ON \((k,l)\)-KERNELS IN \(D\)-JOIN OF DIGRAPHS

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Abstract

In [5] the necessary and sufficient conditions for the existence of \((k,l)\)-kernels in a \(D\)-join of digraphs were given if the digraph \(D\) is without circuits of length less than \(k\). In this paper we generalize these results for an arbitrary digraph \(D\). Moreover, we give the total number of \((k,l)\)-kernels, \(k\)-independent sets and \(l\)-dominating sets in a \(D\)-join of digraphs.

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1. Introduction

For concepts not defined here see [2]. Let \(D\) be a finite, directed graph (for short: a digraph) without loops and multiple arcs, where \(V(D)\) is the set of vertices and \(A(D)\) is the set of arcs of \(D\). By a path from a vertex \(x_1\) to a vertex \(x_n\), \(n \geq 2\), we mean a sequence of vertices \(x_1, \ldots, x_n\) and arcs \((x_i, x_{i+1}) \in A(D)\) for \(i = 1, 2, \ldots, n-1\) and for simplicity we denote it by \(x_1 \ldots x_n\). A circuit is a path with \(x_1 = x_n\). By \(d_D(x_i, x_j)\) we denote the length of the shortest path from \(x_i\) to \(x_j\) in \(D\). If there does not exist a path from \(x_i\) to \(x_j\) in \(D\), then we put \(d_D(x_i, x_j) = \infty\). For any \(X \subseteq V(D)\) and \(x \in (V(D) \setminus X)\) we put \(d_D(x, X) = \min_{y \in X} d_D(x, y)\). By \(C_{\mu}^n_{\alpha} \leq d \leq \mu(x_i)\)
we denote the family of all circuits in $D$ containing the vertex $x_i$ of length $d$, where $\eta \leq d \leq \mu$.

We say that a subset $J \subset V(D)$ is a $(k, l)$-kernel of $D$ if

1. for each $x_i, x_j \in J$ and $i \neq j$, $d_D(x_i, x_j) \geq k$ and
2. for each $x_i \notin J$ there exists $x_j \in J$ such that $d_D(x_i, x_j) \leq l$.

If the set $J$ satisfies the condition in (1) or in (2), then we shall call it a $k$-independent set of $D$ (also called a $k$-stable set of $D$) or an $l$-dominating set of $D$, respectively. We notice that a 2-independent set is an independent set and a 1-dominating set is a dominating set of $D$. In addition, we assume that a subset containing only one vertex and an empty set is also meant as a $k$-independent set. The set $V(D)$ is an $l$-dominating set of $D$. If an $l$-dominating set of $D$ has exactly one vertex, then this vertex we shall call an $l$-dominating vertex of $D$. Moreover, the $l$-dominating vertex of $D$ is also a $(k, l)$-kernel of $D$ for every $k \geq 2$. A digraph $D$ whose every induced subdigrah has a $(k, l)$-kernel is called a $(k, l)$-kernel perfect digraph.

Sufficient conditions for the existence of kernels and $(k, l)$-kernels in digraphs have been investigated, for instance in [1, 3, 4, 5]. By $NkI(D)$, $NlD(D)$ and $NklK(D)$ we mean the number of all $k$-independent sets, $l$-dominating sets and $(k, l)$-kernels of the digraph $D$, respectively. Moreover, by $Nld(D)$ we will denote the number of all $l$-dominating vertices of $D$. The total number of $k$-independent sets and $(k, l)$-kernels in graphs and in some their products were studied in [6] and [8].

Let $D$ be a digraph with $V(D) = \{x_1, \ldots, x_n\}$, $n \geq 2$ and $\alpha = (D_i)_{i \in \{1, \ldots, n\}}$ be a sequence of vertex disjoint digraphs on $V(D_i) = \{y^i_1, \ldots, y^i_{p_i}\}$, $p_i \geq 1$, $i = 1, \ldots, n$. The $D$-join of the digraph $D$ and the sequence $\alpha$ is a digraph $\sigma(\alpha, D)$ such that $V(\sigma(\alpha, D)) = \bigcup_{i=1}^{n}\{\{x_i\} \times V(D_i)\}$ and $A(\sigma(\alpha, D)) = \{((x_s, y^i_s), (x_q, y^i_q)) : x_s = x_q \text{ and } (y^i_s, y^i_q) \in A(D_i) \text{ or } (x_s, x_q) \in A(D))\}$. By $D^c_i$ we mean a copy of the digraph $D_i$ in $\sigma(\alpha, D)$.

