FURTHER RESULTS ON RADIAL GRAPHS

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Abstract

In a graph $G$, the distance $d(u, v)$ between a pair of vertices $u$ and $v$ is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex $u$ is the distance to a vertex farthest from $u$. The minimum eccentricity is called the radius of the graph and the maximum eccentricity is called the diameter of the graph. The radial graph $R(G)$ based on $G$ has the vertex set as in $G$, two vertices $u$ and $v$ are adjacent in $R(G)$ if the distance between them in $G$ is equal to the radius of $G$. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components. The main objective of this paper is to characterize graphs $G$ with specified radius for its radial graph.

Keywords: radius, diameter, radial graph.

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1. Introduction

The graphs considered here are nontrivial and simple. For other graph theoretic notation and terminology, we follow [4, 9]. In a graph $G$, the distance $d(u, v)$ between a pair of vertices $u$ and $v$ is the length of a shortest path joining them. The eccentricity $e(u)$ of a vertex $u$ is the distance to a vertex farthest from $u$. The radius $r(G)$ of $G$ is defined by $r(G) = \min\{e(u) : u \in V(G)\}$ and the diameter $d(G)$ of $G$ is defined by $d(G) = \max\{e(u) : u \in V(G)\}$. A graph for which $r(G) = d(G)$ is called a self-centered graph of radius $r(G)$. A vertex $v$ is called an eccentric vertex of a vertex $u$ if $d(u, v) = e(u)$. A vertex $v$ of $G$ is called an eccentric vertex of $G$ if it is an eccentric vertex of some vertex of $G$. Let $S_i$ denote the subset of vertices of $G$ whose eccentricity is equal to $i$. The concept of antipodal graph was initially introduced by [8] and was further expanded by [2, 3]. The antipodal graph of a graph $G$, denoted by $A(G)$, has the vertex set as in $G$ and two vertices are adjacent if the distance between them is equal to the diameter of $G$. A graph is said to be antipodal if it is the antipodal graph $A(H)$ of some graph $H$. The concept of eccentric graph was introduced by [1]. The eccentric graph based on $G$ is denoted by $G_e$, whose vertex set is $V(G)$ and two vertices $u$ and $v$ are adjacent in $G_e$ if and only if $d(u, v) = \min\{e(u), e(v)\}$. Also Chartrand et al., [5] studied the concept of eccentric graphs. The subgraph of $G$ induced by its eccentric vertices is called the eccentric subgraph of $G$. In [5] a characterization of all graphs that are eccentric subgraph of some connected graph was presented. Kathiresan and Marimuthu [6] introduced a new type of graph called radial graph. Two vertices of a graph $G$ are said to be radial to each other if the distance between them is equal to the radius of the graph. The radial graph of a graph $G$, denoted by $R(G)$, has the vertex set as in $G$ and two vertices are adjacent in $R(G)$ if and only if they are radial in $G$. If $G$ is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of $G$. A graph $G$ is called a radial graph if $R(H) = G$ for some graph $H$. We denote $G_1 = G_2$ if the two graphs $G_1$ and $G_2$ are the same graphs and $G_1 \subset G_2$ if $G_1$ is a proper subgraph of $G_2$. Next we provide some results which will be used to prove some theorems in this paper.

Let $F_{11}, F_{12}, F_{22}, F_{23}, F_{24}$ and $F_3$ denote the set of all connected graphs $G$ for which $r(G) = d(G) = 1$; $r(G) = 1$ and $d(G) = 2$; $r(G) = d(G) = 2$; $r(G) = 2$ and $d(G) = 3$; $r(G) = 2$ and $d(G) = 4$ and $r(G) \geq 3$ respectively. Let $F_4$ denote the set of all disconnected graphs.
Theorem A [4]. If $G$ is a simple graph with diameter at least 3, then $\overline{G}$ has diameter at most 3.

Theorem B [4]. If $G$ is a simple graph with diameter at least 4, then $\overline{G}$ has diameter at most 2.

Theorem C [4]. If $G$ is a simple graph with diameter at least 3, then $\overline{G}$ has radius at most 2.

Theorem D [9]. If $G$ is a self-centered graph with $r(G) \geq 3$, then $\overline{G}$ is a self-centered graph of radius 2.

Theorem E [6]. A graph $G$ is a radial graph if and only if $G$ is the radial graph of itself or the radial graph of its complement.

Theorem F [3]. A graph $G$ is an antipodal graph if and only if $G$ is the antipodal graph of its complement.

