BOUNDS ON THE GLOBAL OFFENSIVE $k$-ALLIANCE NUMBER IN GRAPHS

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Abstract

Let $G = (V(G), E(G))$ be a graph, and let $k \geq 1$ be an integer. A set $S \subseteq V(G)$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $N(v)$ is the neighborhood of $v$. The global offensive $k$-alliance number $\gamma_o^k(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$. We present different bounds on $\gamma_o^k(G)$ in terms of order, maximum degree, independence number, chromatic number and minimum degree.

Keywords: global offensive $k$-alliance number, independence number, chromatic number.

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1. Terminology

Let $G = (V, E) = (V(G), E(G))$ be a finite and simple graph. The open neighborhood of a vertex $v \in V$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N_G[v] = N[v] = N(v) \cup \{v\}$. The degree of $v$, denoted by $d_G(v)$, is $|N(v)|$. By $n(G) = n$, $\Delta(G) = \Delta$ and $\delta(G) = \delta$ we denote the order, the maximum degree and the minimum degree of the graph $G$, respectively. If $A \subseteq V(G)$, then $G[A]$ is the graph induced by the vertex set $A$. We denote by $K_n$ the complete graph of order $n$, and by $K_{r,s}$ the complete bipartite graph with partite sets $X$ and $Y$ such that $|X| = r$ and $|Y| = s$. A set $D \subseteq V(G)$ is a $k$-dominating set of $G$ if every vertex of $V(G) - D$ has at least $k \geq 1$ neighbors in $D$. The $k$-domination number $\gamma_k(G)$ is the cardinality of a minimum $k$-dominating set. The case $k = 1$ leads to the classical domination number $\gamma(G) = \gamma_1(G)$.

In [11], Kristiansen, Hedetniemi and Hedetniemi introduced several types of alliances in graphs, including defensive and offensive alliances. We are interested in a generalization of offensive alliances, namely global offensive $k$-alliances, given by Shaqque and Dutton [14, 15]. A set $S$ of vertices of a graph $G$ is called a global offensive $k$-alliance if $|N(v) \cap S| \geq |N(v) - S| + k$ for every $v \in V(G) - S$, where $k \geq 1$ is an integer. The global offensive $k$-alliance number $\gamma^k_0(G)$ is the minimum cardinality of a global offensive $k$-alliance in $G$. If $S$ is a global $k$-offensive alliance of $G$ and $|S| = \gamma^k_0(G)$, then we say that $S$ is a $\gamma^k_0(G)$-set. A global offensive 1-alliance is a global offensive alliance and a global offensive 2-alliance is a global strong offensive alliance. In [7], Fernau, Rodríguez and Sigarreta show that the problem of finding optimal global offensive $k$-alliances is NP-complete.

If $k \geq 1$ is an integer, then let $L_k(G) = \{x \in V(G) : d_G(x) \leq k - 1\}$. Denote by $\alpha(G)$ the independence number, by $\chi(G)$ the chromatic number, and by $\omega(G)$ the clique number of $G$, respectively. The corona graph $G \circ K_1$ of a graph $G$ is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and a pendant edge $vv'$ are added. Next assume that $G_1$ and $G_2$ are two graphs with disjoint vertex sets. The union $G = G_1 \cup G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ and the join $G = G_1 + G_2$ has $V(G) = V(G_1) \cup V(G_2)$ and

$$E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}.$$
2. Upper Bounds

We begin this section by giving an upper bound on the global offensive $k$-alliance number for an $r$-partite graph $G$ in terms of its order and $|L_k(G)|$.

**Theorem 1.** Let $k \geq 1$ be an integer. If $G$ is an $r$-partite graph, then

$$
\gamma^k_o(G) \leq \frac{(r-1)n(G) + |L_k(G)|}{r}.
$$

**Proof.** Clearly, the set $L_k(G)$ is contained in every $\gamma^k_o(G)$-set. In the case that $|L_k(G)| = |V(G)|$, we are finished. In the remaining case that $|L_k(G)| < |V(G)|$, let $V_1, V_2, \ldots, V_r$ be a partition of the $r$-partite graph $G - L_k(G)$ such that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$, where $V_i = \emptyset$ is possible for $i \geq 2$. Then every vertex of $V_1$ has degree at least $k$ in $G$, and all its neighbors are in $V(G) - V_1$. Thus $V(G) - V_1$ is a global offensive $k$-alliance of $G$. Since

$$
|V_1| \geq \frac{|V_1| + |V_2| + \cdots + |V_r|}{r} = \frac{n(G) - |L_k(G)|}{r},
$$

we obtain

$$
\gamma^k_o(G) \leq n(G) - |V_1| \leq n(G) - \frac{n(G) - |L_k(G)|}{r} = \frac{(r-1)n(G) + |L_k(G)|}{r},
$$

and the proof is complete. ■

The case $k = r = 2$ in Theorem 1 leads to the next result.

**Corollary 2** (Chellali [4]). If $G$ is a bipartite graph, then

$$
\gamma^2_o(G) \leq \frac{n(G) + |L_2(G)|}{2}.
$$

**Observation 3.** If $k \geq 1$ is an integer, then $\gamma^k_o(G) \geq \gamma_k(G)$ for any graph $G$.

**Proof.** If $S$ is any $\gamma^k_o(G)$-set, then every vertex of $V(G) - S$ has at least $k$ neighbors in $S$. Thus $S$ is a $k$-dominating set of $G$ and so $\gamma_k(G) \leq |S| = \gamma^k_o(G)$. ■

Using Theorem 1 for $r = 2$ and Observation 3, we obtain the known theorem by Blidia, Chellali and Volkmann [2].
Corollary 4 (Blidia, Chellali, Volkmann [2] 2006). Let $k$ be a positive integer. If $G$ is a bipartite graph, then

$$\gamma_k(G) \leq \frac{n(G) + |L_k(G)|}{2}.$$ 

Since every graph $G$ is $\chi(G)$-partite and $n(G) \leq \chi(G)\alpha(G)$, we obtain also the following corollaries from Theorem 1.

