ON LOCATING AND DIFFERENTIATING-TOTAL DOMINATION IN TREES

MUSTAPHA CHELLALI

LAMDA-RO Laboratory
Department of Mathematics
University of Blida
B.P. 270, Blida, Algeria

e-mail: m_chellali@yahoo.com

Abstract

A total dominating set of a graph $G = (V, E)$ with no isolated vertex is a set $S \subseteq V$ such that every vertex is adjacent to a vertex in $S$. A total dominating set $S$ of a graph $G$ is a locating-total dominating set if for every pair of distinct vertices $u$ and $v$ in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and $S$ is a differentiating-total dominating set if for every pair of distinct vertices $u$ and $v$ in $V$, $N[u] \cap S \neq N[v] \cap S$. Let $\gamma^L_t(G)$ and $\gamma^D_t(G)$ be the minimum cardinality of a locating-total dominating set and a differentiating-total dominating set of $G$, respectively. We show that for a nontrivial tree $T$ of order $n$, with $\ell$ leaves and $s$ support vertices, $\gamma^L_t(T) \geq \max\{2(n+\ell-s+1)/5, (n+2-s)/2\}$, and for a tree of order $n \geq 3$, $\gamma^D_t(T) \geq 3(n+\ell-s+1)/7$, improving the lower bounds of Haynes, Henning and Howard. Moreover we characterize the trees satisfying $\gamma^L_t(T) = 2(n+\ell-s+1)/5$ or $\gamma^D_t(T) = 3(n+\ell-s+1)/7$.

Keywords: locating-total domination, differentiating-total domination, trees.

2000 Mathematics Subject Classification: 05C69.

1. Introduction

In a graph $G = (V, E)$, the open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$ and the closed neighborhood is $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v$ is the size of its open neighborhood. A leaf of a tree $T$
is a vertex of degree one, while a support vertex of $T$ is a vertex of degree at least two adjacent to a leaf. A strong support vertex is adjacent to at least two leaves. We denote the order of a tree $T$ by $n$, the number of leaves by $\ell$, and the number of support vertices by $s$. A tree $T$ is a double star if it contains exactly two vertices that are not leaves. A double star with, respectively $p$ and $q$ leaves attached at each support vertex is denoted by $S_{p,q}$. A subdivided star $SS_q$ is obtained from a star $K_1, q$ by subdividing each edge by exactly one vertex. A corona of a graph $H$ is the graph $G$ formed from $H$ by adding a new vertex $v'$ for each vertex $v \in V(H)$ and the edge $v'v$. For a subset $S \subseteq V$, we denote by $(S)$ the subgraph induced by the vertices of $S$.

A subset $S$ of vertices of $V$ is a total dominating set of $G$ if every vertex in $V$ is adjacent to a vertex in $S$. The total domination number, $\gamma_t(G)$ is the minimum cardinality of a total dominating set of $G$.

In this paper we are interested in two types of total-dominating sets, namely locating-total dominating sets, and differentiating-total dominating sets defined as follows: A total dominating set $S$ of a graph $G$ is called a locating-total dominating set (LTDS) if for every pair of distinct vertices $u$ and $v$ in $V - S$, $N(u) \cap S \neq N(v) \cap S$, and $S$ is called a differentiating-total dominating set (DTDS) if for every pair of distinct vertices $u$ and $v$ in $V$, $N[u] \cap S \neq N[v] \cap S$. The locating-total domination number, $\gamma^L_t(G)$ is the minimum cardinality of a LTDS of $G$, and the differentiating-total domination number, $\gamma^D_t(G)$ is the minimum cardinality of a DTDS of $G$. A LTDS of minimum cardinality is called a $\gamma^L_t(G)$-set. Likewise we define a $\gamma^D_t(G)$-set. Note that a tree $T$ of order $n$ admits a LTDS (resp., DTDS) if $n \geq 2$ (resp., $n \geq 3$) since the entire vertex set is such a set. Also for every $\gamma^D_t(G)$-set $D$ there is no component of size 2 in the subgraph induced by $D$, for otherwise the two vertices $u, v$ of such a component would satisfy $N[u] \cap D = N[v] \cap D = \{u, v\}$. Locating-total domination and differentiating-total domination were introduced by Haynes, Henning and Howard [4].