The ladder graph $L_n$ [7] with $n$ steps is defined by $L_n = P_n \times K_2$ where $P_n$ is a path on $n$ vertices and $\times$ denotes the Cartesian product of graphs.

2. Graph Equations Involving Radial Graphs

Result 2.1. Let $L_n$ be a ladder with $n$ steps. Then

$$r(L_n) = \begin{cases} \frac{n+2}{2} & \text{if } n \equiv 0, 2 \pmod{4}, \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4}. \end{cases}$$

Result 2.2. Let $L_n$ be a ladder graph with $n$ steps. Then

$$R(L_n) = \begin{cases} 2P_n & \text{if } n \equiv 0, 2 \pmod{4}, \\ C_{2n} & \text{if } n \equiv 1 \pmod{4}, \\ 2C_n & \text{if } n \equiv 3 \pmod{4}. \end{cases}$$

Proof. Let $u_1, u_2, \ldots, u_n$ and let $v_1, v_2, \ldots, v_n$ be the vertices of the ladder $L_n$. The edge set of $L_n$ is $E(L_n) = \{u_iv_i, u_iu_{i+1}, v_iv_{i+1} : i = 1, 2, \ldots, n-1\} \cup \{u_nv_n\}$. 
Case i. \( n \equiv 0 \pmod{4} \).

The radius \( r \) of \( L_n \) is \( \frac{n+2}{2} \). The radial pairs are as follows:

\((u_i, u_{i+1}), (u_i, v_{(i+1)})\) where \( i = 0, 1, \ldots, \frac{n}{2} - 1 \); \((u_{n/2+1}, v_n); (u_{n/2}, v_1)\); 
\((u_j, u_{j-r}), (u_j, v_{j-r+1})\) where \( j = r + 1, r + 2, \ldots, n \).

These radial pairs give the radial graph as the union of two path graphs \( P_1^n \) and \( P_2^n \), where

\[ P_1^n : u_{r-1}, v_{2r-2}, v_{r-2}, u_{2r-3}, u_{r-3}, v_{2r-4}, v_{r-4}, u_{2r-5}, u_{r-5}, v_{2r-6}, v_{r-6}, \ldots, u_{2r-(r-2)}, v_{2r-(r-1)}, v_{r-(r-1)}, u_r \] and

\[ P_2^n : v_{r-1}, u_{2r-2}, u_{r-2}, v_{2r-3}, v_{r-3}, u_{2r-4}, u_{r-4}, v_{2r-5}, v_{r-5}, u_{2r-6}, u_{r-6}, \ldots, v_{2r-(r-2)}, v_{r-(r-1)}, u_{r-(r-1)}, v_r. \]

Figure 1. The graph \( L_4 \) with \( r = 3 \).

Figure 2. The graph \( R(L_4) \).

Case ii. \( n \equiv 2 \pmod{4} \).

We can prove the result as in the Case i.

Case iii. \( n \equiv 3 \pmod{4} \).

The radius \( r \) of \( L_n \) is \( \frac{n+1}{2} \). The radial pairs are as follows:

\((u_i, u_{i+1}), (u_i, v_{(i+1)})\) where \( i = 1, 2, \ldots, r - 1 \); \((u_r, v_1); (u_r, v_n); (u_j, u_{j-r}), (u_j, v_{j-r+1})\) where \( j = r + 1, r + 2, \ldots, n \). The radial graph corresponding
to the above radial pairs is the union of two disjoint cycle graphs $C_n^1$ and $C_n^2$, where
\[ C_n^1 : u_1, u_{r+1}, v_{r-(n-r-1)}, v_{2r-(n-r-1)}, u_{r-(n-r-2)}, v_{2r-(n-r-2)}, \ldots, v_{r-(n-r-3)}, \]
\[ C_n^2 : v_1, v_{r+1}, u_{r-(n-r-1)}, u_{2r-(n-r-1)}, v_{r-(n-r-2)}, v_{2r-(n-r-2)}, \ldots, v_{r-2}, v_{2r-2}, u_{r-1}, u_{2r-1}, v_r, u_1. \]

**Case iv.** $n \equiv 1 \pmod{3}$.
We can prove the result as in Case iii.

**Proposition 2.3.** Let $G$ be a graph of order $n$. Then $R(G) = G$ if and only if $G \in F_{11}$ or $F_{12}$.

**Proof.** Follows from the definition.

To improve the readability of the paper, we offer an outline of the proof of the following two results which are found in [6].

