MULTICOLOR RAMSEY NUMBERS FOR SOME PATHS AND CYCLES

HALINA BIELAK

Institute of Mathematics
UMCS, Lublin, Poland

e-mail: hbiel@golem.umcs.lublin.pl

Abstract

We give the multicolor Ramsey number for some graphs with a path or a cycle in the given sequence, generalizing a results of Faudree and Schelp [4], and Dzido, Kubale and Piwakowski [2, 3].

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1. Introduction

We consider simple graphs with at least two vertices. For given graphs $G_1, G_2, \ldots, G_k$ and $k \geq 2$ multicolor Ramsey number $R(G_1, G_2, \ldots, G_k)$ is the smallest integer $n$ such that in arbitrary $k$-colouring of edges of a complete graph $K_n$ a copy of $G_i$ in the colour $i$ ($1 \leq i \leq k$) is contained (as a subgraph).

Let $ex(n, F)$ be the Turán number for integer $n$ and a graph $F$, defined as the maximum number of edges over all graphs of order $n$ without any subgraph isomorphic to $F$.

Theorems 1, 2 and 3 presented below are very useful for study multicolour Ramsey numbers for paths and cycles. In this paper we generalize the results presented in Theorems 4 and 5.

**Theorem 1** (Faudree and Schelp [4]). If $G$ is a graph with $|V(G)| = kp + r$ ($0 \leq k, 0 \leq r < p$) and $G$ contains no $P_{p+1}$, then $|E(G)| \leq kp(p-1)/2 + r(r-1)/2$ with the equality if and only if $G = kK_p \cup K_r$ or $G = lK_p \cup$
\[(K_{(p-1)/2} + K_{(p+1)/2 + (k-l-1)p+r}) \text{ for some } 0 \leq l < k, \text{ where } p \text{ is odd, and } k > 0, r = (p \pm 1)/2.\]

Let \(c(G)\) be the circumference of \(G\), i.e., the length of the longest cycle in \(G\).

**Theorem 2** (Brandt [1]). Every non-bipartite graph \(G\) of order \(n\) with more than \(\frac{(n-1)^2}{4} + 1\) edges contains cycles of every length \(t\), where \(3 \leq t \leq c(G)\).

For positive integers \(a\) and \(b\), set \(r(a; b) = a \mod b = a - \lfloor \frac{a}{b} \rfloor b\). For integers \(n \geq k \geq 3\), set
\[
\omega(n, k) = \frac{1}{2}(n-1)k - \frac{1}{2}r(k-r-1),
\]
where \(r = r(n-1, k-1)\).

**Theorem 3** (Woodall [7]). Let \(G\) be a graph of order \(n\) and size \(m\) with \(m \geq n\) and \(c(G) = k\). Then \(m \leq \omega(n, k)\) and the result is best possible.

In 1975 Faudree and Schelp published the following results concerning a multicolor Ramsey number for paths.

**Theorem 4** (Faudree and Schelp [4]). If \(r_0 \geq 6(r_1 + r_2)^2\), then \(R(P_{r_0}, P_{r_1}, P_{r_2}) = r_0 + \lceil \frac{r_1}{2} \rceil + \lceil \frac{r_2}{2} \rceil - 2\) for \(r_1, r_2 \geq 2\).

If \(r_0 \geq 6(\sum_{i=1}^{k} r_i)^2\), then \(R(P_{r_0}, P_{2r_1 + \delta}, P_{2r_2}, \ldots, P_{2r_k}) = \sum_{i=0}^{k} r_i - k\) for \(\delta = 0, 1, \ k \geq 1\) and \(r_i \geq 1\) (\(1 \leq i \leq k\)).

Recently, Dzido, Kubale, and Piwakowski published the following results.

**Theorem 5** (Dzido et al. [2, 3]). \(R(P_3, C_k, C_k) = 2k - 1\) for odd \(k \geq 9\), \(R(P_4, C_k) = k + 2\) for \(k \geq 6\), \(R(P_3, P_5, C_k) = k + 1\) for \(k \geq 8\).

Moreover, some asymptotic results are cited below.

**Theorem 6** (Kohayakawa, Simonovits, Skokan [6]). There exists an integer \(n_0\) such that if \(n > n_0\) is odd, then \(R(C_n, C_n, C_n) = 4n - 3\).