In this paper we establish sharp bounds on $\gamma^L_t(T)$, and $\gamma^D_t(T)$ for trees $T$. More precisely, we show that if $T$ is a tree of order $n \geq 2$, with $\ell$ leaves and $s$ support vertices, then $\gamma^L_t(T) \geq \max\{2(n + \ell - s + 1)/5, (n + 2 - s)/2\}$ and if $T$ is a tree of order $n \geq 3$, then $\gamma^D_t(T) \geq 3(n + \ell - s + 1)/7$. Then we give a characterization of trees with $\gamma^L_t(T) = 2(n + \ell - s + 1)/5$, or $\gamma^D_t(T) = 3(n + \ell - s + 1)/7$.

We sometimes consider the removing of an edge of a tree $T$. If $uv$ is an edge of $T$, then we denote by $T_u$ (resp., $T_v$) the subtree of $T$ that contains
u (resp., v) obtained by removing uv. The following notation and fact will be used in the proofs. Let \(n_1, \ell_1, s_1\) be the order, the number of leaves and support vertices of \(T_u\), respectively, and likewise let \(n_2, \ell_2, s_2\) for \(T_v\). Clearly \(n_1 + n_2 = n\), and if \(n_1\) and \(n_2 \geq 3\), then \(\ell_1 + \ell_2 \geq \ell + q\), and \(s_1 + s_2 = s + q\), where \(q\) is the number of new support vertices in \(T_u\) and \(T_v\) with \(0 \leq q \leq 2\). Also if \(D\) is a \(\gamma^L\)-set or \(\gamma^D\)-set, then let \(D_u = D \cap V(T_u), \text{ and } D_v = D \cap V(T_v)\).

2. Lower Bounds on \(\gamma^L(T)\)

In [4], Haynes, Henning and Howard gave two lower bounds on the locating-total domination number for trees and characterized extremal trees for each lower bound. Let \(G = P_n\) be the path on \(n\) vertices.

**Theorem 1** (Haynes, Henning and Howard [4]).

1. If \(T\) is a tree of order \(n \geq 2\), then \(\gamma^L(T) \geq 2(n + 1)/5\).
2. For \(n \geq 2\), \(\gamma^L(P_n) = \gamma(P_n) = \lfloor n/2 \rfloor + \lceil n/4 \rceil - \lfloor n/4 \rfloor\).

**Theorem 2** (Haynes, Henning and Howard [4]). If \(T\) is a tree of order \(n \geq 3\) with \(\ell\) leaves and \(s\) support vertices, then \(\gamma^L(T) \geq (n + 2(\ell - s) + 1)/3\).

Our next result improves the lower bound of Theorem 1 for every nontrivial tree \(T\). It also improves Theorem 2 for trees of order \(n \geq 4\ell - 4s\). Let \(\mathcal{F}\) be the family of trees that can be obtained from \(r\) disjoint copies of \(P_4\) and \(P_3\) by first adding \(r - 1\) edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

**Theorem 3.** If \(T\) is a tree of order \(n \geq 2\), then

\[
\gamma^L(T) \geq 2(n + \ell - s + 1)/5,
\]

with equality if and only if \(T \equiv P_2\) or \(T \in \mathcal{F}\).

**Proof.** We proceed by induction on the order of \(T\). If \(n = 2\), then \(T = P_2\) and \(\gamma^L(P_2) = 2(n + \ell - s + 1)/5 = 2\). Every star \(K_{1,p}\) (\(p \geq 2\)) satisfies \(\gamma^L(K_{1,p}) = p \geq 2(n + \ell - s + 1)/5\) with equality if and only if \(p = 2\), that is \(T = P_3 \in \mathcal{F}\). This establishes the base cases. Assume that every tree \(T'\) of order \(2 \leq n' < n\) satisfies \(\gamma^L(T') \geq 2(n' + \ell' - s' + 1)/5\). Let \(T\) be a
tree of order \( n \). Among all \( \gamma^L(T) \)-sets, let \( D \) be one that contains as few leaves as possible. Note that every vertex \( x \) of \( D \) has at most one private neighbor in \( V - D \) for if it had two private neighbors \( x', x'' \), then we would have \( N(x') \cap D = N(x'') \cap D = \{ x \} \). If \( \ell = 2 \), then \( T \) is a path \( P_n \), and by Theorem 1, \( \gamma^L(P_n) = \lfloor n/2 \rfloor + \lfloor n/4 \rfloor - \lfloor n/4 \rfloor \geq 2(n + \ell - s + 1)/5 \) with equality if and only if \( T = P_3 \in \mathcal{F} \). Thus we may assume that \( \ell \geq 3 \).

