecture. In fact, he proved the following result:

**Theorem 2** (Bataineh). Let \( k \geq 3 \) be a positive integer and \( G \in \mathcal{G}(n; 2k + 1) \). Then for large \( n \), \( E(G) \leq \lfloor (n - 2)^2/4 \rfloor + 3 \).

Furthermore, equality holds if and only if \( G \in \mathcal{G}^*(n) \) where \( \mathcal{G}^*(n) \) is as above.

Let \( \mathcal{G}(n; r, s) \) denote to the class of graphs on \( n \) vertices containing no \( r \) of \( s \)-edge disjoint cycles and

\[
 f(n; r, s) = \max \{ E(G) : G \in \mathcal{G}(n; r, s) \}.
\]

Note that

\[
 \mathcal{G}(n; 2, s) \subseteq \mathcal{G}(n; 3, s) \subseteq \cdots \subseteq \mathcal{G}(n; r, s).
\]

Let \( \Omega(n, r) \) denote to the class of graphs obtained by adding \( r - 1 \) edges to the complete bipartite graphs \( K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil} \). Figure 1(b) displays a member of \( \Omega(n, 2) \).

The Turán-type extremal problem with two odd edge disjoint cycles being the forbidden subgraph, was studied by Bataineh and Jaradat [2]. In fact, they only established partial results by proving the following:

**Theorem 3** (Bataineh and Jaradat). Let \( k = 1, 2 \) and \( G \in \mathcal{G}(n; 2, 2k + 1) \). Then for large \( n \),

\[
 E(G) \leq \lfloor n^2/4 \rfloor + 1.
\]

Furthermore, equality holds if and only if \( G \in \Omega(n, 2) \).

In this paper, we continue the work initiated in [2] by generalizing and extending the above theorem. In fact, we determine \( f(n; r, 2k + 1) \) and characterize the edge maximal members in \( \mathcal{G}(n; r, 2k + 1) \). Now, we state a number of results, which play an important role in proving our result.

**Lemma 4** (Bondy and Murty). Let \( G \) be a graph on \( n \) vertices. If \( E(G) > n^2/4 \), then \( G \) contains a cycle of length \( r \) for each \( 3 \leq r \leq \lfloor (n + 3)/2 \rfloor \).

**Theorem 5** (Brandt). Let \( G \) be a non-bipartite graph with \( n \) vertices and more than \( \lfloor (n - 1)^2/4 \rfloor + 1 \) edges. Then \( G \) contains all cycles of length between 3 and the length of the longest cycle.

In the rest of this paper, \( N_G(u) \) stands for the set of neighbors of \( u \) in the graph \( G \). Moreover, \( G[X] \) denotes to the subgraph induced by \( X \) in \( G \).
2. Edge-Maximal $C_{2k+1}$-edge Disjoint Free Graphs

In this section, we determine $f(n; r, 2k + 1)$ and characterize the edge maximal members in $\mathcal{G}(n; r, 2k + 1)$. Observe that $\Omega(n, r) \subseteq \mathcal{G}(n; r, 2k + 1)$ and every graph in $\Omega(n, r)$ contains $\lfloor n^2/4 \rfloor + r - 1$ edges. Thus, we have established that

\begin{equation}
    f(n; r, 2k + 1) \geq \lfloor n^2/4 \rfloor + r - 1.
\end{equation}

In the following work, we establish that equality holds. Further we characterize the edge maximal members in $\mathcal{G}(n; r, 2k + 1)$.

**Theorem 6.** Let $k \geq 1, r \geq 2$ be two positive integers and $G \in \mathcal{G}(n; r, 2k + 1)$. For large $n$,

\[ \mathcal{E}(G) \leq \lfloor n^2/4 \rfloor + r - 1. \]

Furthermore, equality holds if and only if $G \in \Omega(n, r)$.

**Proof.** We prove the theorem using induction on $r$.

**Step 1.** We show the result for $r = 2$. Note that by Theorem 3, it is enough to prove the result for $k \geq 3$. Let $G \in \mathcal{G}(n, 2, 2k + 1)$. If $G$ does not have a cycle of length $2k + 1$, then by Lemma 4, $\mathcal{E}(G) \leq \lfloor n^2/4 \rfloor$. Thus, $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. So, we need to consider the case when $G$ has cycles of length $2k + 1$. Assume $C = x_1 x_2 \ldots x_{2k+1} x_1$ be a cycle of length $2k + 1$ in $G$. Consider $H = G - \{e_1 = x_1 x_2, e_2 = x_2 x_3, \ldots, e_{2k+1} = x_{2k+1} x_1\}$. Observe that $H$ cannot have $2k + 1$-cycle as otherwise $G$ would have two edge disjoint $2k + 1$-cycles. We now consider two cases according to $H$:

**Case 1.** $H$ is not a bipartite graph. If $k \geq 2$, then by Theorems 1 and 2

\[ \mathcal{E}(H) \leq \lfloor (n - 2)^2/4 \rfloor + 3. \]