Corollary 5. If $G$ is a graph and $k$ a positive integer, then

$$\gamma_k^0(G) \leq \frac{(\chi(G) - 1)n(G) + |L_k(G)|}{\chi(G)}.$$ 

Corollary 6. Let $k \geq 1$ be an integer. If $G$ is a graph with $\delta(G) \geq k$, then

$$\gamma_k^0(G) \leq (\chi(G) - 1)\alpha(G).$$ 

Theorem 7 (Brooks [3] 1941). If $G$ is a connected graph different from the complete graph and from a cycle of odd length, then $\chi(G) \leq \Delta(G)$.

Combining Brooks’ Theorem and Corollary 6, we can prove the following result.

Theorem 8. Let $k \geq 1$ be an integer, and let $G$ be a connected graph with $\delta(G) \geq k$. Then

$$(1) \quad \gamma_k^0(G) \leq (\Delta(G) - 1)\alpha(G)$$

if and only if $G$ is neither isomorphic to the complete graphs $K_{k+1}$ or $K_{k+2}$ nor to a cycle of odd length when $1 \leq k \leq 2$.

Proof. If $G$ is the complete graph $K_n$, then $\Delta(G) = \delta(G) = n - 1 \geq k \geq 1$ and $\alpha(G) = 1$. Since $\gamma_k^0(K_{k+1}) = k$ and $\gamma_k^0(K_{k+2}) = k + 1$, inequality (1) is not true for these two complete graphs. However, in the remaining case that $n \geq k + 3$, we observe that $\gamma_k^0(G) \leq n - 2$, and we arrive at the desired bound

$$\gamma_k^0(G) \leq n - 2 = \Delta(G) - 1 = (\Delta(G) - 1)\alpha(G).$$
Assume next that $1 \leq k \leq 2$. If $G$ is a cycle of odd length, then $\Delta(G) = 2$, $\gamma^{k}_0(G) = \gamma^{2}_0(G) = \lfloor n(G)/2 \rfloor$ and $\alpha(G) = \lfloor n(G)/2 \rfloor$ and thus (1) is not valid in these cases.

For all other graphs inequality (1) follows directly from Brooks' Theorem and Corollary 6. 

**Lemma 9** (Hansberg, Meierling, Volkmann [10]). Let $k \geq 1$ be an integer. If $G$ is a connected graph with $\delta(G) \leq k - 1$ and $\Delta(G) \leq k$, then

$$k\alpha(G) \geq n(G).$$

**Theorem 10.** Let $k \geq 1$ be an integer. If $G$ is a connected $r$-partite graph with $\Delta(G) \geq k$, then

$$\gamma^{k}_\alpha(G) \leq \frac{\alpha(G)}{r}((r - 1)r + k - 1).$$

**Proof.** Assume that $k = 1$. Since $G$ is connected and $\Delta(G) \geq 1$, we note that $|L_1(G)| = 0$. Applying Theorem 1, and using the fact that $r\alpha(G) \geq n(G)$, we receive the desired inequality immediately.

Assume next that $k \geq 2$. Since $G$ is connected and $G - L_k(G)$ is not empty, every component $Q$ of $G[L_k(G)]$ fulfills $\delta(Q) \leq k - 2$ and $\Delta(Q) \leq k - 1$. Hence Lemma 9 implies $(k - 1)\alpha(Q) \geq n(Q)$. If $Q_1, Q_2, \ldots, Q_t$ are the components of $G[L_k(G)]$, we therefore deduce that

$$\alpha(G) \geq \alpha(G[L_k(G)]) = \sum_{i=1}^{t} \alpha(Q_i) \geq \frac{|L_k(G)|}{k - 1}.$$ 

Combining $n(G) \leq r\alpha(G)$ with Theorem 1, we receive the desired inequality as follows:

$$\gamma^{k}_\alpha(G) \leq \frac{(r - 1)n(G) + |L_k(G)|}{r} \leq \frac{(r - 1)r\alpha(G) + (k - 1)\alpha(G)}{r} = \frac{\alpha(G)}{r}((r - 1)r + k - 1).$$

The case $r = 2$ in Theorem 10 leads to the next result.
Corollary 11. Let \( k \geq 1 \) be an integer. If \( G \) is a connected bipartite graph with \( \Delta(G) \geq k \), then
\[
\gamma^k_\alpha(G) \leq \frac{(k+1)\alpha(G)}{2}.
\]
Using Observation 3, we obtain the following known bounds on the 2-domination number.

Corollary 12 (Fujisawa, Hansberg, Kubo, Saito, Sugita, Volkmann [9] 2008). If \( G \) is a connected bipartite graph of order at least 3, then
\[
\gamma_2(G) \leq \frac{3\alpha(G)}{2}.
\]

Corollary 13 (Blidia, Chellali, Favaron [1] 2005). If \( T \) is a tree of order at least 3, then
\[
\gamma_2(T) \leq \frac{3\alpha(T)}{2}.
\]

Theorem 14 (Favaron, Hansberg, Volkmann [6] 2008). Let \( G \) be a graph. If \( r \geq 1 \) is an integer, then there is a partition \( V(G) = V_1 \cup V_2 \cup \cdots \cup V_r \) of \( V(G) \) such that
\[
|N_G(u) \cap V_i| \leq \frac{d_G(u)}{r}
\]
for each \( i \in \{1, 2, \ldots, r\} \) and each \( u \in V_i \).

Theorem 15. Let \( k \geq 1 \) be an integer. If \( G \) is a graph of order \( n \) with minimum degree \( \delta \geq k \), then
\[
\gamma^k_\alpha(G) \leq \frac{k+1}{k+2}n,
\]
and the bound given in (3) is best possible.