Assume that \( T \) contains a strong support vertex \( y \) adjacent to at least three leaves. Then \( D \) contains \( y \) and all its leaves except possibly one. Let \( y' \in D \) be any leaf adjacent to \( y \), and let \( T' = T - \{ y \} \). Clearly \( D - \{ y' \} \) is a LTDS of \( T', n' = n - 1, \ell' = \ell - 1 \), and \( s' = s \). By induction on \( T' \), we have \( |D - \{ y' \}| \geq \gamma^L(T') \geq 2(n' + \ell' - s' + 1)/5 \), implying that \( |D| > 2(n + \ell - s + 1)/5 \). Thus every support vertex is adjacent to at most two leaves.

Assume that \( \langle D \rangle \) contains a connected component \( \langle D_i \rangle \) of diameter at least 3. Then there exists an edge \( uw \), such that \( \langle D_i - uv \rangle \) contains no isolated vertices. Clearly each of \( T_u \) and \( T_v \) has order at least three, \( D_u \) and \( D_v \) are two LTDS of \( T_u \) and \( T_v \), respectively. Recall that \( n_1 + n_2 = n, \ell_1 + \ell_2 = \ell + q, \) and \( s_1 + s_2 = s + q \), where \( 0 \leq q \leq 2 \) is defined above. Applying the inductive hypothesis to \( T_u \) and \( T_v \), we obtain

\[
|D| = |D_u| + |D_v| \geq 2(n_1 + \ell_1 - s_1 + 1)/5 + 2(n_2 + \ell_2 - s_2 + 1)/5 > 2(n + \ell - s + 1)/5.
\]

Thus every component of \( \langle D \rangle \) has diameter one or two.

If each \( w \in V - D \) is a leaf, then \( D \) contains for each support vertex all its leaves except possibly one. Hence \( |D| \geq n - s \). Since \( \ell \geq 3 \) and \( n - s \geq \ell \), it follows \( |D| \geq n - s > 2(n + \ell - s + 1)/5 \). Thus there exists a vertex \( w \in V - D \) such that \( w \) is not a leaf. Assume now that \( w \) has a neighbor \( v \in V - D \). Then each of \( T_w \) and \( T_v \) has order at least three, \( D_w \) and \( D_v \) are two LTDS of \( T_w \) and \( T_v \), respectively. By induction on \( T_w \) and \( T_v \), we obtain

\[
|D| = |D_w| + |D_v| \geq 2(n_1 + \ell_1 - s_1 + 1)/5 + 2(n_2 + \ell_2 - s_2 + 1)/5 > 2(n + \ell - s + 1)/5.
\]

Hence we may assume that \( V - D \) is an independent set and so every private neighbor of a vertex of \( D \) is a leaf. Suppose now that \( w \) has degree at least three and let \( z \) be any vertex of \( N(w) \cap D \). By removing \( wz \), then \( T_w \) has order at least three. If \( V(T_z) = \{ z, z' \} \) then \( z' \) is a leaf of \( T \) and so \( \{ w \} \cup D - \{ z' \} \) is a \( \gamma^L(T) \)-set with less leaves than \( D \), contradicting our assumption on \( D \). Thus \( T_z \) has order at least three. Also \( D_w \) and \( D_z \) are
two LTDS of $T_w$ and $T_z$, respectively. The rest of the proof is similar to as shown above, which leads to $|D| > 2(n + \ell - s + 1)/5$. Thus every vertex of $V - D$ is either a leaf or has degree two. Note that all cases considered until now do not lead to extremal trees because $|D| > 2(n + \ell - s + 1)/5$.

Let $W$ be the set of vertices of $V - D$ having degree two. Since $T$ is a tree, $|W| = k - 1$ where $k$ is the number of connected components of $\langle D \rangle$. Let $T'$ be the forest induced by the vertices of $V(T) - W$ and let $T_1, T_2, \ldots, T_k$ the components of $T'$. Then $n_1 + \cdots + n_k = n - |W|$, $\ell_1 + \cdots + \ell_k \geq \ell + q$, and $s_1 + \cdots + s_k = s + q$, where $q$ is the number of new support vertices. Also $D \cap V(T_i) = D_i$ is a LTDS of $T_i$, for every $i = 1, \ldots, k$. By induction on each $T_i$, we obtain

$$|D| = \sum_{i=1}^{k} |D_i| \geq \sum_{i=1}^{k} 2(n_i + \ell_i - s_i + 1)/5 \geq 2(n - |W| + \ell - s + k)/5 = 2(n + \ell - s + 1)/5.$$ 