**Proposition 2.4.** If $r(G) > 1$, then $R(G) \subseteq \overline{G}$.

**Proof.** If two vertices $u$ and $v$ are adjacent in $R(G)$, then they are non-adjacent in $G$, since $r(G) > 1$.

**Lemma 2.5.** Let $G$ be a graph of order $n$. Then $R(G) = \overline{G}$ if and only if either $S_2(G) = V(G)$ or $G$ is disconnected in which each component is complete.

**Proof.** If $S_2(G) = V(G)$, then $R(G) = \overline{G}$. If $G$ is the union of complete graphs, that is $G = K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_t}$, then its radial graph $R(G)$ is the complete multipartite graph $K_{n_1,n_2,\ldots,n_t}$, and consequently, $R(G) = \overline{G}$.

If $R(G) = \overline{G}$, then $d(u,v) = 1$ or $r$ in $G$ for any two distinct vertices $u$ and $v$ where $r$ is the radius of $G$. By Proposition 2.3, $r(G) \neq 1$. If $2 < r < \infty$, then there exist at least two vertices $x$ and $y$ in $G$ such that they are nonadjacent in both $R(G)$ and $G$, which contradicts $R(G) = \overline{G}$.

If there is a pair of vertices $x$ and $y$ such that $d(x,y) > 2$, then they are nonadjacent in $G$ and $R(G)$. If $G$ has a noncomplete component then $R(G) \neq \overline{G}$.

Now, we provide some graph equations involving radial graphs.
Proposition 2.6. Let $G$ be a graph. Then $R(G) = R(\overline{G})$ if $G$ is any one of the following graphs.

1. $G$ or $\overline{G}$ is complete.
2. $G$ or $\overline{G}$ is disconnected with each component complete out of which one is an isolated vertex.

Proof. If $G$ is complete, then by Proposition 2.3, $R(G) = G$. Also $R(\overline{G}) = \overline{G}$.

If $\overline{G}$ is complete, then by Proposition 2.3, $R(\overline{G}) = \overline{G}$. Also $R(G) = G$.

If $G$ is disconnected with each component complete out of which one is an isolated vertex, then $R(G) = \overline{G}$ by Lemma 2.5. But $R(\overline{G}) = \overline{G}$ since $\overline{G}$ has a vertex of degree $n - 1$.

If $\overline{G}$ is disconnected with each component complete out of which one is an isolated vertex, then $R(\overline{G}) = G$ and $R(G) = G$.

 Lemma 2.7. If $G$ and $\overline{G}$ are members of $F_{22}$, then $R(G) = R(\overline{G})$.

Proof. Since $G$ and $\overline{G}$ are members of $F_{22}$, by Lemma 2.5, $R(G) = \overline{G}$ and $R(\overline{G}) = G$ and hence $R(G) = R(\overline{G})$.

Lemma 2.8. Let $G$ be a disconnected graph with each component complete and has no isolates. Then $R(G) = R(\overline{G})$.

Proof. Since each component of $G$ is complete, by Lemma 2.5, $R(G) = \overline{G}$ and $R(\overline{G}) = G$. Thus $R(G) = R(\overline{G})$.

Lemma 2.9. Let $G$ be a connected graph such that $R(G) = R(\overline{G})$. Then either $G$ or $\overline{G}$ is a member of $F_{22}$.

Proof. We prove this lemma by assuming that $G \notin F_{22}$. It suffices to show that $\overline{G} \in F_{22}$. If not, then $\overline{G} \in A = F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$. It is impossible that $G$ is connected if $\overline{G} \in F_{11} \cup F_{12}$.

From Theorems A and C, if $\overline{G} \in F_{23}$, then $G \in F_{22} \cup F_{23}$. But, since $G \notin F_{22}$, only the case $G \in F_{23}$ has to be considered. By Proposition 2.4, $R(\overline{G}) \subseteq G$. If $G \in F_{23}$, then $R(G) \subseteq \overline{G}$ and hence $G \subseteq R(\overline{G})$. Thus $R(\overline{G}) \subseteq R(G)$, a contradiction. Since $G$ is connected $G \notin F_4$.

If $\overline{G} \in F_{23} \cup F_3$, then by Theorem B, $G \notin F_{22}$, a contradiction.

Now let $\overline{G} \in F_4$. If $\overline{G}$ has at least one isolated vertex, then $R(G) = G$. But $R(\overline{G}) \subseteq G$. Thus $R(\overline{G}) \subseteq R(G) \subseteq \overline{G}$. This contradicts the fact that
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If $\overline{G}$ has no isolated vertices, then $G \in F_{22}$, a contradiction. The above argument force us to conclude that $\overline{G}$ is a member of $F_{22}$. ■

**Lemma 2.10.** Let $G$ be a disconnected graph such that $\overline{R(G)} = R(\overline{G})$. Then each component of $G$ is a complete graph and has no isolates.