**Theorem 7**. (Figaj, Luczak [5]). For even \(n\), \(R(C_n, C_n, C_n) = 2n + o(n)\).
2. Results

First we prove the following theorem, extending the result of Dzido et al. (see Theorem 5).

**Theorem 8.** Let \( t, q \ (t \geq q \geq 2) \) be positive integers and \( m \) be odd integer. Let for even \( q \) either \( t > \frac{3}{2}q^2 - 2q + 2 \) and \( m = t + \lfloor \frac{q}{2} \rfloor \) or \( t > \frac{5}{8}(3q^2 - 10q + 16) \) and \( m \leq t + \lfloor \frac{q}{2} \rfloor - 1 \). Let for odd \( q \), \( t > \frac{1}{2}(3q^2 - 14q + 21) \) and \( m \leq t + \lfloor \frac{q}{2} \rfloor - 1 \). Then \( R(P_q, P_t, C_m) = 2t + 2\lfloor \frac{q}{2} \rfloor - 3 \).

**Proof.** Let \( n = 2t + 2\lfloor \frac{q}{2} \rfloor - 3 \) and \( a = t + \lfloor \frac{q}{2} \rfloor - 2 \). First we prove that \( R(P_q, P_t, C_m) \geq 2t + 2\lfloor \frac{q}{2} \rfloor - 3 \). Let \( K_a \) be (red, blue)-coloured without red \( P_q \) and without blue \( P_t \). It is possible by \( R(P_q, P_t) = a + 1 \). So there exists the critical colouring of the graph \( H = K_a \cup K_m \). Let the edges of \( H \) be coloured with green. Since \( H \) is bipartite graph it does not contain any \( C_m \).

Now we prove that \( R(P_q, P_t, C_m) \leq 2t + 2\lfloor \frac{q}{2} \rfloor - 3 \).

Note that \( |E(K_a)| = (2t + 2\lfloor \frac{q}{2} \rfloor - 3)(t + \lfloor \frac{q}{2} \rfloor - 2) \) and \( |E(K_{a,0})| = (t + \lfloor \frac{q}{2} \rfloor - 2)^2 \).

Let \( d = |E(K_a)| - |E(K_{a,0})| = (t + \lfloor \frac{q}{2} \rfloor - 2)(t + \lfloor \frac{q}{2} \rfloor - 1) \).

So

\[
(2) \quad d = (t-1)(t+q-4) + \left\lfloor \frac{q}{2} \right\rfloor \left( \left\lfloor \frac{q}{2} \right\rfloor - 1 \right) + 2(t-1) - (t-1) \left( \left\lfloor \frac{q}{2} \right\rfloor - \left\lfloor \frac{q}{2} \right\rfloor \right).
\]

Suppose that we can colour \( E(K_n) \) with three colours (red, blue, green) without red \( P_q \), blue \( P_t \) and green \( C_m \). So the red subgraph of \( K_n \) has at most \( \text{ex}(n, P_q) \) edges and the blue subgraph of \( K_n \) has at most \( \text{ex}(n, P_t) \) edges. Now we apply Theorem 1 for \( p = t - 1 \). We have two cases. If \( 2|q \) and \( t = q \) then set \( k = 3, r = 0 \). In the opposite case, set \( k = 2 \) and \( r = 2\lfloor \frac{q}{2} \rfloor - 1 \). Thus, we can write \( \text{ex}(n, P_t) \leq (t-1)(t-2) + (2\lfloor \frac{q}{2} \rfloor - 1)\left( \left\lfloor \frac{q}{2} \right\rfloor - 1 \right) \).

Moreover, by Theorem 1 for \( p = q - 1 \), we get \( \text{ex}(n, P_q) \leq \frac{n(q-2)}{2} \). So \( \text{ex}(n, P_q) \leq (t-1)(q-2) + \frac{1}{2}(2\lfloor \frac{q}{2} \rfloor - 1)(q-2) \).