But, $\mathcal{E}(G) = \mathcal{E}(H) + 2k + 1 \leq \lfloor (n-2)^2/4 \rfloor + 2k + 4 \leq \lfloor n^2/4 \rfloor - n + 2k + 5$. Thus, for $n \geq 2k + 5$, we have $\mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1$. If $k = 1$, then by Theorems 5 $\mathcal{E}(H) \leq \lfloor (n-1)^2/4 \rfloor + 1$. And so, by using the same argument as in the above, we get that for $n \geq 7$,

\[ \mathcal{E}(G) < \lfloor n^2/4 \rfloor + 1. \]

**Case 2.** $H$ is a bipartite graph. Let $X$ and $Y$ be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \lfloor n/2 \rfloor$ and $|Y| = \lceil n/2 \rceil$. Thus, $\mathcal{E}(H) \leq \lfloor n^2/4 \rfloor$. Restore the edges of the cycle $C$. We now consider the following subcases:

(2.1). One of $X$ and $Y$ contains two edges of the cycle, say $e_i$ and $e_j$ belong to $X$. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X - \{x_i, x_{i+1}, x_j, x_{j+1}\}$. We split this subcase into two subcases:
2.1.1. \( i \) and \( j \) are not consecutive. Then \(|N_G(x_i) \cap N_G(x_{i+1}) \cap N_G(x_j) \cap N_G(x_{j+1}) \cap N_G(y_1) \cap N_G(y_2) \cap \cdots \cap N_G(y_{k-1})| \leq k + 2\), as otherwise \( G \) contains two edge disjoint \( 2k + 1 \)-cycles. Thus,

\[
\mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k + 2)|Y| + k + 2.
\]

So,

\[
\mathcal{E}(G) = \mathcal{E}_G(X - \{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \mathcal{E}_G(\{x_i, x_{i+1}, x_j, x_{j+1}, y_1, y_2, \ldots, y_{k-1}\}, Y) + \mathcal{E}(G[X]) + \mathcal{E}(G[Y])
\]

\[
\leq (|X| - k - 3)|Y| + (k + 2)|Y| + k + 2 + 2k + 1
\]

\[
\leq |X||Y| - |Y| + 3k + 3 \leq (|X| - 1)|Y| + 3k + 3.
\]

Observe that \(|X| + |Y| = n\). The maximum of the above equation is when \(|Y| = \left\lfloor \frac{n - 1}{2} \right\rfloor \) and \(|X| = 1 = \left\lfloor \frac{n - 1}{2} \right\rfloor \). Thus,

\[
\mathcal{E}(G) \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + 3k + 3.
\]

Hence, for \( n \geq 6k + 7 \), we have \( \mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \).

2.1.2. \( i \) and \( j \) are consecutive, say \( j = i + 1 \). Then by following, word by word, the same arguments as in (2.1.1) and by taking into the account that \(|N_G(x_i) \cap N_G(x_{i+1}) \cap N_G(x_{j+2}) \cap N_G(y_1) \cap N_G(y_2) \cap \cdots \cap N_G(y_{k-1})| \leq k + 1\) and so \( \mathcal{E}(\{x_i, x_{i+1}, x_{i+2}, y_1, y_2, \ldots, y_{k-1}\}, Y) \leq (k + 1)|Y| + k + 1 \), we get the same inequality.

\[
(2.2). \quad \mathcal{E}(G[X]) = 1 \quad \text{and} \quad \mathcal{E}(G[Y]) = 0 \quad \text{or} \quad \mathcal{E}(G[X]) = 0 \quad \text{and} \quad \mathcal{E}(G[Y]) = 1,
\]

say \( e_1 \in E(G[X]) \). Then \( G - e_1 \) is a bipartite graph and so as in the above \( \mathcal{E}(G - e_1) \leq \left\lfloor \frac{n^2}{4} \right\rfloor \). Thus, \( \mathcal{E}(G) = \mathcal{E}(G - e_1) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \).

One can observe from the above arguments that for \( r = 2 \) the only time we have equality is in case when \( G \) is obtained by adding an edge to the complete bipartite graph \( K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil} \). This gives rise to the class \( \Omega(n, 2) \).

**Step 2.** Assume that the result is true for \( r - 1 \). We now show the result is true for \( r \geq 3 \). To accomplish that we use similar arguments to those in Step 1. Let \( G \in \mathcal{G}(n; r, 2k + 1) \). If \( G \) contains no \( r - 1 \) edge disjoint cycles of length \( 2k + 1 \), then by the inductive step \( \mathcal{E}(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 2 \). Thus, \( \mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1 \).