It may be noted that if all digraphs from the sequence $\alpha$ have the same vertex set, then from the $D$-join we obtain the generalized lexicographic product of the digraph $D$ and the sequence of the digraphs $D_i$, i.e., $\sigma(\alpha, D) = D[D_1, \ldots, D_n]$. If all digraphs from the sequence $\alpha$ are isomorphic to the same digraph $H$, then from the $D$-join we obtain the composition $D[H]$ of the digraphs $D$ and $H$.

The existence of $(k, l)$-kernels in the lexicographic product $D[D_1, \ldots, D_n]$ was studied in [7]. Moreover, in [8] the total number of $k$-independent sets of a lexicographic product of graphs were determined using the concept of
the Fibonacci polynomial of graphs. In [5] the necessary and sufficient conditions for the existence of \((k,l)\)-kernels in \(D\)-join were given, where \(D\) is a digraph without circuits of length less than \(k\). It was proved:

**Theorem 1** [5]. Let \(D\) be a digraph without circuits of length less than \(k\). A subset \(S^* \subseteq V(\sigma(\alpha, D))\) is a \(k\)-independent set of \(\sigma(\alpha, D)\) if and only if there exists a \(k\)-independent set \(S \subseteq V(D)\) such that \(S^* = \bigcup_{i \in I} S_i\), where \(I = \{i : x_i \in S\}\), \(S_i \subseteq V(D_i)\) and \(S_i\) is a \(k\)-independent set of \(D_i\) for every \(i \in I\).

**Theorem 2** [5]. Let \(Q \subseteq V(D), I = \{i : x_i \in Q\}\) and \(Q_i \subseteq V(D_i)\). If \(Q\) is an \(l\)-dominating set of \(D\) and \(Q_i\) is an \(l\)-dominating set of \(D_i\) for every \(i \in I\), then \(Q^* = \bigcup_{i \in I} Q_i\) is an \(l\)-dominating set of \(\sigma(\alpha, D)\).

**Theorem 3** [5]. Let \(k \geq 2, l \leq k - 1\) be integers. Let \(D\) be a digraph without circuits of length less than \(k\). The subset \(J^*\) is a \((k,l)\)-kernel of the \(\sigma(\alpha, D)\) if and only if there exists a \((k,l)\)-kernel \(J \subseteq V(D)\) of the digraph \(D\) such that \(J^* = \bigcup_{i \in I} J_i,\) where \(I = \{i : x_i \in J\}\), \(J_i \subseteq V(D_i)\) and \(J_i\) is a \((k,l)\)-kernel of \(D_i\) for every \(i \in I\).

In this paper, we generalize these results for an arbitrary digraph \(D\). Moreover, we determine the total number of \(k\)-independent sets, \(l\)-dominating sets and \((k,l)\)-kernels in \(\sigma(\alpha, D)\).

2. **THE EXISTENCE OF \((k,l)\)-KERNELS IN D-JOIN**

In this section, we give the necessary and sufficient conditions for the existence of \((k,l)\)-kernels in \(D\)-join if \(D\) is an arbitrary digraph on \(n, n \geq 2\) vertices and \(\alpha = (D_i)_{i \in \{1, \ldots, n\}}\) is an arbitrary sequence of vertex disjoint digraphs on \(p_i, p_i \geq 1\) vertices.

**Theorem 4.** Let \((x_i, y_i), (x_j, y_j) \in V(\sigma(\alpha, D))\). Then

\[
d_{\sigma(\alpha, D)}((x_i, y_i), (x_j, y_j)) = \begin{cases} d_D(x_i, x_j) & \text{for } i \neq j, \\ \min\{d_D(x_i, y_i), d_D(x_i)\} & \text{for } i = j, \end{cases}
\]

where \(d_D(x_i)\) denotes the length of the shortest circuit containing the vertex \(x_i\) in \(D\).
Proof. Assume that \((x_i, y_p^i), (x_j, y_q^j)\) are two different vertices of \(V(\sigma(\alpha, D))\) and distinguish two possible cases:

1. \(i \neq j\). Then the theorem follows immediately from the definition of \(\sigma(\alpha, D)\).