Let \( s = \text{ex}(n, P_t) + \text{ex}(n, P_q) \). So the red-blue subgraph of \( K_n \) has at most \( s \) edges and

\[
s \leq (t-1)(t+q-4) + (q-1)(q-2) - \begin{cases} 
0, & 2|q, \\
3(q-2), & 2 \not| q.
\end{cases}
\]
By the above fact and (2) we note that 
\[ d - s \geq h(q, t), \]
where
\[ h(q, t) = \left\lfloor \frac{q}{2} \right\rfloor \left( \left\lfloor \frac{q}{2} \right\rfloor - 1 \right) - (q - 1)(q - 2) + (t - 1) + \begin{cases} (t - 1), & 2|q, \\ \frac{3(q-2)}{2}, & 2 \nmid q. \end{cases} \]

Moreover, \( h(q, t) > 0 \) if and only if
\[ t > \begin{cases} \frac{1}{8} (3q^2 - 10q + 16), & 2|q, \\ \frac{1}{4} (3q^2 - 14q + 21), & 2 \nmid q. \end{cases} \]

So for \( t \) satisfying the above condition the green subgraph \( G' \) of \( K_n \) has more edges than the graph \( K_{a,a} \). Namely, \( |E(G')| > |E(K_{a,a})| + h(q, t) \). Note that \( G' \) is not a bipartite graph. In the opposite case we have at least 
\[ t + \left\lfloor \frac{q}{2} \right\rfloor - 1 = R(P_t, P_q) \]
vertices in a part of the bipartite graph and the proof is done since we get a red \( P_q \) or a blue \( P_t \).

By definition (1), we get
\[ \omega(n, m - 1) = \omega(2t + 2 \left\lfloor \frac{q}{2} \right\rfloor - 3, m - 1) = (t + \left\lfloor \frac{q}{2} \right\rfloor - 2)(m - 1) - \frac{1}{2} r(m - 2 - r), \]
where \( r = r(n - 1, m - 2) \). So \( \omega(n, m - 1) \leq (t + \left\lfloor \frac{q}{2} \right\rfloor - 2)(m - 1) \).

We would like to apply the theorems of Woodall and Brandt. We look for a lower bound of the longest cycle in the green graph \( G' \). Thus let \( b \geq 0 \) be maximum integer \( b \geq 0 \) such that the following inequalities hold
\begin{enumerate}[(i)]  
  \item \( b \cdot a < h(q, t) \) 
  \item \( \omega(n, m - 1) \leq (t + \left\lfloor \frac{q}{2} \right\rfloor - 2)(t + \left\lfloor \frac{q}{2} \right\rfloor - 2 + b) < |E(G')| \).
\end{enumerate}

Evidently \( b < 2 \), else we get a contradiction to the first of the above inequalities. Moreover, if \( 2|q \) and \( t > \frac{1}{4} (3q^2 - 8q + 8) \), then \( b = 1 \). For other cases \( b = 0 \).

Then, by Theorem 3, we get \( c(G') \geq (t + \left\lfloor \frac{q}{2} \right\rfloor - 1 + b) \). Thus we get a cycle of order at least \( (t + \left\lfloor \frac{q}{2} \right\rfloor - 1 + b) \) in the green graph \( G' \).

Moreover, \( \frac{(n-1)^2}{4} + 1 = (t + \left\lfloor \frac{q}{2} \right\rfloor - 2)^2 + 1 < |E(G')| \). So, by Theorem 2, the green graph \( G' \) is weakly pancyclic. Hence we get a green cycle \( C_m \) for \( m \leq t + \left\lfloor \frac{q}{2} \right\rfloor - 1 + b \), a contradiction. Therefore each (red, blue, green)-colouring of \( E(K_n) \) contains a red \( P_q \), a blue \( P_t \) or a green \( C_m \). So we get the upper bound for \( R(P_q, P_t, C_m) \). The proof is done.

In general case we get the following theorem.
Theorem 9. \( R(P_q, P_t, C_m) \geq \left\lfloor \frac{n}{q} \right\rfloor - 2 + \max \{ t + \left\lfloor \frac{m}{q} \right\rfloor, m + \left\lfloor \frac{n}{q} \right\rfloor \} \).