Proof. Choose \( r = k + 2 \) in Theorem 14, and let \( V_1, V_2, \ldots, V_r \) be a partition of \( V(G) \) as in Theorem 14 such that \( |V_1| \geq |V_2| \geq \cdots \geq |V_r| \). If \( D = V_2 \cup V_3 \cup \cdots \cup V_r \), then it follows from (2) and the hypothesis that \( \delta \geq k \) for each \( v \in V_1 = V(G) - D \) that
\[
|N_G(v) \cap D| \geq \left\lfloor \frac{k+1}{k+2}d_G(v) \right\rfloor \geq \left\lfloor \frac{d_G(v)}{k+2} \right\rfloor + k
\]
\[
\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k.
\]
Thus $D$ is a global offensive $k$-alliance of $G$ such that $|D| \leq (k+1)n/(k+2)$, and (3) is proved.

Let $H$ be a connected graph, and let $G_k = H \circ K_{k+1}$. Then it is easy to see that $\gamma^k_o(G_k) = (k+1)n(G_k)/(k+2)$, and therefore (3) is the best possible.

**Corollary 16** (Favaron, Fricke, Goddard, Hedetniemi, Hedetniemi, Kristiansen, Laskar, Skaggs [5] 2004). Let $G$ be graph of order $n$ and minimum degree $\delta$.

If $\delta \geq 1$, then $\gamma^1_o(G) \leq 2n/3$.

If $\delta \geq 2$, then $\gamma^2_o(G) \leq 3n/4$.

In the case that $\delta \geq k+2$, we obtain the following bound, improving the bound of Theorem 15.

**Theorem 17.** Let $k \geq 2$ be an integer, and let $G$ be a graph of order $n$ with minimum degree $\delta \geq k+2$. Then

$$
\gamma^k_o(G) \leq \frac{k}{k+1} n.
$$

**Proof.** Choose $r = k+1$ in Theorem 14, and let $V_1, V_2, \ldots, V_r$ be a partition of $V(G)$ as in Theorem 14 such that $|V_1| \geq |V_2| \geq \cdots \geq |V_r|$. If $D = V_2 \cup V_3 \cup \cdots \cup V_r$, then it follows from (2) and the hypothesis $\delta \geq k+2$ for each $v \in V_1 = V(G) - D$ that

$$
|N_G(v) \cap D| \geq \left\lceil \frac{k}{k+1} d_G(v) \right\rceil \geq \frac{d_G(v)}{k+1} + k
$$

$$
\geq |N_G(v) \cap V_1| + k = |N_G(v) - D| + k.
$$

Thus $D$ is a global offensive $k$-alliance of $G$ such that $|D| \leq kn/(k+1)$, and (4) is proved.

**Theorem 18.** Let $k \geq 1$ be an integer, and let $G$ be a connected non-complete graph such that $\delta(G) \geq k$ and $\gamma^k_o(G) = (\Delta(G) - 1)\alpha(G)$. Then $\Delta(G) \leq k+2$, $\Delta(G) - \delta(G) \leq 1$ and if $k \geq 2$, then $\delta(G) \leq k+1$.

**Proof.** Because of $\chi(G)\alpha(G) \geq n(G)$, Corollary 5 and the hypothesis imply that

$$
(\Delta(G) - 1)\alpha(G) = \gamma^k_o(G) \leq \frac{(\chi(G) - 1)n(G)}{\chi(G)} \leq (\chi(G) - 1)\alpha(G).
$$
Since \( G \) is neither a complete graph nor a cycle of odd length, it follows from Brooks’ Theorem that \( \Delta(G) = \chi(G), \chi(G)\alpha(G) = n(G) \) and

\[
(5) \quad \gamma_0^k(G) = \frac{(\chi(G) - 1)n(G)}{\chi(G)} = \frac{(\Delta(G) - 1)n(G)}{\Delta(G)}.
\]

If we suppose on the contrary that \( \Delta(G) \geq k + 3 \), then it follows from \( (5) \) and Theorem 15 that

\[
\frac{\Delta(G) - 1}{\Delta(G)} n(G) = \gamma_0^k(G) \leq \frac{k + 1}{k + 2} n(G) \leq \frac{\Delta(G) - 2}{\Delta(G) - 1} n(G).
\]

This contradiction shows that \( \Delta(G) \leq k + 2 \).

If we suppose on the contrary that \( \Delta(G) - \delta(G) \geq 2 \), then we deduce that \( \delta(G) = k \) and \( \Delta(G) = k + 2 = \chi(G) \). Since \( \chi(G)\alpha(G) = n(G) \), there exists a partition of \( V(G) \) in \( \chi = \chi(G) \) colour classes \( U_1, U_2, \ldots, U_\chi \) such that \( |U_1| = |U_2| = \cdots = |U_\chi| = \alpha(G) \). Let \( v \) be a vertex of minimum degree \( \delta(G) = k \), and assume, without loss of generality, that \( v \in U_1 \). As \( d_G(v) = k \) and \( \chi(G) = k + 2 \), there exists a colour class \( U_j \) with \( 2 \leq j \leq \chi \) such that \( v \) is not adjacent to any vertex in \( U_j \). Therefore \( U_j \cup \{v\} \) is an independent set. This is a contradiction to the fact that \( |U_j| = \alpha(G) \), and the desired inequality \( \Delta(G) - \delta(G) \leq 1 \) is proved.