Assume now that $\gamma^T_i(T) = 2(n + \ell - s + 1)/5$. Then we have equality throughout the above inequality chain. In particular, $\gamma^T_i(T_i) = 2(n_i + \ell_i - s_i + 1)/5$ for each $i$, and $\ell_1 + \cdots + \ell_k \geq \ell + q$, and $s_1 + \cdots + s_k = s + q$. This means that $T'$ has a new leaf if and only if it has a new support vertex. So each $T_i$ has order at least three. Recall that every component $\langle D_i \rangle$ has diameter one or two. Suppose that for some $i$, $\langle D_i \rangle$ has diameter two, that is $\langle D_i \rangle$ is a star of center vertex say, $x$ and leaves $y_1, y_2, \ldots, y_t$ with $t \geq 2$. We distinguish between three cases. If $x$ is a not a support vertex neither in $T$ nor in $T_i$, then each $y_i$ is support vertex of $T_i$, and so $T_i$ is a subdivided star but $\gamma^T_i(T_i) > 2(n_i + \ell_i - s_i + 1)/5$, a contradiction. If $x$ is a not a support vertex of $T$ but $x$ is a support vertex of $T_i$, then $T_i$ is a corona of $K_{1, t-1}$, where $|D_i| = t + 1 > 2(n_i + \ell_i - s_i + 1)/5$. Now if $x$ is a support vertex of $T$ with at most two leaves, then every $y_j$ is a support vertex in $T_i$ for either $1 \leq j \leq t$ or $2 \leq j \leq t$, but then $\gamma^T_i(T_i) > 2(n_i + \ell_i - s_i + 1)/5$, a contradiction.

Finally, assume that each connected subgraph $\langle D_i \rangle$ is of diameter one. Then $T_i = P_3$ or $P_4$, and the leaves of $T_i$ are leaves in $T$. Thus every component of $T'$ is either a path $P_3$ or $P_4$ where every vertex of $W$ joins two support vertices of any different components $T_i, T_j$.

Conversely, let $T \in \mathcal{F}$ be a tree obtained from $k_1$ disjoint copies of $P_3$ and $k_2$ disjoint copies of $P_4$ with $k_1 + k_2 \geq 1$, by adding $k_1 + k_2 - 1$ new vertices, where each new vertex is adjacent to exactly two support vertices.
Clearly the set of all support vertices plus one leaf from each copy of $P_3$
forms a minimum LTDS of $T$ of size $2(n + \ell - s + 1)/5$. So extremal trees $T$
achieving $\gamma^l_T(T) = 2(n + \ell - s + 1)/5$ are precisely those of $\mathcal{F}$.

Note that in [2], Chellali and Haynes showed that every nontrivial tree satis-
sifies $\gamma^l(T) \geq (n + 2 - \ell)/2$. Since every LTDS is a total dominating set,
$\gamma^l_T(T) \geq (n + 2 - \ell)/2$. Our next result improves this lower bound.

**Theorem 4.** If $T$ is a tree of order $n \geq 2$, then $\gamma^l_T(T) \geq (n + 2 - s)/2$.

**Proof.** We proceed by induction on the order of $T$. It is a routine matter to
to check that the result holds if $\text{diam}(T) \in \{1, 2\}$. Assume that every tree
$T'$ of order $2 \leq n' < n$ satisfies $\gamma^l_T(T') \geq (n' + 2 - s')/2$. Let $T$ be a tree of
order $n$ and $S$ a $\gamma^l_T(T)$-set that contains leaves as few as possible.

If the subgraph induced by $V - S$ contains some edge $xy$, then let $T_x$
and $T_y$ be the trees obtained by removing the edge $xy$ where $x \in T_x$ and
$y \in T_y$. Clearly each of $T_x$ and $T_y$ has order at least three, $S_x$ and $S_y$
are two LTDS of $T_x$ and $T_y$, respectively. Also $n_1 + n_2 = n$, and $s_1 + s_2 = s + q$,
where $q$ is the number of new support vertices with $0 \leq q \leq 2$. By induction
on $T_x$ and $T_y$, we have $|S| = |S_x| + |S_y| \geq (n_1 + 2 - s_1)/2 + (n_2 + 2 - s_2)/2 =
(n + 4 - s - q) \geq (n + 2 - s)/2$. Thus $V - S$ is independent.