2. \(i = j\). Using the definition of \(\sigma(\alpha, D)\) we have that there exists a path from \((x_i, y_p^i)\) to \((x_i, y_q^j)\) in \(\sigma(\alpha, D)\) of the same length as the path from \(y_p\) to \(y_q\) in \(D_i\). Moreover, if there exists a circuit in \(D\) which includes a vertex \(x_i\), then by the definition of \(\sigma(\alpha, D)\) it follows that there also exists a path from \((x_i, y_p^i)\) to \((x_i, y_q^j)\) of length \(d_D(x_i)\) equal to the length of the shortest circuit in \(D\), which includes a vertex \(x_i\). Otherwise, if there does not exist a circuit in \(D\) which includes a vertex \(x_i\), then we put \(d_D(x_i) = \infty\). Evidently \(d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_i, y_q^j)) = \min\{d_D(y_p, y_q), d_D(x_i)\}\).

Thus the theorem is proved.

Theorem 5. A subset \(S^* \subset V(\sigma(\alpha, D))\) is a \(k\)-independent set of \(\sigma(\alpha, D)\) if and only if \(S \subset V(D)\) is a \(k\)-independent set of \(D\) such that \(S^* = \bigcup_{i \in I} S_i\), where \(I = \{i : x_i \in S\}, S_i \subseteq V(D_i^c)\) and for every \(i \in I\)

(a) \(S_i\) is a \(k\)-independent set of \(D_i^c\) if \(C_{D_i}^{d \leq k - 1}(x_i) = \emptyset\) or

(b) \(S_i\) is 1-element set containing an arbitrary vertex from \(V(D_i^c)\), otherwise.

Proof. 1. Let \(S^*\) be a \(k\)-independent set of the \(D\)-join \(\sigma(\alpha, D)\). Denote \(S = \{x_i \in V(D) : S^* \cap V(D_i^c) \neq \emptyset\}\). First, we shall prove that \(S\) is a \(k\)-independent set of \(D\). Let \(x_i, x_j \in S\) be two different vertices. Then by the definition of the set \(S\) there exist \(1 \leq r \leq p_i\) and \(1 \leq s \leq p_j\) such that \((x_i, y_r^i), (x_j, y_s^j) \in S^*\). By Theorem 4 and from the assumption of the set \(S^*\) we obtain that \(d_D(x_i, x_j) = d_{\sigma(\alpha, D)}((x_i, y_r^i), (x_j, y_s^j)) \geq k\). The definition of the set \(S\) implies that \(S^* = \bigcup_{i \in I} S_i\), where \(I = \{i : x_i \in S\}\). We consider the following cases.

I.1. Let \(C_{D_i}^{d \leq k - 1}(x_i) = \emptyset\).

Because \(S^*\) is \(k\)-independent so by the definition of \(\sigma(\alpha, D)\) and by assumption it follows immediately that \(S_i\) is a \(k\)-independent set of \(D_i^c\).

I.2. Let \(C_{D_i}^{d \leq k - 1}(x_i) \neq \emptyset\).

We shall prove that \(S_i\) contains exactly one arbitrary vertex from \(V(D_i^c)\). By Theorem 4 we obtain that for arbitrary two vertices from \(V(D_i^c)\) the distance between them in \(\sigma(\alpha, D)\) is less than \(k\). Consequently, \(S_i\) must contain exactly one arbitrary vertex from \(V(D_i^c)\).
Hence from the above cases we obtain that $S_i$ is a $k$-independent set of $D_i^c$ if there does not exist in $D$ a circuit containing $x_i$ of length less than $k$ or $S_i$ contains exactly one arbitrary vertex from $V(D_i^c)$, otherwise.

