PARTITIONS OF A GRAPH INTO CYCLES CONTAINING A SPECIFIED LINEAR FOREST

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Abstract

In this note, we consider the partition of a graph into cycles containing a specified linear forest. Minimum degree and degree sum conditions are given, which are best possible.

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1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. We will generally follow notation and terminology of [2]. For a vertex $x$ of a graph $G$, the neighborhood of $x$ is denoted by $N_G(x)$ and $d_G(x) = |N_G(x)|$ is the degree of $x$ in $G$. For a subgraph $H$ of $G$ and

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a vertex \( x \in V(G) - V(H) \), we also denote \( N_H(x) = N_G(x) \cap V(H) \) and \( d_H(x) = |N_H(x)| \). For a subset \( S \) of \( V(G) \), we write \( \langle S \rangle \) for the subgraph induced by \( S \). For a subgraph \( H \) of \( G \) and a subset \( S \) of \( V(G) \), \( d_H(S) = \sum_{x \in S} d_H(x) \), \( N_H(S) = \bigcup_{x \in S} N_H(x) \) and define \( G - H = \langle V(G) - V(H) \rangle \) and \( G - S = \langle V(G) - S \rangle \). For a graph \( G \), \(|G| = |V(G)|\) is the order of \( G \), \( \delta(G) \) is the minimum degree of \( G \), and 

\[
\sigma_2(G) = \min \{d_G(x) + d_G(y) \mid xy \notin E(G), x, y \in V(G), x \neq y\}
\]

is the minimum degree sum of nonadjacent vertices. (When \( G \) is complete, we define \( \sigma_2(G) = \infty \).)

A forest is a graph each of whose components is a tree and a linear forest is a forest consisting of paths. We regard a single vertex as a path of order 1. For a path \( P = v_1v_2 \cdots v_p \), we call \( v_i \) an internal vertex for \( 2 \leq i \leq p - 1 \). If \( P \) is contained in a cycle \( C \) as a subgraph, we denote it by \( P \subset C \).

For graphs \( G \) and \( H \), \( G \cup H \) is the union of \( G \) and \( H \), and \( G + H \) is the join of \( G \) and \( H \). \( K_n \) is a complete graph of order \( n \).

Suppose that \( H_1, \ldots, H_k \) are vertex-disjoint subgraphs such that \( V(G) = \bigcup_{i=1}^{k} V(H_i) \). Then we say \( G \) can be partitioned into \( H_1, \ldots, H_k \) and \( \{H_1, \ldots, H_k\} \) is a partition of \( G \).

Research on partitions of a graph into cycles with a specified number of components was started by Brandt et al.

**Theorem 1** (Brandt et al. [1]). Suppose that \(|G| \geq 4k\) and \( \sigma_2(G) \geq |G| \). Then \( G \) can be partitioned into \( k \) cycles.

In this paper, we consider partitions into cycles each of which contains exactly one component of a specified linear forest as a subgraph. In the following, \( n \) always denotes the order of a graph \( G \), and ‘disjoint’ means ‘vertex-disjoint’ because we only deal with partitions of the vertex set.

The special cases where each component of a specified linear forest is a vertex or an edge were considered in several papers [3–11]. In particular, the following theorem was obtained in [7].

**Theorem 2** (Enomoto and Matsumura [7]). Suppose that \( n \geq 10p + 10q \), \( p + q \geq 1 \) and either

\[
\delta(G) \geq \max \Bigg\{ \frac{n + q}{2}, \frac{n + p + 2q - 3}{2} \Bigg\},
\]

...
or

\[ \sigma_2(G) \geq \max\{n + q, n + 2p + 2q - 2\}. \]

Then for any linear forest with components \( P_1, \ldots, P_{p+q} \) such that \( |P_i| = 1 \) for \( 1 \leq i \leq p \) and \( |P_i| = 2 \) for \( p + 1 \leq i \leq p + q \), \( G \) can be partitioned into cycles \( H_1, \ldots, H_{p+q} \) such that \( P_i \subset H_i \).

In this paper, we consider a more general case, that is, we specify not only vertices and edges but also paths of order at least 3. The main result of this paper is the following.