Let $w$ be a vertex of $V - S$ different to a leaf. If $w$ does not exist,
then $|S| \geq n - s \geq (n + 2 - s)/2$, since $s \geq 2$. Thus $w$ exists and has at
least two neighbors in $S$. Let $z$ be any neighbor of $N(w) \cap S$, and consider
the trees $T_w$ and $T_z$ obtained by removing the edge $wz$ where $w \in T_w$ and
$z \in T_z$. If $V(T_z) = \{z, z'\}$, then $z'$ is a leaf of $T$ and $\{w\} \cup S - \{z'\}$ is
a $\gamma^l_T(T)$-set with less leaves than $S$, a contradiction with our choice of $S$.

Thus $T_z$ has order at least three. Also $S_w$ and $S_z$ are two LTDS of $T_w$ and
$T_z$, respectively. Hence by induction on $T_w$ and $T_z$ and since $n_1 + n_2 = n$,
and $s_1 + s_2 = s + q$, where $0 \leq q \leq 2$ is defined as above, we obtain $|S| =
|S_w| + |S_z| \geq (n_1 + 2 - s_1)/2 + (n_2 + 2 - s_2)/2 = (n + 4 - s - q) \geq (n + 2 - s)/2$.
This achieves the proof.

The lower bound of Theorem 4 is sharp for the path $P_n$ with $n \equiv 0(\text{mod } 4)$
and improves Theorem 3 for nontrivial trees with $n > 4\ell + s - 6$.

3. **Lower Bound on $\gamma^D_T(T)$**

In [4], Haynes, Henning and Howard gave a lower bound of the differentiating-
total domination number of any tree with at least three vertices.
Theorem 5 (Haynes, Henning and Howard [4]).

(1) If $T$ is a tree of order $n \geq 3$, then $\gamma^D_l(T) \geq 3(n + 1)/7$.

(2) For $n \geq 3$, $\gamma^D_l(P_n) = \lceil 3n/5 \rceil + 1$ if $n \equiv 3(\text{mod } 5)$ and $\gamma^D_l(P_n) = \lceil 3n/5 \rceil$, otherwise.

Note that as mentioned in [4], since total dominating set is an identifying code, Theorem 2 is also a lower bound for code contains at least $3(n + 1)/7$ vertices. Theorem 5(2), thus $T$ is a path $P_n$ with $n \geq 5$, then by Theorem 5(2), $\gamma^D_l(P_n) > 3(n + \ell - s + 1)/7$. Thus we assume that $\ell \geq 3$.

A subset $S$ of vertices of $V$ is an identifying code (or a differentiating domination set as defined in [3]) if for every pair of distinct vertices $u$ and $v$ in $V$, $N[u] \cap S \neq N[v] \cap S \neq \emptyset$.

In [1], Bldia et al. showed for trees of order $n \geq 4$ that every identifying code contains at least $3(n + \ell - s + 1)/7$ vertices. Since every differentiating-total dominating set is an identifying code, $3(n + \ell - s + 1)/7$ is a lower bound for $\gamma^D_l(T)$ which improves Theorem 5. Note that the lower bound $3(n + \ell - s + 1)/7$ is better than $(n + 2(\ell - s) + 1)/3$ for trees with $2n > 5\ell - 5s - 2$. For the purpose of characterizing extremal trees we give here a proof of $\gamma^D_l(T) \geq 3(n + \ell - s + 1)/7$, by using a similar argument to that used in the proof of Theorem 3.

Let $\mathcal{G}$ be the family of trees that can be obtained from $r$ disjoint copies of a corona of $P_3$, a double star $S_{2,1}$ and a star $K_{1,3}$ by first adding $r - 1$ edges so that they are incident only with support vertices and the resulting graph is connected, and then subdividing each new edge exactly once.

Theorem 6. If $T$ is a tree of order $n \geq 3$, then

$$\gamma^D_l(T) \geq 3(n + \ell - s + 1)/7,$$

with equality if and only if $T \in \mathcal{G}$.