**Proof.** Suppose at least one of the components of $G$ is not complete. Then $R(G) \subset \overline{G}$. This implies that $G \subset \overline{R(G)}$. ■

The following examples show that the notions of radial graph and antipodal graph are independent.

![Figure 3. A radial graph but not an antipodal graph.](image)

![Figure 4. An antipodal graph but not a radial graph.](image)

Next we characterize those antipodal which are radial graphs.

**Theorem 2.11.** A graph $G$ is both radial and antipodal if and only if either $\overline{G} \in F_{22}$ or each component of $\overline{G}$ is complete.

**Proof.** If either $\overline{G} \in F_{22}$ or each component of $\overline{G}$ is complete, then $A(\overline{G}) = R(\overline{G}) = G$, by Lemma 2.5.

Conversely assume that $G$ is both radial and antipodal. Assume that $\overline{G} \notin F_{22}$. We must claim that each component of $\overline{G}$ is complete. Suppose $\overline{G}$ has at least one noncomplete component. Then $A(\overline{G}) \neq G$. This is a contradiction. ■
3. The Radius of Radial Graphs

Theorem G [6]. If both $G$ and $\overline{G}$ are of self-centered graphs of radius 2, then so is $R(G)$.

Proposition 3.1. Let $G$ be a graph of order $n$. Then $r(R(G)) = 1$ if and only if either $\Delta(G) = n - 1$ or $G$ is disconnected with at least one isolated vertex.

Corollary 3.2.
(a) Let $G$ be a connected graph. Then $R(G) \in F_{11}$ if and only if $G \in F_{11}$.
(b) Let $G$ be a connected graph. Then $R(G) \in F_{12}$ if and only if $G \in F_{12}$.
(c) Let $G$ be a disconnected graph. Then $R(G) \in F_{11}$ if and only if $G = K_n$.
(d) Let $G$ be a disconnected graph. Then $R(G) \in F_{12}$ if and only if $G$ has at least one isolated vertex and has a nontrivial component.

Proposition 3.3. Let $G$ be a disconnected graph. Then $R(G) \in F_{22}$ if and only if $G$ has no isolated vertex.

Proof. Follows from the definition.

Next we provide a characterization theorem for $R(G)$ and $R(\overline{G})$ to be members of $F_{22}$.

Theorem 3.4. Let $G$ be a connected graph. Then $R(G)$ and $R(\overline{G})$ are members of $F_{22}$ if and only if $G$ and $\overline{G}$ are members of $F_{22}$.

Proof. If $G$ and $\overline{G}$ are members of $F_{22}$, then by Lemma 2.5, $R(G) = \overline{G}$ and $R(\overline{G}) = G$.

Conversely assume that $R(G)$ and $R(\overline{G})$ are members of $F_{22}$. Assume that $G \notin F_{22}$. Then $G \in A = F_{11} \cup F_{12} \cup F_{23} \cup F_{24} \cup F_3 \cup F_4$.

If $G \in F_{11}$, then by Theorem 2.3, $R(G) = G$, a contradiction to the fact that $R(G) \in F_{22}$. If $G \in F_{12}$, then $R(G) = G$, a contradiction. If $G \in F_{23}$, then by Theorem A, $\overline{G} \in F_{22}$ or $\overline{G} \in F_{23}$. Then $R(\overline{G}) = G$ or $R(\overline{G}) \subset G$, a contradiction, since $R(\overline{G}) \in F_{22}$. It is easy to obtain a contradiction if $G \in F_{24}$ or $G \in F_3$. $G \in F_4$ will not hold since $G$ is connected. Therefore $G \in F_{22}$. Similarly we can prove that $\overline{G} \in F_{22}$. ■

Theorem 3.5. Let $G$ be a connected graph. Then $R(G) \in F_{23}$ if $G \in F_{22}$ and $\overline{G} \in F_{23}$. 
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Proof. By Lemma 2.5, $R(G) = \overline{G}$. Also $\overline{G} \in F_{23}$ and hence $R(G) \in F_{23}$. \hfill $\blacksquare$

Theorem 3.6. Let $G$ be a connected graph. Then $R(G) \in F_3$ if $G \in F_{22}$ and $\overline{G} \in F_3$.

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