**Theorem 3.** Suppose that \( n \geq 10p + 10q' \), \( p + q \geq 1 \), \( p \geq 0 \), \( q' \geq q \geq 0 \), and either

\[ \delta(G) \geq \max\left\{ \frac{n + q'}{2}, \frac{n + p + q + q' - 3}{2} \right\}, \]

or

\[ \sigma_2(G) \geq \max\{n + q', n + 2p + q + q' - 2\}. \]

Then for any linear forest with components \( P_1, \ldots, P_{p+q} \) such that \( |P_i| = 1 \) for \( 1 \leq i \leq p \), \( |P_i| \geq 2 \) for \( p + 1 \leq i \leq p + q \) and \( \sum_{i=p+1}^{p+q} |E(P_i)| = q' \), \( G \) can be partitioned into cycles \( H_1, \ldots, H_{p+q} \) such that \( P_i \subset H_i \).

The minimum degree condition in Theorem 3 is sharp in the following sense. (In the following five examples, we let \( m \) be a sufficiently large integer.)

**Example 1.** Suppose that \( q' \geq q \geq 1 \) and \( p + q \geq 2 \). Let \( G_1 = (K^1_m \cup K^2_m) + K_{p+q+q'-2} \), where \( K^i_m \) is a complete graph of order \( m \) for \( i = 1, 2 \). Take \( p \) distinct vertices \( P_1, \ldots, P_p \) and \( q - 1 \) disjoint paths \( P_{p+1}, \ldots, P_{p+q-1} \) in \( K_{p+q+q'-2} \) such that \( |E(P_i)| \geq 1 \) and \( \sum_{i=p+1}^{p+q-1} |E(P_i)| = q_0 < q' \). Moreover, we take a path \( P_{p+q} \) which connects \( K^1_m \) and \( K^2_m \), \( |E(P_{p+q})| = q' - q_0 \) and all internal vertices are contained in \( K_{p+q+q'-2} \). (If \( q' - q_0 = 1 \), we add an edge \( e \) which connects \( K^1_m \) and \( K^2_m \) directly and let \( P_{p+q} = e \).) Then we cannot take a cycle passing through \( P_{p+q} \) without using vertices in \( \bigcup_{i=1}^{p+q-1} V(P_i) \). Hence \( G_1 \) cannot have the desired partition, while \( \delta(G_1) = (|G_1| + p + q + q' - 4)/2 \).
Example 2. Suppose that \( q = 0 \) and let \( G_2 = K_{m,m+1} \), a complete bipartite graph with partite sets of order \( m \) and \( m + 1 \). Clearly, \( G_2 \) cannot have the desired partition, while \( \delta(G) = (|G_2| - 1)/2 \).

Example 3. Suppose that \( p = 0 \) and \( q' \geq q \geq 1 \) and let \( G_3 = K_{m+q'} + (m+1)K_1 \). Take \( q \) disjoint paths \( P_1, \ldots, P_q \) in \( K_{m+q'} \) so that \( |E(P_i)| \geq 1 \) and \( \sum_{i=1}^{q} |E(P_i)| = q' \). Then \( G_3 \) does not have the desired partition, while \( \delta(G_3) = (|G_3| + q' - 1)/2 \).

Example 4. Suppose that \( p \geq 1 \). Let \( G_4 = (K_p \cup K_m) + K_{2p+q'+q'-1} \). Take \( p \) distinct vertices \( P_1, \ldots, P_p \) in \( K_p \) and \( q \) disjoint paths \( P_{p+1}, \ldots, P_{p+q} \) in \( K_{2p+q'+q'-1} \) so that \( \sum_{i=p+1}^{p+q} |E(P_i)| = q' \). To make a cycle through \( P_i \) for
1 ≤ i ≤ p, we have to use at least 2 vertices in $K_{2p+q+q'-1}$ but only $2p - 1$ vertices are available. Then $G_4$ cannot have the desired partition, while $\sigma_2(G_4) = |G_4| + 2p + q + q' - 3$.