Proof. We use an induction on the order of $T$. If $\text{diam}(T) = 2$, then $T = K_{1,p}(p \geq 2)$. Thus $\gamma^D_l(K_{1,2}) = 3 > 3(n + \ell - s + 1)/7$ and for $p \geq 3$, $\gamma^D_l(K_{1,p}) = p > 3(n + \ell - s + 1)/7$ with equality if and only if $p = 3$, that is $T = K_{1,3} \in \mathcal{G}$. If $\text{diam}(T) = 3$, then $T = S_{p,q}$. Thus $\gamma^D_l(S_{1,1}) = 3 > 3(n + \ell - s + 1)/7$ and for max{$p, q$} $\geq 2$, $\gamma^D_l(S_{p,q}) = p + q \geq 3(n + \ell - s + 1)/7$ with equality if and only if $p + q = 3$, that is $T = S_{2,1} \in \mathcal{G}$. This establishes the base cases. Assume that every tree $T'$ of diameter at least 4 and order $n'$, $5 \leq n' < n$ satisfies $\gamma^D_l(T') \geq 3(n' + \ell' - s' + 1)/7$. Let $T$ be a tree of order $n$, and $D$ a $\gamma^D_l(T)$-set. If $T$ is a path $P_n$ with $n \geq 5$, then by Theorem 5(2), $\gamma^D_l(P_n) > 3(n + \ell - s + 1)/7$. Thus we assume that $\ell \geq 3$. 

If any strong support vertex \( y \) is adjacent to at least four leaves, then let \( T' = T - \{y'\} \), where \( y' \) is any leaf adjacent to \( y \). Without loss of generality \( y' \in D \), and then \( D - \{y'\} \) is a DTDS of \( T' \). Hence by induction on \( T' \) we have \( |D| - 1 \geq \gamma^D_0(T') \geq 3(n' + \ell' - s' + 1)/7 \). Since \( n' = n - 1, \ell' = \ell - 1, \) and \( s' = s \), we obtain \( |D| > 3(n + \ell - s + 1)/7 \). For the next we assume that each support vertex is adjacent to at most three leaves.

Assume, the subgraph \( \langle D \rangle \) contains a connected component \( \langle D_i \rangle \) of diameter at least 5. Thus there exists an edge \( uv \), such that each connected component of \( \langle D_i - uv \rangle \) has diameter at least 2. Then \( D_u \) and \( D_v \) are two DTDS of \( T_u \) and \( T_v \), respectively. Since \( n_1 + n_2 = n, \ell_1 + \ell_2 \geq \ell + q, \) and \( s_1 + s_2 = s + q \), then by induction on \( T_u \) and \( T_v \), we obtain

\[
|D| = |D_u| + |D_v| \geq 3(n_1 + \ell_1 - s_1 + 1)/7 + 3(n_2 + \ell_2 - s_2 + 1)/7 > 3(n + \ell - s + 1)/7.
\]

Thus every component of \( \langle D \rangle \) has diameter two, three or four.

Suppose that \( \langle V - D \rangle \) contains some edge \( uv \). Then by removing the edge \( uv \), each of \( T_u \) and \( T_v \) has order at least four, \( D_u \) and \( D_v \) are two DTDS of \( T_u \) and \( T_v \), respectively. By using the induction on \( T_u \) and \( T_v \), it follows that \( \gamma^D_0(T) > 3(n + \ell - s + 1)/7 \). Thus \( V - D \) is independent and hence every private neighbor of a vertex of \( D \) is a leaf.

Let \( w \) be any vertex of \( V - D \) different to a leaf. If \( w \) does not exist, then \( |D| \geq n - s \geq 3(n + \ell - s + 1)/7 \) with equality only if \( T \) is a corona of a path \( P_3 \) or a double star \( S_{2,1} \). Thus \( T \in \mathcal{G} \). Now if \( w \) has degree at least three, then let \( z \) be any vertex of \( N(w) \cap D \). Then by removing \( wz \), \( T_w \) has order at least seven and \( T_z \) has order at least three, \( D_w \) and \( D_z \) are two DTDS of \( T_w \) and \( T_z \), respectively. The rest of the proof is similar to as shown above and so \( |D| > 3(n + \ell - s + 1)/7 \). Thus every vertex of \( V - D \) is either a leaf or has degree two.