Next assume that \( k \geq 2 \), and suppose on the contrary that \( \delta(G) \geq k + 2 \). Then \( k \leq \Delta(G) - 2 \) and \( (5) \) and Theorem 17 lead to the contradiction

\[
\frac{\Delta(G) - 1}{\Delta(G)} n(G) = \gamma_0^k(G) \leq \frac{k}{k + 1} n(G) \leq \frac{\Delta(G) - 2}{\Delta(G) - 1} n(G).
\]

Thus \( \delta(G) \leq k \leq \delta(G) + 1 \) when \( k \geq 2 \), and the proof of Theorem 18 is complete. \( \blacksquare \)

**Example 19.**

1. Let \( H_1, H_2, \ldots, H_t \) be \( t \geq 2 \) copies of the complete graph \( K_{k+1} \), and let \( u_i, v_i \in E(H_i) \) for \( 1 \leq i \leq t \). Define the graph \( G \) as the disjoint union \( H_1 \cup H_2 \cup \cdots \cup H_t \) together with the edge set \( \{v_1u_2, v_2u_3, \ldots, v_{t-1}u_t\} \). Then it is easy to verify that \( \Delta(G) = k + 1, \delta(G) = k, \alpha(G) = t, \gamma_0^k(G) = tk \) and thus \( \gamma_0^k(G) = (\Delta(G) - 1)\alpha(G) \).

2. Let \( F_1 \) and \( F_2 \) be \( 2 \) copies of the complete graph \( K_{k+1} \) with the vertex sets \( V(F_1) = \{x_1, x_2, \ldots, x_{k+1}\} \) and \( V(F_2) = \{y_1, y_2, \ldots, y_{k+1}\} \). Define the graph \( H \) as the disjoint union \( F_1 \cup F_2 \) together with the edge set \( \{x_1y_1, x_2y_2, \ldots, x_ky_k\} \). If \( H_1, H_2, \ldots, H_t \) are \( t \geq 2 \) copies of \( H \), then let
$u_{2i-1}$ and $u_{2i}$ be the vertices of degree $k$ in $H_i$ for all $i \in \{1, 2, \ldots, t\}$. Define the graph $G$ as the disjoint union $H_1 \cup H_2 \cup \cdots \cup H_t$ together with the edge set $\{u_2u_3, u_4u_5, \ldots, u_{2t}u_1\}$. Then $G$ is a $(k + 1)$-regular graph with $\alpha(G) = 2t$, $\gamma_0^k(G) = 2kt$ and thus $\gamma_0^k(G) = (\Delta(G) - 1)\alpha(G)$.

3. Let $k \geq 2$, and let $F_1$ and $F_2$ be 2 copies of the complete graph $K_k$ such that $V(F_1) = \{x_1, x_2, \ldots, x_k\}$ and $V(F_2) = \{y_1, y_2, \ldots, y_k\}$. Define the graph $H$ as the disjoint union $F_1 \cup F_2$ together with the edge set $\{x_1y_1, x_2y_2, \ldots, x_{k-1}y_{k-1}\}$. If $H_1, H_2, \ldots, H_t$ are $t \geq 2$ copies of $H$, then let $u_{2i-1}$ and $u_{2i}$ be the vertices of degree $k - 1$ in $H_i$ for all $i \in \{1, 2, \ldots, t\}$. Define the graph $G$ as the disjoint union $H_1 \cup H_2 \cup \cdots \cup H_t$ together with the edge set $\{u_2u_3, u_4u_5, \ldots, u_{2t}u_1\}$. Then $G$ is a $k$-regular graph with $\alpha(G) = 2t$, $\gamma_0^k(G) = 2(k - 1)t$ and thus $\gamma_0^k(G) = (\Delta(G) - 1)\alpha(G)$.

4. Let $H_1$ and $H_2$ be 2 copies of the complete graph $K_{k+2}$, and let $x \in E(H_1)$ and $y \in E(H_2)$. Define the graph $G'$ as the disjoint union $H_1 \cup H_2$ together with the edge $xy$. Then $\Delta(G') = k + 2$, $\delta(G') = k + 1$, $\alpha(G') = 2$, $\gamma_0^k(G') = 2(k + 1)$ and thus $\gamma_0^k(G') = (\Delta(G') - 1)\alpha(G')$.

These four examples show that $\Delta = k + 1$ and $\delta = k$, $\Delta = \delta = k + 1$, $\Delta = \delta = k + 2$ and $\delta = k + 1$ in Theorem 18 are possible.

Theorem 20. If $G$ is a graph and $k$ an integer such that $1 \leq k \leq \delta(G) - 1$, then

$$\gamma_0^{k+1}(G) \leq \frac{\gamma_0^k(G) + n(G)}{2}.$$

Proof. Let $S$ be a $\gamma_0^k(G)$-set, and let $A$ be the set of isolated vertices in the subgraph induced by the vertex set $V(G) - S$. Then the subgraph induced by $V(G) - (S \cup A)$ contains no isolated vertices. If $D$ is a minimum dominating set of $G[V(G) - (S \cup A)]$, then the well-known inequality of Ore [12] implies

$$|D| \leq \frac{|V(G) - (S \cup A)|}{2} \leq \frac{|V(G) - S|}{2} = \frac{n(G) - \gamma_0^k(G)}{2}.$$

Since $\delta(G) \geq k + 1$, every vertex of $A$ has at least $k + 1$ neighbors in $S$, and therefore $D \cup S$ is a global offensive $(k + 1)$-alliance of $G$. Thus we obtain the desired bound as follows:

$$\gamma_0^{k+1}(G) \leq |S \cup D| \leq \gamma_0^k(G) + \frac{n(G) - \gamma_0^k(G)}{2} = \frac{\gamma_0^k(G) + n(G)}{2}. \quad \blacksquare$$
The graphs $G$ of even order and without isolated vertices with $\gamma(G) = n/2$ have been characterized independently by Payan and Xuong [13] and Fink, Jacobson, Kinch and Roberts [8].

**Theorem 21** (Payan, Xuong [13] 1982 and Fink, Jacobson, Kinch, Roberts [8] 1985). Let $G$ be a graph of even order $n$ without isolated vertices. Then $\gamma(G) = n/2$ if and only if each component of $G$ is either a cycle $C_4$ or the corona of a connected graph.