So, we need to consider the case when \( G \) has \( r - 1 \) edge disjoint cycles of length \( 2k + 1 \). Assume that \( \{C_i = x_{i1}, x_{i2}, \ldots, x_{i(2k+1)}, x_{i1}\}_{i=1}^{r-1} \) be the set of cycles of length \( 2k + 1 \). Consider \( H = G - \cup_{i=1}^{r-1} E(C_i) \). Observe that \( H \) cannot have \( 2k + 1 \)-cycles as otherwise \( G \) would have \( r \) of edges disjoint \( 2k + 1 \)-cycles. As in Step 1, we consider two cases:
Case I. $H$ is not a bipartite graph. If $k \geq 2$, then by Theorems 1 and 2 $\mathcal{E}(H) \leq \left\lfloor \frac{(n - 2)^2}{4} \right\rfloor + 3$. Thus, $\mathcal{E}(G) = \mathcal{E}(H) + (r - 1)(2k + 1) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + (r - 1) - n + 4 + 2k(r - 1)$. Hence, for $n > 4 + 2k(r - 1)$, we have $\mathcal{E}(G) < \left\lceil \frac{n^2}{4} \right\rceil + r - 1$.

If $k = 1$, then by Theorems 5 $\mathcal{E}(H) \leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + 1$.

By using the same argument as in the above, we get that for $n \geq 4(r - 1) + 1$,

$$\mathcal{E}(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + 1.$$

Case II. $H$ is a bipartite graph. Let $X$ and $Y$ be the partition of $V(H)$. Thus, $\mathcal{E}(H) \leq |X||Y|$. Observe $|X| + |Y| = n$. The maximum of the above is when $|X| = \left\lceil \frac{n}{2} \right\rceil$ and $|Y| = \left\lfloor \frac{n}{2} \right\rfloor$. Thus, $\mathcal{E}(H) \leq \left\lceil \frac{n^2}{4} \right\rceil$. Now, we consider the following two subcases:

(II.I) There is $1 \leq m \leq r - 1$ such that $C^m$ contains at least two edges, say $e_i = x_{mi}x_{m(i+1)}$ and $e_j = x_{mj}x_{m(j+1)}$, joining vertices of one of $X$ and $Y$, say $X$. Let $y_1, y_2, \ldots, y_{k-1}$ be a set of vertices in $X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}\}$. To this end we have two subcases:

(II.I.1) $i$ and $j$ are not consecutive. Then $|N_Y(x_{mi}) \cap N_Y(x_{m(i+1)}) \cap N_Y(x_{mj}) \cap N_Y(x_{m(j+1)}) \cap N_Y(y_1) \cap N_Y(y_2) \cap \cdots \cap N_Y(y_{k-1})| \leq k+2$, as otherwise $H \cup \{e_i, e_j\}$ contains two edges disjoint $2k + 1$-cycles and so $G$ contains $r$ edge disjoint cycles of length $2k + 1$. Thus, as in (2.1.1) of Step 1,

$$\mathcal{E}_H\left(\{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y\right) \leq (k + 2)|Y| + k + 2.$$ And so,

$$\mathcal{E}(G) = \mathcal{E}(H) + \left| \bigcup_{i=1}^{r-1} E(C^i) \right|$$

$$= \mathcal{E}_H\left( X - \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y \right) + \mathcal{E}_H\left( \{x_{mi}, x_{m(i+1)}, x_{mj}, x_{m(j+1)}, y_1, y_2, \ldots, y_{k-1}\}, Y \right) + \left| \bigcup_{i=1}^{r-1} E(C^i) \right|$$

$$\leq (|X| - k - 3)|Y| + (k + 2)|Y| + k + 2 + (r - 1)(2k + 1)$$

$$= (|X| - 1)|Y| + k + 2 + (r - 1)(2k + 1).$$

Moreover, the maximum of the above inequality is when $|Y| = \left\lceil \frac{n - 1}{2} \right\rceil$ and $|X| - 1 = \left\lceil \frac{n - 1}{2} \right\rceil$. Thus,

$$\mathcal{E}(G) \leq \left\lceil \frac{(n - 1)^2}{4} \right\rceil + k + 2 + (r - 1)(2k + 1).$$

Hence, for $n \geq 6k(r - 1) + 7$, we have $\mathcal{E}(G) < \left\lceil \frac{n^2}{4} \right\rceil + (r - 1)$. 

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i and j are consecutive, say \( j = i + 1 \). Then by following word by word the same arguments as in (2.1.2) of Step 1 and (II.1) of Step 2, we get the same inequality

\[ E(G) < \left\lfloor \frac{n^2}{4} \right\rfloor + (r - 1). \]

(II.2). Each \( 1 \leq m \leq r - 1 \), \( C^m \) has exactly one edge belonging to one of \( X \) and \( Y \). Let \( e \) be the edge of \( C^1 \) that belongs to one of \( X \) and \( Y \). Then \( G - e \in \mathcal{G}(n; r - 1, 2k + 1) \) and so by inductive step, \( E(G) = E(G - e) + 1 \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 2 + 1 = \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1. \)

We now characterize the extremal graphs. Throughout the proof, we notice that the only time we have equality is in case when \( G \) obtained by adding \( r - 1 \) edges to the complete bipartite graph \( K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil} \). This gives rise to the class \( \Omega(n, r) \). This completes the proof of the theorem.

From Theorem 6 and the inequality (1), we get that for \( k \geq 1 \), \( r \geq 2 \) and large \( n \),

\[ f(n; r, 2k + 1) = \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1. \]

References


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