![Figure 3. The graph $G_4$.](image)

**Example 5.** Suppose that $p = 0$ and let $G_5 = (K_1 \cup K_m) + K_{q+q'-1}$. Take $q - 1$ disjoint paths $P_1, \ldots, P_{q-1}$ in $K_{q+q'-1}$ so that $\sum_{i=1}^{q-1} |E(P_i)| = q' - 1$ and an edge $P_q$ connecting $K_1$ and $K_{q+q'-1}$. Then we cannot take a cycle through $P_q$ without using the vertices of other specified paths. Hence $G_5$ cannot be partitioned into cycles $H_1, \ldots, H_{p+q}$ such that $P_i \subset H_i$, while $\sigma_2(G_5) = |G_5| + q + q' - 3$.

![Figure 4. The graph $G_5$.](image)

The graphs $G_2$ and $G_3$ show that the condition ‘$\sigma_2(G) \geq n + q''$ cannot be dropped because $\sigma_2(G_2) = |G_2| - 1$ and $\sigma_2(G_3) = |G_3| + q' - 1$.

For the case where each component of a specified linear forest is a path of order at least 3, the degree sum condition of Theorem 3 is not sharp and we prove the following.
Theorem 4. Suppose that \( n \geq 3q + q', q \geq 1, q' \geq 2q \) and
\[
\sigma_2(G) \geq \max\{n + q', n + q + q' - 3\}.
\]
Then for any disjoint paths of order at least 3 \( P_1, \ldots, P_q \) such that \( \sum_{i=1}^{q} |E(P_i)| = q' \), \( G \) can be partitioned into cycles \( H_1, \ldots, H_q \) such that \( P_i \subset H_i \).

The graph \( G_1 \) shows the sharpness of the degree sum condition in Theorem 4, because \( \sigma_2(G_1) = |G_1| + p + q + q' - 4 \).

To prove Theorem 4, we prove the following theorem, which deals with the case where all paths are of order 3.

Theorem 5. Suppose that \( n \geq 5q, q \geq 1 \) and
\[
\sigma_2(G) \geq \max\{n + 2q, n + 3q - 3\}.
\]
Then for any disjoint paths of order 3 \( P_1, \ldots, P_q \), \( G \) can be partitioned into cycles \( H_1, \ldots, H_q \) such that \( P_i \subset H_i \).

We can prove Theorems 3 and 4 similarly. The proof of Theorem 3 is given in the next section. Before proving Theorem 4, we will give a proof of Theorem 5 in Section 3. We will prove Theorem 4 in Section 4.

2. Proof of Theorem 3

Let \( \{p_i\} = V(P_i) \) for \( 1 \leq i \leq p \) and \( x_i \) and \( y_i \) be endvertices of \( P_i \) for \( p + 1 \leq i \leq p + q \).

We generate a new graph \( G' \) from \( G \) by deleting all internal vertices of \( P_i \) and adding the edge \( x_iy_i \) if \( x_iy_i \notin E(G) \) for \( p + 1 \leq i \leq p + q \). Then
\[
\delta(G') \geq \max\left\{ \frac{n + q'}{2}, \frac{n + p + q + q' - 3}{2} \right\} - (q' - q)
\]
\[
= \max\left\{ \frac{(n - q' + q) + q}{2}, \frac{(n - q' + q) + p + 2q - 3}{2} \right\}
\]
\[
= \max\left\{ \frac{|G'| + q}{2}, \frac{|G'| + p + 2q - 3}{2} \right\}.
\]
and
\[
\sigma_2(G') \geq \max\{n + q', n + 2p + q + q' - 2\} - 2(q' - q) \\
= \max\{(n - q' + q) + q, (n - q' + q) + 2p + 2q - 2\} \\
= \max\{|G'| + q, |G'| + 2p + 2q - 2\}.
\]

Moreover, \(|G'| \geq 10p + 10q' - (q' - q) = 10p + 9q' + q \geq 10p + 10q\). Hence by Theorem 2, \(G'\) can be partitioned into cycles \(H_1', \ldots, H_{p+q}'\) such that \(p_i \in V(H_i')\) for \(1 \leq i \leq p\) and \(x_i, y_i \in E(H_i')\) for \(p + 1 \leq i \leq p + q\).