**Proof.** Let \( r = \left\lfloor \frac{n}{q} \right\rfloor - 3 + \max \{ t + \left\lfloor \frac{m}{q} \right\rfloor, m + \left\lfloor \frac{n}{q} \right\rfloor \} \) and \( x = \left\lfloor \frac{n}{q} \right\rfloor - 1 \). Let \( K_{r-x} \) be subgraph of \( K_r \) (blue, green)-coloured without blue \( P_t \) and without green \( C_m \). Such critical colouring exists by \( R(P_t, P_m) = r - x + 1 \). Let other edges of \( K_r \) be coloured with red. The red subgraph does not contain any \( P_q \).

The proof is done.

Now we extend the result of Faudree and Schelp presented above in Theorem 4.

**Proposition 10.** Let \( t_0 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 2 \), \( k \geq 2 \) be integers and \( n = t_0 + \sum_{i=1}^{k} (\left\lfloor \frac{t_i}{2} \right\rfloor - 1) \). Let \( x = 2 \) if \( t_0 = t_1 = t_2 \) and \( 2 \nmid t_0 \), and \( x = 0 \) in the opposite case. Then \( R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) \geq n + x \).

**Proof.** Let \( t_0 = t_1 = t_2 = 2 \) \( \text{and} \ 2 \nmid t_0 \). We define the critical colouring of the graph \( K_{n+x-1} \), with \( x = 2 \). Let \( A, B, C, D, E_j, (j = 3, \ldots, k) \) be sets with \( |A| = |B| = |C| = |D| = \left\lfloor \frac{n}{2} \right\rfloor \) and \( |E_j| = \left\lfloor \frac{t_j}{2} \right\rfloor - 1, (j = 3, \ldots, k) \).

Let the edges with ends in the sets \( A \cup B \) and \( C \cup D \) be coloured with the colour 0, the edges with one end in the set \( A \) and the second one in the set \( C \) be coloured with the colour 1, the edges with one end in the set \( B \) and the second one in the set \( D \) be coloured with the colour 1. Other edges with ends in \( A \cup B \cup C \cup D \) colour with the colour 2. Let \( V_j = A \cup B \cup C \cup D \cup \bigcup_{i=3}^{k} E_i, (j = 3, \ldots, k) \). Let colour the edges with both ends in \( E_j \) or one end in \( E_j \) and the second one in the set \( V_j \) with the colour \( j, (j = 3, \ldots, k) \). Note that the colouring contains no monochromatic \( P_{t_i} \) in the colour \( i \).

If the condition \( t_0 = t_1 = t_2 = 2 \) \( \text{and} \ 2 \nmid t_0 \) does not hold we define the critical colouring of the graph \( K_{n+x-1} \), with \( x = 0 \). Namely, let \( |A| = t_0 + \left\lfloor \frac{t_1}{2} \right\rfloor - 2 \), \( |E_j| = \left\lfloor \frac{t_j}{2} \right\rfloor - 1, (j = 2, \ldots, k) \) and \( V_j = A \cup \bigcup_{i=2}^{k} E_i, (j = 2, \ldots, k) \). Let colour the edges with both ends in \( E_j \) or one end in \( E_j \) and the second one in the set \( V_j \) with the colour \( j, (j = 2, \ldots, k) \). The edges with ends in the set \( A \) colour critically with colours 0 and 1 (it is possible by \( R(P_{t_0}, P_{t_1}) = t_0 + \left\lfloor \frac{t_1}{2} \right\rfloor - 1 \)).

The proof is done.

Now we show some sufficient conditions for \( R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) = n + x \) with \( x = 0 \) or \( x = 2 \) and \( n = t_0 + \sum_{i=1}^{k} (\left\lfloor \frac{t_i}{2} \right\rfloor - 1) \).

**Theorem 11.** Let \( t_0 \geq t_1 \geq t_2 \geq \cdots \geq t_k \geq 2 \), \( k \geq 2 \) be integers and \( n = t_0 + \sum_{i=1}^{k} (\left\lfloor \frac{t_i}{2} \right\rfloor - 1) \). Let \( x = 2 \) if \( t_0 = t_1 = t_2 \) and \( 2 \nmid t_0 \), and \( x = 0 \)
in the opposite case, and let \( r_i = (n + x) \mod (t_i - 1) \) \((i = 0, 1, \ldots, k)\). The sufficient conditions for \( R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) = n + x \) are as follows:

(i) \( t_0 > t_1, \) \( 2|t_i \) for each \( i \geq 1 \) and 
\[
 t_0 > \max \left\{ \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \right)^2 - \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \cdot \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + 2 \right\},
\]