A graph is $P_4$-free if and only if it contains no induced subgraph isomorphic to the path $P_4$ of order four. A graph is $(K_4-e)$-free if and only if it contains no induced subgraph isomorphic to the graph $K_4-e$, where $e$ is an arbitrary edge of the complete graph $K_4$. The graph $\overline{G}$ denotes the complement of the graph $G$. Next we give a characterization of some special graphs attaining equality in Theorem 20.

**Theorem 22.** Let $G$ be a connected $P_4$-free graph such that $\overline{G}$ is $(K_4-e)$-free. If $k$ is an integer with $1 \leq k \leq \delta(G) - 1$, then $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$ if and only if

1. $G = K_{k+3}$ or
2. $\overline{G} = H \cup 2K_{1,1}$ such that $n(H) = k + 2$ and all components of $H$ are isomorphic to $K_{1,1}$, to $K_{3,3}$, to $K_{3,4}$ or to $K_{4,4}$ or
3. $G = (Q_1 \cup Q_2) + F$, where $Q_1, Q_2$ and $F$ are three pairwise disjoint graphs such that $1 \leq |V(F)| \leq k + 1$, $\alpha(F) \leq 2$, and $Q_1$ and $Q_2$ are cliques with $|V(Q_1)| = |V(Q_2)| = k + 3 - |V(F)|$ such that $|V(F)| \leq 2$ or $\alpha(F) = 1$ and $|V(F)| = k + 1$ or $\alpha(F) = 2$ and $F = K_{k+1} - M$, where $M$ is a matching of $F$ or $\alpha(F) = 2$ and $F = K_k - M$, where $M$ is a perfect matching of $F$ or $\alpha(F) = 2$ and $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ with $k \geq 3t + 3$ and all components of $F$ are isomorphic to $K_{t+2,t+2}$, to $K_{t+2,t+3}$ or to $K_{t+3,t+3}$.

**Proof.** Assume that $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$. Following the same notation as used in the proof of Theorem 20, we obtain $|D| = \frac{|V(G)| - S}{2}$, and we observe that $S \cup D$ is a $\gamma_o^{k+1}(G)$-set. It follows that $G[V(G) - S]$ has no isolated vertices and so by Theorem 21, each component of $G[V(G) - S]$ is either a cycle $C_4$ or the corona of some connected graph. Using the
hypothesis that \(G\) is \(P_4\)-free, we deduce that each component of \(G[V(G) - S]\) is isomorphic to \(K_2\) or to \(C_4\). Since \(G\) is \((K_4 - e)\)-free, there remain exactly the three cases that \(G[V(G) - S]\) is isomorphic to \(K_2\), to \(C_4\) or to \(2K_2\).

**Case 1.** First assume that \(G[V(G) - S] = K_2\). Suppose that \(G\) has an independent set \(Q\) of size at least two. Then the hypothesis \(\delta(G) \geq k + 1\) implies that \(V(G) - Q\) is a global offensive \((k + 1)\)-alliance of \(G\) of size \(n - |Q| < |S \cup D| = n - 1\), a contradiction. Therefore \(\alpha(G) = 1\) and thus \(G = K_{k+3}\).

**Case 2.** Second assume that \(G[V(G) - S]\) is a cycle \(C_4 = x_0x_1x_2x_3x_0\). In the following the indices of the vertices \(x_i\) are taken modulo 4. Recall that \(S \cup D\) is a \(\gamma_o^{i+1}(G)\)-set, and \(D\) contains two vertices of the cycle \(C_4\). Clearly, since \(S\) is a \(\gamma_o^i(G)\)-set, every vertex of the cycle \(C_4\) has degree at least \(k + 4\). Suppose that \(d_G(x_i) \geq k + 5\) for an \(i \in \{0, 1, 2, 3\}\). Then \(\{x_{i+2}\} \cup S\) is a global offensive \((k + 1)\)-alliance of \(G\) of size \(|S| + 1 < |S \cup D| = |S| + 2\), a contradiction. Thus \(d_G(x_i) = k + 4\) for every \(i \in \{0, 1, 2, 3\}\). Now if \(Q\) is an \(\alpha(G)\)-set, then \(|Q| \leq 2\), for otherwise the hypothesis \(\delta(G) \geq k + 1\) implies that \(V(G) - Q\) is a global offensive \((k + 1)\)-alliance of \(G\) of size \(|V(G) - Q| < |S \cup D| = n(G) - 2\), a contradiction too. Since there are two non-adjacent vertices on the cycle \(C_4\) and \(G\) is \(P_4\)-free, it follows that every vertex of \(S\) has at least three neighbors on the cycle \(C_4\).

**Subcase 2.1.** Assume that \(\alpha(G[S]) = 1\). Then the subgraph induced by \(S\) is complete and \(|S| \geq k + 2\). If \(|S| = k + 2\), then we observe that every vertex of \(S\) has exactly four neighbours on the cycle \(C_4\). Thus, in each case, we deduce that \(d_G(y) \geq k + 5\) for every \(y \in S\). But then for any subset \(W\) of \(S\) of size three, the set \(V(G) - W\) is a global offensive \((k + 1)\)-alliance of \(G\) of size less than \(|S \cup D|\), a contradiction.

**Subcase 2.2.** Assume that \(\alpha(G[S]) = 2\). Suppose that there exists a vertex \(w \in S\) with at least \(k + 1\) neighbors in \(S\). Then, since \(|N(w) \cap V(C_4)| \geq 3\), say \(\{x_0, x_1, x_2\} \subseteq N(w) \cap V(C_4)\), we observe that \((S - \{w\}) \cup \{x_0, x_2\}\) is a global offensive \((k + 1)\)-alliance of \(G\) of size \(|S| + 1 < |S \cup D|\), a contradiction. Thus every vertex of \(S\) has at most \(k\) neighbors in \(S\).