II. Let $S \subseteq V(D)$ be a $k$-independent set of the digraph $D$. Let $I = \{i : x_i \in S\}$ and let $S_i$ be as in the assumption. We shall prove that $S^* = \bigcup_{i \in I} S_i$ is a $k$-independent set of the $D$-join $\sigma(\alpha,D)$. Let $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ be two distinct vertices. Consider the possible cases:

II.1. $(x_i, y_p^i) \in S_i$ and $(x_j, y_q^j) \in S_j$, where $i \neq j$.

Since $S$ is $k$-independent in $D$, so by Theorem 4 it follows that $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) = d_D(x_i, x_j) \geq k$.

II.2. $(x_i, y_p^i), (x_j, y_q^j) \in S_i$, where $p \neq q$ for some $i \in I$.

Since $S_i$ contains at least two vertices, so by the assumption, $S_i$ is $k$-independent of $D_i^c$ and $c_D^{<k}(x_i) = \emptyset$. To prove that $S^*$ is a $k$-independent set of $\sigma(\alpha,D)$ assume on the contrary that $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) < k$. If $k = 2$, then a contradiction with the independence of $S_i$ in $D_i^c$. Let $k \geq 3$.

This means that there exists a path $(x_i, y_p^i) \ldots (x_j, y_q^j)$ in $\sigma(\alpha,D)$ of length less than $k$ such that at least one inner vertex of this path does not belong to $V(D_i^c)$. Hence there exists in $D$ a circuit containing the vertex $x_i$ of length less than $k$, a contradiction to the assumption.

Taking the two above cases into considerations we obtain that for distinct $(x_i, y_p^i), (x_j, y_q^j) \in S^*$ there holds $d_{\sigma(\alpha,D)}((x_i, y_p^i), (x_j, y_q^j)) \geq k$, hence $S^*$ is a $k$-independent set of $\sigma(\alpha,D)$.

Thus the theorem is proved.

If $D$ is a digraph without circuits of length less than $k$, then we obtain Theorem 1.

**Theorem 6.** A subset $Q^* \subseteq V(\sigma(\alpha,D))$ is an $l$-dominating set of $\sigma(\alpha,D)$ if and only if $Q \subseteq V(D)$ is an $l$-dominating set of $D$ such that $Q^* = \bigcup_{i \in I} Q_i$, where $I = \{i; x_i \in Q\}$, $Q_i \subseteq V(D_i^c)$ and for every $i \in I$

(a) $Q_i$ is an $l$-dominating set of $D_i^c$ if $c_D^{\leq l}(x_i) = \emptyset$ and for each $j \in I$ and $j \neq i$, there holds $d_D(x_i, x_j) > l$ or

(b) $Q_i$ is an arbitrary nonempty subset of $V(D_i^c)$, otherwise.

**Proof.** 1. Let $Q^*$ be an $l$-dominating set of the $D$-join $\sigma(\alpha,D)$. Denote $Q = \{x_i \in V(D); Q^* \cap V(D_i^c) \neq \emptyset\}$. First, we shall prove that $Q$ is an
l-dominating set of $D$. Let $x_j \notin Q$. By the definition of the set $Q$ we have that for each $1 \leq r \leq p_j$ there holds $(x_j, y^r_j) \notin Q^*$. Since $Q^*$ is $l$-dominating so there exists $(x_i, y^r_i) \in Q^*$, where $i \neq j$ such that $d_{\sigma(\alpha, D)}((x_j, y^r_j), (x_i, y^r_i)) \leq l$. Evidently, $x_i \in Q$, so using Theorem 4 we obtain that $d_D(x_j, x_i) = d_{\sigma(\alpha, D)}((x_j, y^r_j), (x_i, y^r_i)) \leq l$. The definition of the set $Q$ implies that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$, where $\mathcal{I} = \{i \mid x_i \in Q\}$. Consider the following cases:

I.1. Assume that $C^d_{\sigma(D)}(x_i) = \emptyset$ and for each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$.

Since $Q^*$ is $l$-dominating so from the definition of $\sigma(\alpha, D)$ and by our assumptions immediately follows that $Q_i$ is an $l$-dominating set of $D_i$.

I.2. Assume that case I.1 does not hold.