(ii) \( t_0 > t_1, \) \( 2 \not| t_i \) for exactly one \( i \geq 1 \) and 
\[
 t_0 > \max \left\{ 2 \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + 1 \right)^2 - \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \cdot \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + 2 \right\},
\]

(iii) \( t_0 \in \{4, 6, 8\} \), \( t_0 = t_1 > t_2 \) and \( t_i = 2 \) for each \( i = 2, \ldots, k \),

(iv) \( t_0 \in \{3, 5\} \), \( t_0 = t_1 > t_2 \) and \( t_i = 2 \) for each \( i = 2, \ldots, k \),

(v) \( t_0 = t_1 = t_2 = 3 > t_3 \) and \( t_i = 2 \) for each \( i = 3, \ldots, k \) or \( t_0 = t_1 = t_2 = t_3 = 3 \) and \( t_i = 2 \) for each \( i = 4, \ldots, k \),

(vi) \( t_i = 2 \) for each \( i = 0, \ldots, k \).

Proof. By Proposition 10 we get the lower bound \( n + x \leq R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) \). Now we prove the upper bound. Evidently, \( 0 \leq r_i < t_i - 1 \). By definition of \( n \) and \( r_0 \) we have

\[
(3) \quad \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) = w \cdot (t_0 - 1) + r_0,
\]

where \( w \geq 0 \) and \( 0 \leq r_0 \leq t_0 - 2 \) are integers.

By Theorem 1 we get \( \sum_{i=0}^{k} ex(n + x, P_i) \leq s \), where
\[
s = \frac{n + x}{2} \sum_{i=0}^{k} (t_i - 2) - \frac{1}{2} \sum_{i=0}^{k} r_i(t_i - 1 - r_i).
\]

Let \( g = \left( \left\lfloor \frac{n + x}{2} \right\rfloor \right) - s \). Evidently,

\[
(4) \quad g = \frac{n + x}{2} \left( n + x - 1 - \sum_{i=0}^{k} t_i + 2k + 2 \right) + \frac{1}{2} \sum_{i=0}^{k} r_i(t_i - 1 - r_i).
\]

Note that, \( g > 0 \) is a sufficient condition for \( R(P_{t_0}, P_{t_1}, P_{t_2}, \ldots, P_{t_k}) \leq n + x \).

Let \( y \) be the number of odd \( t_i \), for \( i = 1, \ldots, k \). So

\[
(5) \quad y = \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - \left\lfloor \frac{t_i}{2} \right\rfloor \right).
\]
Let
\[ a = r_0 - \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + y - 1 - x \right). \]

(6)

Then by the definition of \( n \) we have
\[
\begin{align*}
g &= (a - r_0) \frac{t_0 + \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + x}{2} + \frac{1}{2} r_0(t_0 - 1 - r_0) \\
&\quad + \frac{1}{2} \sum_{i=1}^{k} r_i(t_i - 1 - r_i).
\end{align*}
\]

(7)

Hence, by (7) and (6), we get
\[
g = \frac{a}{2} t_0 - \frac{1}{2} \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + x \right)^2 + \frac{1}{2} \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + x \right)(2x + 1 - y)
\]

(8)

\[ - \frac{1}{2} r_0(r_0 + 1) + \frac{1}{2} \sum_{i=1}^{k} r_i(t_i - 1 - r_i). \]

If \( a > 0 \) and \( g > 0 \) then we can find some additional restriction on \( t_i \) to obtain the upper bound of Ramsey number for the sequence of paths.

By (6), the assumption \( a > 0 \) gives
\[
r_0 \geq \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + y - x.
\]

(9)

Let us consider three cases.

**Case 1.** Suppose that \( t_0 > t_1 \). So \( x = 0 \). Thus, by the value of \( n \), we get
\[
r_0 = \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + 1.
\]

(10)

By (6), (10) and the assumption \( a > 0 \), we have \( y = 0 \) or \( y = 1 \). Moreover, if \( y = 0 \) then \( a = 2 \) and if \( y = 1 \) then \( a = 1 \).

By (8),
\[
t_0 > \frac{1}{a} \left( r_0(r_0 + 1) + \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \right)^2 - (1 - y)\sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right).
\]
is a sufficient condition for \( g > 0 \).