Let \(S = X \cup Y\) such that every vertex of \(X\) has exactly three and every vertex of \(Y\) exactly 4 neighbors on \(C_4\). We shall show that \(X = \emptyset\). If \(X \neq \emptyset\), then let \(S_{x_i} \subseteq X\) be the set of vertices such that each vertex of \(S_{x_i}\) is not adjacent to \(x_{i+2}\) for \(i \in \{0, 1, 2, 3\}\). Because of \(\alpha(G) = 2\), we observe that
the set $S_{x_i} \cup \{x_i\}$ induces a complete graph for each $i \in \{0,1,2,3\}$. In addition, since $G$ is $P_4$-free it is straightforward to verify that all vertices of $X \cup C_4$ are adjacent to all vertices of $Y$ and that $S_{x_i} \cup S_{x_{i+1}} \cup \{x_i, x_{i+1}\}$ induces a complete graph for each $i \in \{0,1,2,3\}$. Now assume, without loss of generality, that $S_{x_0} \neq \emptyset$, and let $w \in S_{x_0}$. On the one hand we have seen above that $d_G(w) \leq k + 3$. On the other hand, we observe that $d_G(w) = d_G(x_0)$. But since $d_G(x_0) = k + 4$, we have a contradiction.

Hence we have shown that $X = \emptyset$, and this leads to $|S| = k + 2$. If we define $H = G[S]$, then $\omega(H) = 2$, $\delta(H) \geq 1$ and $\Delta(H) \leq 4$. Since $H$ is also $P_4$-free, $H$ does not contain an induced cycle of odd length. Using $\omega(H) = 2$, we deduce that $H$ is a bipartite graph. Now let $H_1$ be a component of $H$. If $H_1$ is not a complete bipartite graph, then $H_1$ contains a $P_4$, a contradiction. Thus the components of $H$ consists of $K_{1,1}$, $K_{1,2}$, $K_{1,3}$, $K_{1,4}$, $K_{2,2}$, $K_{2,3}$, $K_{2,4}$, $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$.

If $K_{1,2}$ is a component of $H$, then $V(G) - V(K_{1,2})$ is a global offensive $(k + 1)$-alliance of $G$ of size $n - 3$, a contradiction.

If $K_{1,3}$ is a component of $H$ with a leaf $u$, then $(V(G) - V(K_{1,3})) \cup \{u\}$ is a global offensive $(k + 1)$-alliance of $G$ of size $n - 3$, a contradiction.

If $K_{1,4}$ is a component of $H$ and $u,v$ are two leaves of this star, then $(V(G) - V(K_{1,3})) \cup \{u,v\}$ is a global offensive $(k + 1)$-alliance of $G$ of size $n - 3$, a contradiction.

If $K_{2,2}$ is a component of $H$, then $V(G) - V(K_{2,2})$ is a global offensive $(k + 1)$-alliance of $G$ of size $n - 4$, a contradiction.

Next let $K_{2,3}$ be a component of $H$ with the bipartition $\{v_1, v_2, v_3\}$ and $\{u_1, u_2\}$. Then $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k + 1)$-alliance of $G$ of size $n - 3$, a contradiction.

Finally, let $K_{2,4}$ be a component of $H$ with the bipartition $\{v_1, v_2, v_3, v_4\}$ and $\{u_1, u_2\}$. Then $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k + 1)$-alliance of $G$ of size $n - 3$, a contradiction.

Case 3. Third assume that $G[V(G) - S] = 2K_2$. Let $2K_2 = J_1 \cup J_2 = J$ such that $V(J_1) = \{u_1, u_2\}$ and $V(J_2) = \{u_3, u_4\}$. If $\alpha(G) \geq 3$, then we obtain the contradiction $\gamma^{k+1}_0(G) \leq n - 3$. Thus $\alpha(G) = 2$. Since $S$ is a $\gamma^k_0(G)$-set, every vertex of $J$ has degree at least $k + 2$. Suppose that $d_G(u_1) \geq k + 3$ and $d_G(u_2) \geq k + 3$. Then $\{u_3\} \cup S$ is a global offensive $(k + 1)$-alliance of $G$ of size $|S| + 1 < |S \cup D| = |S| + 2$, a contradiction. Thus $J_1$ contains at least one vertex of degree $k + 2$, and for reason of symmetry, also $J_2$ contains a vertex of degree $k + 2$. Since $\alpha(G) = 2$, every vertex of
$S$ has at least two neighbors in $J_1$ or in $J_2$. Now let $x \in S$. If $x$ has two neighbors in $J_i$ and one neighbor in $J_{3-i}$ for $i = 1, 2$, then the hypothesis that $G$ is $P_4$-free implies that $x$ is adjacent to each vertex of $J$. Consequently, $S$ can be partitioned in three subsets $S_1, S_2$ and $A$ such that all vertices of $S_1$ are adjacent to all vertices of $J_1$ and there is no edge between $S_1$ and $J_2$, all vertices of $S_2$ are adjacent to all vertices of $J_2$ and there is no edge between $S_2$ and $J_1$, all vertices of $A$ are adjacent to all vertices of $J$. Since $G$ is $P_4$-free, it follows that there is no edge between $S_1$ and $S_2$, and that all vertices of $S_i$ are adjacent to all vertices of $A$ for $i = 1, 2$. Furthermore, $\alpha(G) = 2$ shows that $G[S_1]$ and $G[S_2]$ are cliques. Altogether we see that $G[S_1]$ and $G[S_2]$ are cliques. Altogether we see that $d_G(u_i) = k + 2$ for each $i \in \{1, 2, 3, 4\}$ and therefore $|S_1| + |A| = |S_2| + |A| = k + 1$. It follows that $|S_1| = |S_2|$ and $|S| + |A| = 2k + 2$. Since $G$ is connected, we deduce that $|A| \geq 1$ and so $1 \leq |A| \leq k + 1$. If we define $F = G[A]$ and $Q_i = G[S_i \cup V(J_i)]$ for $i = 1, 2$, then we derive the desired structure, since $\alpha(G[A]) \leq 2$.