If we replace \(x_i, y_i\) by \(P_i\), then we get a cycle \(H_i\) from \(H_i'\) for \(p + 1 \leq i \leq p + q\) and \(\{H_1, \ldots, H_{p+q}\}\) is the desired partition of \(G\).

3. Proof of Theorem 5

To prove Theorem 5, we first prove the following theorem.

**Theorem 6.** Suppose that \(n \geq 5q, q \geq 1\) and \(\sigma_2(G) \geq n + 3q - 3\). Then for any disjoint paths of order 3 \(P_1, \ldots, P_q\), \(G\) contains \(q\) disjoint cycles \(C_1, \ldots, C_q\) such that \(P_i \subset C_i\) and \(|C_i| \leq 5\).

To complete the proof of Theorem 5, we use the following theorem.

**Theorem 7** (Egawa et al. [4]). Suppose that \(q \geq 1\), \(\sigma_2(G) \geq n + q\) and \(C_1, \ldots, C_q\) are disjoint subgraphs such that \(C_i\) is a cycle or \(K_2\) and \(e_i \in E(C_i)\) for \(1 \leq i \leq q\). Then there exist disjoint subgraphs \(H_1, \ldots, H_q\) such that \(V(G) = \bigcup_{i=1}^{q} V(H_i), e_i \in E(H_i)\) and \(H_i\) is a cycle if \(C_i\) is a cycle and \(H_i\) is a cycle or \(K_2\) if \(C_i\) is \(K_2\) for \(1 \leq i \leq q\).

3.1. Proof of Theorem 6

A cycle \(C\) is called *admissible* if \(P_i \subset C\) for some \(i, 1 \leq i \leq q\), \(|V(C) \cap \bigcup_{i=1}^{q} V(P_i)| = 3\) and \(|C| \leq 5\). For \(1 \leq r \leq q\), a set of cycles \(\{C_1, \ldots, C_r\}\) is *admissible* if each \(C_i\) is admissible, and \(C_i\) and \(C_j\) are disjoint if \(i \neq j\).

If we say ‘\(r\) admissible cycles’, then it means that the set of these \(r\) cycles is admissible. A set of admissible cycles \(\{C_1, \ldots, C_r\}\) is *minimal* if there exist no \(r\) admissible cycles \(P_1, \ldots, P_r\) such that \(|\bigcup_{i=1}^{r} V(P_i)| < |\bigcup_{i=1}^{r} V(C_i)|\).

Let \(G\) be an edge-maximal counterexample and \(P_i = x_i, y_i, z_i\) for \(1 \leq i \leq q\). Clearly, \(G\) is not complete. Let \(x\) and \(y\) be nonadjacent vertices of \(G\) and
define $G' = G + xy$, the graph obtained from $G$ by adding the edge $xy$.

Then $G'$ is no longer a counterexample and $G'$ has $q$ admissible cycles.

Since $G$ is a counterexample, the edge $xy$ is contained in some admissible cycle. This implies that $G$ contains $q - 1$ admissible cycles and we take minimal admissible cycles $C_1, \ldots, C_{q-1}$. Without loss of generality, we may assume that $P_i \subset C_i$ for $1 \leq i \leq q - 1$. Let $L = (\bigcup_{i=1}^{q-1} V(C_i))$, $M = G - L$ and $D = M - V(P_q)$. Note that $x_q z_q \notin E(G)$ and $N_D(x_q) \cap N_D(z_q) = \emptyset$. If possible, we take $C_1, \ldots, C_{q-1}$ so that $d_D(x_q) > 0$ and $d_D(z_q) > 0$.