Let \( W \) be the set of vertices of \( V - D \) having degree two. Since \( T \) is a tree, \( |W| = k - 1 \) where \( k \) is the number of connected components of \( \langle D \rangle \). Let \( T' \) be the forest induced by the vertices of \( V(T) - W \) and let \( T_1, T_2, \ldots, T_k \) the components of \( T' \). Then \( n_1 + \cdots + n_k = n - |W|, \ell_1 + \cdots + \ell_k \geq \ell + q, \) and \( s_1 + \cdots + s_k = s + q \), where \( q \) is the number of new support vertices. Also \( D \cap V(T_i) = D_i \) is a DTDS of \( T_i \), for every \( i = 1, \ldots, k \). By induction on each \( T_i \), we obtain

\[
|D| = \sum_{i=1}^{k} |D_k| \geq \sum_{i=1}^{k} 3(n_i + \ell_i - s_i + 1)/7 \geq 3(n - |W| + \ell - s + k)/7
\]

\[
= 3(n + \ell - s + 1)/7.
\]
Assume now that $\gamma^D_i(T) = 3(n + \ell - s + 1)/7$. Then we have equality throughout this inequality chain. In particular, $\gamma^D_i(T_i) = 3(n_i + \ell_i - s_i + 1)/7$ for each $i$, and $\ell_1 + \cdots + \ell_k = \ell + q$, and $s_1 + \cdots + s_k = s + q$. Thus $T'$ contains a new leaf if and only if it has a new support vertex. So each $T_i$ has order at least four. Recall that each component of $\langle D \rangle$ has diameter two, three or four. We first assume that the subgraph $\langle D_i \rangle$ has diameter three or four. We will show that no leaf of $T_i$ is contained in $\langle D_i \rangle$. Assume to the contrary that a leaf $y \in V(T_i) \cap D_i$ and let $z \in D_i$ be its support vertex. Note that $y$ may be a new leaf in $T_i$. Consider the tree $T'_i = T_i - \{y\}$. Then $\langle D_i - \{y\} \rangle$ has diameter at least two and $D_i - \{y\}$ is a DTDS of $T'_i$, with $n'_i = n_i - 1, \ell'_i \geq \ell_i - 1$, and $s'_i \leq s_i$. It follows that $|D_i - \{y\}| \geq \gamma^D_i(T'_i) \geq 3(n'_i + \ell'_i - s'_i + 1)/7$ and so $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction since $|D_i| = 3(n_i + \ell_i - s_i + 1)/7$. Thus $\langle D_i \rangle$ contains no leaf of $T_i$ and hence every support vertex of $T_i$ is adjacent to exactly one leaf. Now let $k_1$ be the number of support vertices of $T_i$. Thus $T_i$ has $k_1$ leaves. Let $k_2 = n_i - 2k_1$. Clearly $k_1 + k_2 \geq 4$ since $\langle D_i \rangle$ is a component of diameter three or four, but then $|D_i| = k_1 + k_2 > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction.

Thus for each $i = 1, \ldots, k$, the subgraph $\langle D_i \rangle$ has diameter two, and so $\langle D_i \rangle$ is a star of center vertex $x$ and leaves $y_1, y_2, \ldots, y_t$ with $t \geq 2$. Note that $|D_i| = t + 1$. If $x$ is not a support vertex neither in $T$ nor in $T_i$, then each $y_i$ is support vertex of $T_i$. Hence $T_i$ is a subdivided star with $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$, a contradiction. If $x$ is not a support vertex of $T$ but it is a vertex support of $T_i$, then $T_i$ is a corona of $K_{1,t-1}$, where $|D_i| > 3(n_i + \ell_i - s_i + 1)/7$. Now if $x$ is a support vertex of $T$ with at most three leaves, then every $y_j$ is a support vertex in $T_i$ for either $1 \leq j \leq t$, $2 \leq j \leq t$, or $3 \leq j \leq t$, but then $\gamma^L_i(T_i) = 3(n_i + \ell_i - s_i + 1)/7$ if and only if $T_i = K_{1,3}, S_{2,1}$ or $T_i$ is a corona of a path $P_3$. Thus every component of $T'$ is either a path $K_{1,3}, S_{2,1}$ or corona of $P_3$ where every leaf of $W$ joins two support vertices. Therefore extremal trees $T$ achieving $\gamma^L_i(T) = 3(n + \ell - s + 1)/7$ are precisely those of $G$.

The converse is easy to show.

Acknowledgment

I would like to thank the referees for their remarks and suggestions that helped improve the manuscript.
References


Received 20 April 2006
Revised 14 March 2008
Accepted 9 May 2008