We shall prove that $Q_i$ is an arbitrary nonempty subset of $D_i^c$. If $C^d_{\sigma(D)}(x_i) \neq \emptyset$, then there exists in $D$ a circuit which includes the vertex $x_i$ of length less than or equal to $l$. So for arbitrary two vertices $(x_i, y^r_i), (x_i, y^s_i) \in V(D_i^c)$ there holds $d_{\sigma(\alpha, D)}((x_i, y^r_i), (x_i, y^s_i)) \leq l$. If there exists $j \in \mathcal{I}$ and $j \neq i$ such that there exists in $D$ a path $x_i \ldots x_j$ of length less than or equal to $l$, then for an arbitrary vertex $(x_i, y^r_i) \in V(D_i^c)$ holds $d_{\sigma(\alpha, D)}((x_i, y^r_i), Q_j) \leq l$. Hence $d_{\sigma(\alpha, D)}((x_i, y^r_i), Q^*) \leq l$. All this implies that $Q_i$ is an arbitrary nonempty subset of $V(D_i^c)$.

II. Let $Q \subseteq V(D)$ be an l-dominating set of the digraph $D$, where $\mathcal{I} = \{i : x_i \in Q\}$ and let $Q_i$ be as in the theorem. We shall prove that $Q^* = \bigcup_{i \in \mathcal{I}} Q_i$ is an $l$-dominating set of the $D$-join. We distinguish the following cases:

II.1. Let $(x_j, y^r_j) \notin Q^*$ and $j \notin \mathcal{I}$.

Then by the definition of the set $Q$ we have that $x_j \notin Q$. Since $Q$ is an $l$-dominating set of $D$, so there exists $i \in \mathcal{I}$ such that $x_i \in Q$ and $d_D(x_j, x_i) \leq l$. Hence there is $1 \leq q \leq p_i$ such that $(x_i, y^q_i) \in Q^*$. By Theorem 4 we obtain that $d_{\sigma(\alpha, D)}((x_j, y^r_j), Q^*) \leq l$.

II.2. Let $(x_j, y^r_j) \notin Q^*$ and $j \in \mathcal{I}$.

If $Q_j$ is an $l$-dominating set of $D_j^c$, then $d_{D_j^c}((x_j, y^r_j), Q_j) \leq l$. So $d_{\sigma(\alpha, D)}((x_j, y^r_j), Q^*) \leq l$. If $Q_j$ is a nonempty subset of $V(D_j^c)$, then from the assumption of the theorem we have that there exists $t \in \mathcal{I}$ and $t \neq j$ such that there exists a path $x_j \ldots x_t$ in $D$ of length less than or equal to $l$ or $C^d_{\sigma(D)}(x_j) \neq \emptyset$. Consequently, $d_{\sigma(\alpha, D)}((x_j, y^r_j), Q_j) \leq l$ or
\(d_{\sigma(\alpha,D)}((x_j, y^D_p), Q_j) \leq l\), respectively. Hence \(d_{\sigma(\alpha,D)}((x_j, y^D_p), Q^*) \leq l\), so \(Q^*\) is an \(l\)-dominating set of \(\sigma(\alpha,D)\).

Thus the theorem is proved.

\textbf{Theorem 7.} Let \(k \geq 2, 1 \leq l \leq k - 1\) be integers. The subset \(J^* \subset V(\sigma(\alpha,D))\) is a \((k,l)\)-kernel of the \(D\)-join \(\sigma(\alpha,D)\) if and only if there exists a \((k,l)\)-kernel \(J \subset V(D)\) such that \(J^* = \bigcup_{i \in I} J_i\), where \(I = \{i : x_i \in J\}\), \(J_i \subseteq V(D_i^c)\) and for every \(i \in I\)

(a) \(J_i\) is a \((k,l)\)-kernel of \(D_i^c\) if \(C_{D_i}^{d \leq k-1}(x_i) = \emptyset\) or

(b) \(J_i\) is 1-element set containing an arbitrary vertex of \(V(D_i^c)\) if \(C_{D_i}^{d \leq l}(x_i) \neq \emptyset\) or

(c) \(J_i\) is 1-element set containing an \(l\)-dominating set of \(D_i^c\), otherwise.