**Claim 1.** We have $d_D(x_q) > 0$ and $d_D(z_q) > 0$.

**Proof.** We first remark that we can take $C_1, \ldots, C_{q-1}$ so that $d_D(x_q) > 0$. To see this, suppose that $d_D(x_q) = 0$ and take any $y \in V(D)$. Since

$$d_M(x_q) + d_M(y) \leq 1 + |M| - 2 = |M| - 1,$$

we have

$$d_L(x_q) + d_L(y) \geq n + 3q - 3 - (|M| - 1) = |L| + 3q - 2$$

$$= \sum_{i=1}^{q-1} |C_i| + 3q - 2 > \sum_{i=1}^{q-1} (|C_i| + 3).$$

Hence

$$d_{C_i}(x_q) + d_{C_i}(y) \geq |C_i| + 4$$

holds for some $i$, $1 \leq i \leq q - 1$.

If $|C_i| = 3$, then this inequality cannot hold. Hence $|C_i| \geq 4$. Without loss of generality, we may assume that $i = 1$.

Suppose that $|C_1| = 4$ and let $C_1 = x_1 y_1 z_1 v x_1$. Note that $N_{C_1}(x_q) = N_{C_1}(y) = V(C_1)$. If we take $D_1 = x_1 y_1 z_1 y x_1$ and let $D_i = C_i$ for $2 \leq i \leq q - 1$, then $\{D_1, \ldots, D_{q-1}\}$ is also minimal admissible and $x_q$ can have a neighbor in $G - \bigcup_{i=1}^{q-1} V(D_i)$ because $x_q v \in E(G)$.

Next suppose that $|C_1| = 5$ and let $C_1 = x_1 y_1 z_1 v x_1$. If $\{x_1, z_1\} \subset N_{C_1}(y)$, then we can find a shorter admissible cycle passing through $P_1$. Hence we have $d_{C_1}(y) = 4$. By symmetry, we may assume that $N_{C_1}(y) = \{y_1, z_1, v, u\}$. Then $N_{C_1}(x_q) = V(C_1)$. If we take $D_1 = z_1 y_1 x_1 u y z_1$ and let $D_i = C_i$ for $2 \leq i \leq q - 1$, then $\{D_1, \ldots, D_{q-1}\}$ is minimal admissible and
can have a neighbor in $G - \bigcup_{i=1}^{q-1} V(D_i)$ because $x_qv \in E(G)$. Hence we may assume that $d_D(x_q) > 0$.

Now suppose that the claim is false. In view of the remark made at the beginning of the proof, we may assume that $d_D(x_q) > 0$ and $d_D(z_q) = 0$. Take $z \in N_D(x_q)$ and $y \in V(D) - \{z\}$. Arguing as above, we see that there exists $j$ such that $d_{C_j}(z_q) + d_{C_j}(y) \geq |C_j| + 4$ and we can take admissible cycles $D_1, \ldots, D_{q-1}$ so that $\{D_1, \ldots, D_{q-1}\}$ is minimal admissible and $z_q$ can have a neighbor in $G - \bigcup_{i=1}^{q-1} V(D_i)$. But this contradicts the choice of $C_1, \ldots, C_{q-1}$ mentioned immediately before the statement of Claim 1.

Take any $z \in N_D(x_q)$ and $w \in N_D(z_q)$. Note that $\{zw, xqw, zqw\} \cap E(G) = \emptyset$, $N_D(x_q) \cap N_D(w) = \emptyset$, and $N_D(z_q) \cap N_D(z) = \emptyset$. (It may occur $\{yqz, yqw\} \cap E(G) \neq \emptyset$.)

Let $S = \{x_q, z_q, z, w\}$. Since

$$d_M(S) \leq 8 + 2(|M| - 5) = 2|M| - 2,$$

we have

$$d_L(S) \geq 2(n + 3q - 3) - (2|M| - 2) = 2|L| + 6q - 4$$

$$= \sum_{i=1}^{q-1} 2|C_i| + 6q - 4 > \sum_{i=1}^{q-1} (2|C_i| + 6).$$

This means that

$$d_{C_i}(S) \geq 2|C_i| + 7$$

for some $i$, $1 \leq i \leq q$.

If $|C_i| = 3$, then this inequality cannot hold. Hence $|C_i| \geq 4$.