Assume that $|V(F)| \geq 3$ and $\alpha(F) = 1$. If $x_1, x_2, x_3$ are three arbitrary vertices in $F$, then let $S_0 = V(G) - \{x_1, x_2, x_3\}$. If $d_G(x_i) \geq k + 5$ for each $i = 1, 2, 3$, then $S_0$ is a global offensive $(k+1)$-alliance of $G$, a contradiction. Otherwise, we have $n - 1 = d_G(x_i) \leq k + 4$ for at least one $i \in \{1, 2, 3\}$ and so $n \leq k + 5$ and thus $|V(F)| = k + 1$.

Assume next that $|V(F)| \geq 3$ and $\alpha(F) = 2$. As we have seen in Case 2, all components of $\overline{F}$ are complete bipartite graphs.

**Subcase 3.1.** Assume that $K_{1, 1}$ is the greatest component of $\overline{F}$. Let $u$ and $v$ be the two vertices of the complete bipartite graph $K_{1, 1}$. If $n \geq k + 7$, then let $w$ be a further vertex in $F$, and it is easy to verify that $V(G) - \{u, v, w\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n - 3$, a contradiction. If $n = k + 6$ and there exists a vertex $w$ in $F$ of degree $k + 5$, then $V(G) - \{u, v, w\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n - 3$, a contradiction.

**Subcase 3.2.** Assume that $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ and $\overline{F}$ contains a component $K_{p, q}$ with $1 \leq p \leq q$ and $p + q \geq 3$. Let $\{v_1, v_2, \ldots, v_q\}$ and $\{u_1, u_2, \ldots, u_p\}$ be a partition of $K_{p, q}$.

If $K_{1, s} \subseteq \overline{F}$ with $s \geq t + 4$, then $\delta(G) \leq k$, a contradiction to $\delta(G) \geq k + 1$. Thus $q \leq t + 3$.

If $q \leq t + 1$ or $q = t + 2$ and $p \leq t + 1$, then it is easy to see that $V(G) - \{u_1, v_1, v_2\}$ is a global offensive $(k+1)$-alliance of $G$ of size $n - 3$, a contradiction.
Conversely, if $G = K_{k+3}$, then obviously $\gamma_o^k(G) = k + 1$, $\gamma_o^{k+1}(G) = k + 2$ and so $\gamma_o^{k+1}(G) = (\gamma_o^k(G) + n(G))/2$.

Now let $G = H \cup 2K_{1,1}$ such that $n(H) = k + 2$ and the components of $H$ are complete bipartite graphs $K_{1,1}$, $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$. Thus $k + 1 \leq d_G(z) \leq k + 4$ for every $z \in V(G)$, and $G$ contains a cycle $C$ on four vertices, where each vertex of $C$ has degree at least two. This implies that $\gamma_o^k(G) = n(G) - 2$, $\gamma_o^{k+1}(G) \geq n(G) - 4$. Let $D$ be a $\gamma_o^{k+1}(G)$-set. First, assume that $\gamma_o^{k+1}(G) = n(G) - 4$.

Now let us prove that $\gamma_o^{k+1}(G) = n(G) - 2$. Clearly, $\gamma_o^{k+1}(G) \geq \gamma_o^k(G) \geq n(G) - 4$. Let $D$ be a $\gamma_o^{k+1}(G)$-set. First, assume that $\gamma_o^{k+1}(G) = n(G) - 4$. Then, since $n(G) = k + 6$ and $\alpha(G) = 2$, the induced subgraph $G[V(G) - D]$ is isomorphic to $2K_{1,1}$, say $ab$ and $cd$, and every vertex of $V(G) - D$ is adjacent to all vertices of $D$. Since $d_G(x) = k + 3$ for every $x \in \{a, b, c, d\}$ it follows that $a, b, c, d$ lie in one component $C_4$ of $H$, a contradiction. Second, assume that $\gamma_o^{k+1}(G) = n(G) - 3$. Since every vertex has degree at most $k + 4$, no vertex of $V(G) - D$ has two neighbors in $V(G) - D$. Moreover, since $\alpha(G) = 2$, $G[V(G) - D]$ is formed by two adjacent vertices $x, y$ plus an isolated vertex $w$. Since $w$ has degree at least two in $G$, the vertices $w, x, y$ lie in one component in $H$ and so belong to $K_{3,3}$, $K_{3,4}$ or $K_{4,4}$. Thus each of $x$ and $y$ has at least two non-neighbors in $D$ and hence $|N(x) \cap D| \leq k + 1$, a contradiction to the fact $D$ is a $\gamma_o^{k+1}(G)$-set. Thus $|D| \geq n(G) - 2$ and the equality follows from the fact that $V(G)$ minus any two non-adjacent vertices of $C$ is a global offensive $(k + 1)$-alliance of $G$. Therefore $\gamma_o^{k+1}(G) = n(G) - 2 = (\gamma_o^k(G) + n(G))/2$.

Finally, let $G = (Q_1 \cup Q_2) + F$, where $Q_1, Q_2$ and $F$ are three pairwise disjoint graphs such that $1 \leq |V(F)| \leq k + 1$, $\alpha(F) \leq 2$, and $Q_1$ and $Q_2$ are cliques with $|V(Q_1)| = |V(Q_2)| = k + 3 - |V(F)|$ such that $|V(F)| \leq 2$ or $\alpha(F) = 1$ and $|V(F)| = k + 1$ or $\alpha(F) = 2$ and $F = K_{k+1} - M$, where $M$ is matching of $F$ or $\alpha(F) = 2$ and $F = K_k - M$, where $M$ is a perfect matching of $F$ or $\alpha(F) = 2$ and $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ with $k \geq 3t + 3$ and all components of $F$ are isomorphic to $K_{t+2,t+2}$, to $K_{t+3,t+3}$ or to $K_{t+3,t+3}$.

