Note

PARITY VERTEX COLORINGS OF BINOMIAL TREES

Petr Gregor
Department of Theoretical Computer Science and Math. Logic
Charles University
Malostranské nám. 25, 118 00 Prague, Czech Republic
e-mail: gregor@ktiml.mff.cuni.cz

and

Riste Škrekovski
Department of Mathematics
University of Ljubljana
Jadranska 21, 1000 Ljubljana, Slovenia

Abstract

We show for every $k \geq 1$ that the binomial tree of order $3k$ has a vertex-coloring with $2k + 1$ colors such that every path contains some color odd number of times. This disproves a conjecture from [1] asserting that for every tree $T$ the minimal number of colors in a such coloring of $T$ is at least the vertex ranking number of $T$ minus one.

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1. Introduction

A parity vertex coloring of a graph $G$ is a vertex coloring such that each path in $G$ contains some color odd number of times. For a study of parity vertex and (similarly defined) edge colorings, the reader is referred to [1,2]. A vertex ranking of $G$ is a proper vertex coloring by a linearly ordered set of colors such that every path between vertices of the same color contains some vertex of a higher color. The minimum numbers of colors in a parity vertex coloring and a vertex ranking of $G$ are denoted by $\chi_p(G)$ and $\chi_r(G)$, respectively.

Clearly, every vertex ranking is also parity vertex coloring, so $\chi_p(G) \leq \chi_r(G)$ for every graph $G$. Borowiecki, Budajová, Jendrol’, and Krajčí [1] conjectured that for trees these parameters behave almost the same.

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Conjecture 1. For every tree $T$ it holds $\chi_r(T) - \chi_p(T) \leq 1$.

In this note we show that the above conjecture is false for every binominal tree of order $n \geq 5$. A binominal tree $B_n$ of order $n \geq 0$ is a rooted tree defined recursively. $B_0 = K_1$ with the only vertex as its root. The binominal tree $B_n$ for $n \geq 1$ is obtained by taking two disjoint copies of $B_{n-1}$ and joining their roots by an edge, then taking the root of the second copy to be the root of $B_n$.

Binominal trees have been under consideration also in other areas. For example, $B_n$ is a spanning tree of the $n$-dimensional hypercube $Q_n$ that has been conjectured [3] to have the minimum average congestion among all spanning trees of $Q_n$. In [1] it was shown, in our notation, that $\chi_r(B_n) = n + 1$ for all $n \geq 0$.

We show that $\chi_p(B_{3k}) \leq 2k + 1$ for every $k \geq 1$, which hence disproves the above conjecture. More precisely, for the purpose of induction we prove a stronger statement in the below theorem. Let us say that a color $c$ on a vertex-colored path $P$ is

- inner, if $c$ does not appear on the endvertices of $P$,
- single, if $c$ appears exactly once on $P$.

Moreover, we say that a vertex of $B_n$ is even (resp. odd) if its distance to the root is even (resp. odd).

**Theorem 2.** For every $k \geq 1$ the binominal tree $B_{3k}$ has a parity vertex coloring with $2k + 1$ colors such that every path of length at least 2 has an inner single color.

**Proof.** For $k = 1$ we define the coloring $f : V(B_3) \rightarrow \{1, 2, 3\}$ by $f = g_{(1,2,3)}$ where $g_{(a,b,c)}$ is defined on Figure 1(a). Observe that $f$ satisfies the statement. In what follows, we assume $k \geq 2$.

The binominal tree $B_{3k+3}$ can be viewed as $B_{3k}$ with a copy of $B_3$ hanged on each vertex. See Figure 1 for an illustration. For a vertex $v \in V(B_{3k})$, let us denote...
Figure 2. The constructed coloring of $B_6$ with 5 colors.

the copy of $B_3$ hanged on $v$ by $B_3(v)$. Let $f'$ be the coloring of $B_{3k}$ with colors \{1, 2, \ldots, 2k+1\} obtained by induction and let $i = 2k+2$, $j = 2k+3$ be the new colors. We define the coloring $f : V(B_{3k+3}) \rightarrow \{1, 2, \ldots, j\}$ by

$$f(B_3(v)) = \begin{cases} g(f'(v), i, j) & \text{if } v \text{ is even}, \\ g(f'(v), j, i) & \text{if } v \text{ is odd}. \end{cases}$$

for every vertex $v \in V(B_{3k})$. See Figure 1 for an illustration. Obviously, it is a proper coloring.

Now we show that the constructed coloring $f$ satisfies the statement. Let $P$ be a path in $B_{3k+3}$ with endvertices in subtrees $B_3(u)$ and $B_3(v)$, respectively. We distinguish three cases.

Case 1. $u = v$. Then $P$ is inside $B_3(u)$ and we are done since the statement holds for $k = 1$.

Case 2. $uv \in E(B_{3k+3})$. Without lost of generality, we assume that $u$ is odd and $u$ is a child of $v$, see Figure 1(b). Clearly, the path $P$ contains the vertices $u$ and $v$. Moreover, if none of the colors $a = f'(u)$, $b = f'(v)$ is inner and single on $P$, then both endvertices of $P$ are in $\{u, v, x, y\}$ where $x$, $y$ are the vertices as on Figure 1(b). Observe that then in all possible cases, $i$ or $j$ is an inner single color on $P$ or $P = (u, v)$. Case 3. $u \neq v$ and $uv \notin E(B_{3k+3})$. Let $P = (P_1, P_2, P_3)$ where $P_1$, $P_2$, and $P_3$ are subpaths of $P$ in $B_3(u)$, $B_{3k}$, and $B_3(v)$ respectively.
As the length of $P_2$ is at least 2, it contains an inner single color $d$ by induction. Since $d$ is inner, it does not appear neither on $P_1$ nor $P_2$. Therefore, the color $d$ is also inner and single on $P$.

From Theorem 2 we obtain the following upper bound.

**Corollary 3.** $\chi_p(B_n) \leq \lceil \frac{2n+3}{3} \rceil$ for every $n \geq 0$.

**Proof.** It is enough to show that $\chi_p(B_{n+1}) \leq \chi_p(B_n)+1$ for every $n \geq 0$. To this end, if we color both copies of $B_n$ in $B_{n+1}$ by (the same) parity vertex coloring with $\chi_p(B_n)$ colors, and we give the root of $B_{n+1}$ a new color, we obtain a parity vertex coloring of $B_{n+1}$ with $\chi_p(B_n)+1$ colors.

On the other hand, Borowiecki et al. [1] showed that $\chi_p(P_n) = \lceil \log_2(n+1) \rceil$ for every $n$-vertex path $P_n$. This gives us a trivial lower bound $\chi_p(B_n) \geq \lceil \log_2(2n+1) \rceil$ as $B_n$ contains a $2n$-vertex path. We ask if the following linear upper bound holds.

**Question 4.** Is it true that $\chi_p(B_n) \geq \frac{n}{2}$ for every $n \geq 0$?

**References**


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