Suppose that $|C_i| = 4$ and let $C_i = x_iy_i z_i z_i x_i$. By symmetry, we may assume that $N_{C_i}(x_q) = N_{C_i}(z) = V(C_i)$. Then $v \notin N_{C_i}(z_q) \cup N_{C_i}(w)$, because otherwise we can find two admissible cycles. But this means that $d_{C_i}(S) \leq 14$, a contradiction.

Next, suppose that $|C_i| = 5$ and let $C_i = x_i y_i z_i v v x_i$. If $d_{C_i}(z) = 5$, then we can find an admissible cycle $x_i y_i z_i z_i x_i$, which is shorter than $C_i$. Hence $d_{C_i}(z) \leq 4$. Similarly, $d_{C_i}(w) \leq 4$. If $(N_{C_i}(x_q) \cap N_{C_i}(z_q)) \cap \{v, w\} \neq \emptyset$, we can also find shorter admissible cycle passing through $P_q$. Hence $d_{C_i}(x_q) + d_{C_i}(z_q) \leq 8$. But this implies that $d_{C_i}(S) \leq 16$, a contradiction.

This completes the proof of Theorem 6.
3.2. Proof of Theorem 5

By Theorem 6, there exist disjoint cycles \( C_1, \ldots, C_q \) such that \( P_i \subseteq C_i \). Let \( P_i = x_iz_iy_i \) for \( 1 \leq i \leq q \).

We make \( G' \) from \( G \) by deleting \( \{y_1, \ldots, y_q\} \) and adding the edge \( x_iz_i \) for \( 1 \leq i \leq q \) if \( x_iz_i \notin E(G) \). Then we have disjoint subgraphs \( C'_1, \ldots, C'_q \) of \( G' \) such that \( x_iz_i \in E(C'_i) \), and \( C'_i \) is a cycle if \( |C_i| \geq 4 \), and \( C'_i \) is \( K_2 \) if \( |C_i| = 3 \). Moreover,

\[
\sigma_2(G') \geq \max\{n + 3q - 3, n + 2q\} - 2q \\
= \max\{(n - q) + 2q - 3, (n - q) + q\} \\
= \max\{|G'| + 2q - 3, |G'| + q\} \geq |G'| + q.
\]

Hence by Theorem 7, there exist disjoint subgraphs \( H'_1, \ldots, H'_q \) satisfying \( V(G') = \bigcup_{i=1}^q V(H'_i) \), \( x_iz_i \in E(H'_i) \) for \( 1 \leq i \leq q \) and \( H'_i \) is a cycle if \( C'_i \) is a cycle and \( H'_i \) is a cycle or \( K_2 \) if \( C'_i \) is \( K_2 \).

By replacing the edge \( x_iz_i \) by \( P_i \), we make a cycle \( H_i \) from \( H'_i \) for \( 1 \leq i \leq q \). Then \( \{H_1, \ldots, H_k\} \) is the desired partition of \( G \).

This completes the proof of Theorem 5.

4. Proof of Theorem 4

Let \( P_i = x_iz_iy_i \) for \( 1 \leq i \leq q \). We make \( G' \) from \( G \) by deleting all internal vertices except \( z_i \) of \( P_i \) and adding the edge \( z_iz_iy_i \) if \( z_iz_iy_i \notin E(G) \) for \( 1 \leq i \leq q \). Then

\[
\sigma_2(G) \geq \max\{n + q', n + q + q' - 3\} - 2(q' - 2q) \\
\geq \max\{(n - q') + 2q, (n - q') + 2q + 3q - 3\} \\
\geq \max\{|G'| + 2q, |G'| + 3q - 3\}.
\]

Moreover, \( |G'| \geq 3q + q' - (q' - 2q) = 5q \). Hence by Theorem 5, \( G' \) can be partitioned into cycles \( H'_1, \ldots, H'_q \) such that \( P'_i \subseteq H'_i \) for \( 1 \leq i \leq q \), where \( P'_i = x_iz_iz_iy_i \).

We replace \( P'_i \) by \( P_i \) and get a cycle \( H_i \) from \( H'_i \) for \( 1 \leq i \leq q \). Then \( \{H_1, \ldots, H_k\} \) is the desired partition of \( G \).

This completes the proof of Theorem 4.
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