Let $D$ be a global offensive $(k + 1)$-alliance of $G$. Since each vertex of $Q_i$ has degree $k + 2$, the set $V(G) - D$ contains at most one vertex of $Q_i$ for every $i = 1, 2$. Moreover, if $(V(G) - D) \cap V(Q_i) \neq \emptyset$, then $V(F) \subseteq D$.  

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Now suppose that $\gamma_{\alpha}^{k+1}(G) \leq n - 3$, and assume, without loss of generality, that $V(G) - D = \{u, v, w\}$. Then as noted above $V(Q_1) \cup V(Q_2) \subseteq D$, and hence the vertices $u, v, w$ belong to $V(F)$. It follows that $|V(F)| \geq 3$.

Obviously, we obtain a contradiction when $\alpha(F) = 1$ and $|V(F)| = k + 1$.

Assume next that $\alpha(F) = 2$. This implies that at least two vertices of $V(G) - D$ are adjacent in $G$.

First assume that $F = K_k - M$, where $M$ is a perfect matching of $F$. Note that every vertex of $V(F)$ has degree $k + 4$. Since $M$ is perfect, $\{u, v, w\}$ induces either a path $P_3$ or a clique $K_3$ with center vertex, say $v$, in $G$. But then $v$ has a non-neighbor in $D$ for which it is matched in $M$, and so $v$ has exactly $k + 2$ neighbors in $D$ against two in $V(G) - D$, a contradiction.

Second assume that $F = K_{k+1} - M$, where $M$ is a matching of $F$. Note that $n = k + 5$ and $|D| = k + 2$. As above, $\{u, v, w\}$ induces either a path $P_3$ or a clique $K_3$ with center vertex, say $v$, in $G$. But then $v$ has at most $k + 2$ neighbors in $D$ against two in $V(G) - D$, a contradiction.

Assume now that $\alpha(F) = 2$ and $|V(F)| = k + 1 - t$ for $0 \leq t \leq k - 2$ with $k \geq 3t + 3$ and all components of $\overline{F}$ are isomorphic to $K_{t+2,t+2}$, to $K_{t+2,t+3}$ or to $K_{t+3,t+3}$. Note that in this case $n = k + 5 + t$ and so $|D| = n - 3 = k + 2 + t$. Assume, without loss of generality, that $u$ and $v$ are adjacent in $G$. This leads to $|N_G(u) \cap D| \leq (k + 5 + t) - (t + 2 + 2) = k + 1$, a contradiction to the assumption that $D$ is a global offensive $(k + 1)$-alliance of $G$.

Altogether, we have shown that $\gamma_{\alpha}^{k+1}(G) = n - 2$. Finally, it is a simple matter to obtain $\gamma_{\alpha}^{k}(G) = n - 4$, and the proof of Theorem 22 is complete.

# 3. Lower Bounds

Our aim in this section is to give lower bounds on the global offensive $k$-alliance number of a graph in terms of its order $n$, minimum degree $\delta$ and maximum degree $\Delta$.

**Theorem 23.** Let $k$ be a positive integer. If $G$ is a graph of order $n$, minimum degree $\delta$ and maximum degree $\Delta$, then

$$\gamma_{\alpha}^{k}(G) \geq \frac{n(\delta + k)}{2\Delta + \delta + k}.$$  

**Proof.** If $S$ is any $\gamma_{\alpha}^{k}(G)$-set, then

$$\Delta \gamma_{\alpha}^{k}(G) = \Delta |S| \geq \sum_{v \in S} d_G(v) \geq \sum_{v \in V(G) - S} \frac{d_G(v) + k}{2}.$$
This leads to
\[\gamma_0^k(G)(2\Delta + \delta + k) \geq n(\delta + k),\]
and (6) is proved.

**Theorem 24.** Let \( k \geq 1 \) be an integer, and let \( G \) be a graph of order \( n \), minimum degree \( \delta \) and maximum degree \( \Delta \). If \( \delta \) is even and \( k \) odd or \( \delta \) odd and \( k \) even, then
\[\gamma_0^k(G) \geq \frac{n(\delta + k + 1)}{2\Delta + \delta + k + 1}.\]

**Proof.** If \( S \) is any \( \gamma_0^k(G) \)-set, then
\[
\Delta\gamma_0^k(G) = \Delta |S| \geq \sum_{v \in S} d_G(v)
\]
\[
\geq \sum_{v \in V(G) - S, d_G(v) = \delta} \frac{d_G(v) + k + 1}{2} + \sum_{v \in V(G) - S, d_G(v) > \delta} \frac{d_G(v) + k}{2}
\]
\[
\geq |V(G) - S| \frac{\delta + k + 1}{2} = (n - \gamma_0^k(G)) \frac{\delta + k + 1}{2}.
\]
This leads to
\[\gamma_0^k(G)(2\Delta + \delta + k + 1) \geq n(\delta + k + 1),\]
and (7) is proved.

**Example 25.** Let \( G \) be a \( k \)-regular bipartite graph of order \( n \) with the partite sets \( X \) and \( Y \). Then
\[
\gamma_0^k(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + k)}{2\Delta + \delta + k}
\]
and
\[
\gamma_0^{k-1}(G) = |X| = |Y| = \frac{n}{2} = \frac{n(\delta + (k - 1) + 1)}{2\Delta + \delta + (k - 1) + 1}
\]
for \( k \geq 2 \). This family of graphs demonstrate that the bounds in Theorems 23 and 24 are best possible.
References


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