Thus we get \( t_0 > r_0^2 - (r_0 - 1) \) for \( y = 0 \) and \( t_0 > r_0(2r_0 - 1) + 1 \) for \( y = 1 \).

Elementary counting leads to the condition (i) and (ii), respectively.

Case 2. Suppose that \( t_0 = t_1 > t_2 \). Thus \( x = 0 \) and by (8) we get

\[
(11) \quad g = \frac{a + r_0}{2}t_0 - \frac{1}{2} \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \right)^2 + \frac{1}{2} (1 - y) \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) - r_0(r_0 + 1) + \frac{1}{2} \sum_{i=2}^{k} r_i(t_i - 1 - r_i).
\]

If \( a + r_0 > 0 \) and \( g > 0 \) then we can find some further restriction on \( t_i \) to obtain the above Ramsey number for the sequence of paths.

First, by (6) and the assumption \( a + r_0 > 0 \), we note that

\[
(12) \quad r_0 > \frac{1}{2} \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + y - 1 \right).
\]

Moreover, by (11), if

\[
(13) \quad t_0 > \frac{1}{a + r_0} \left( 2r_0(r_0 + 1) + \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \right)^2 - (1 - y) \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \right)
\]

then \( g > 0 \).

By definition of \( r_0 \), (3) and (12), we get

\[
(14) \quad t_0 - 2 \geq r_0 \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) - w \cdot (t_0 - 1) > \frac{1}{2} \left( \sum_{i=1}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + y - 1 \right).
\]

Let us assume that \( w > 0 \). Then, by \( t_0 = t_1 \), we get

\[
(15) \quad \frac{1}{2} \left( \sum_{i=2}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + 2 - y \right) > \left\lfloor \frac{t_0}{2} \right\rfloor + \frac{1}{2} \left\lfloor \frac{t_0}{2} \right\rfloor
\]

\[
> \frac{1}{2} \left( \sum_{i=2}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) + y + 2 \right).
\]
The left-side inequality in (15) follows by the right-side inequality from (14). The right-side inequality in (15) follows by the most left and the most right relation in (14). Hence we get a contradiction.

Let us assume that \( w = 0 \). Then, by (3) and \( t_0 = t_1 \), we get \( r_0 = \left\lfloor \frac{t_0}{2} \right\rfloor + \sum_{i=2}^{k} \left( \left\lfloor \frac{t_i}{2} \right\rfloor - 1 \right) \). By (14) we get \( y = 0 \) or \( y = 1 \). So, by (13) and (6), we get \( t_0 > \frac{1}{r_0+1} \left( 2r_0(r_0+1) + (r_0-1)(r_0-2+y) \right) \).

Considering the case we get \( t_0 > 3r_0 - 7 + 16/(r_0+2) \) for \( y = 0 \) and \( t_0 > 3r_0 - 3 + 4/(r_0+1) \) for \( y = 1 \). Elementary counting leads to the condition (iii) and (iv), respectively.

Case 3. Suppose that \( t_0 = t_1 = t_2 \). If the condition (v) holds then \( n = 3, x = 2 \). If the condition (vi) holds then \( n = 2, x = 0 \). Thus, by (4), we get \( g > 0 \) for these cases and the result holds. The proof is done.

We conclude with the following result for three paths.

**Corollary 12.** Let \( m, t, q \) \((m \geq t \geq q \geq 2)\) be positive integers. Let either \( m > \frac{1}{2}((t+q)^2 - 7(t+q)+14) \) and 2 \( \nmid (t+q) \) or \( m > \frac{1}{2}((t+q)^2 - 6(t+q)+12) \) and 2 \| \( t \) and 2 \| \( q \). Then \( R(P_q, P_t, P_m) = m + \left\lfloor \frac{t}{2} \right\rfloor + \left\lfloor \frac{q}{2} \right\rfloor - 2 \).

**Proof.** If 2 \( \nmid (t+q) \) then we apply Theorem 11 (ii). If 2 \| \( t \) and 2 \| \( q \) then we apply Theorem 11 (i) for \( m > 2 \) and Theorem 11 (vi) for \( m = q = t = 2 \). □

**References**


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