\textbf{Proof.} 1. Let \(k \geq 2, 1 \leq l \leq k - 1\) be integers. Let \(J^*\) be a \((k,l)\)-kernel of the \(D\)-join \(\sigma(\alpha,D)\). Denote \(J = \{x_i \in V(D); J^* \cap V(D_i^c) \neq \emptyset\}\). First, we shall prove that \(J\) is a \((k,l)\)-kernel of \(D\). Let \(x_i, x_j \in J\) and \(i \neq j\). Then from the definition of the set \(J\) we have that there exists 1 \(\leq p \leq p_i\) and \(1 \leq q \leq p_j\) such that \((x_i, y^D_p), (x_j, y^D_q) \in J^*\). By Theorem 4 we have that \(d_D(x_i, x_j) = d_{\sigma(\alpha,D)}((x_i, y^D_p), (x_j, y^D_q)) \geq k\). So, \(J\) is a \(k\)-independent set of \(D\). Now, we will show that \(J\) is an \(l\)-dominating set of \(D\). Let \(x_j \notin J\). Using the definition of the set \(J\) for each 1 \(\leq r \leq p_j\) holds \((x_j, y^D_r) \notin J^*\). Since \(J^*\) is \(l\)-dominating, hence there exists \((x_i, y^D_s) \in J^*\), where \(j \neq i\) such that \(d_{\sigma(\alpha,D)}((x_i, y^D_s), (x_i, y^D_j)) \leq l\).

From the definition of the set \(J\) we have that \(x_i \in J\), so by Theorem 4 there holds \(d_D(x_j, x_i) = d_{\sigma(\alpha,D)}((x_j, y^D_j), (x_i, y^D_s)) \leq l\). Consequently, \(J\) is an \(l\)-dominating set of \(D\), hence \(J\) is a \((k,l)\)-kernel of \(D\). The definition of the set \(J\) implies that \(J^* = \bigcup_{i \in I} J_i\), where \(I = \{i : x_i \in J\}\). Consider the possible cases:

I.1. Let \(C_{D_i}^{d \leq k-1}(x_i) = \emptyset\).

We shall prove that \(J_i\) is a \((k,l)\)-kernel of \(D_i^c\) in this case. From Theorem 5(a) we obtain that \(J_i\) is a \(k\)-independent set of \(D_i^c\). Next we shall show that \(J_i\) is \(l\)-dominating. Since \(J\) is a \(k\)-independent set of \(D\) and \(l \leq k - 1\), then for each \(j \in I\) and \(j \neq i\) there holds \(d_D(x_i, x_j) \geq k \geq l + 1\). So, there does not exist in \(D\) a path \(x_i, \ldots, x_j\) of length less than or equal to \(l\). Moreover, \(C_{D_i}^{d \leq k-1}(x_i) = \emptyset\) and \(l \leq k - 1\), hence \(C_{D_i}^{d \leq l}(x_i) = \emptyset\).

From the above and by Theorem 6(a) we obtain that \(J_i\) is an \(l\)-dominating set of \(D_i^c\). Consequently, \(J_i\) is a \((k,l)\)-kernel of \(D_i^c\) in this case.
I.2. Let \( C_D^{d \leq k-1}(x_i) \neq \emptyset \).

Then by Theorem 5(b) the set \( J_i \) contains exactly one arbitrary vertex from \( V(D_i^c) \). So \( J_i \) is a \( k \)-independent set of \( D_i^c \). Because \( l \leq k - 1 \), then for each \( j \in I \) and \( j \neq i \) there holds \( d_D(x_i, x_j) \geq k \geq l + 1 \). Hence there does not exist in \( D \) a path \( x_i \ldots x_j \) of length \( d_D(x_i, x_j) \leq l \). From the assumption there exists in \( D \) a circuit containing the vertex \( x_i \) of length less than \( k \). We distinguish the following possibilities:

I.2.1. \( C_D^{d \leq l}(x_i) \neq \emptyset \).

Then by Theorem 6(b) it follows immediately that \( J_i \) is a 1-element set containing an arbitrary vertex of \( V(D_i^c) \).

I.2.2. \( C_D^{d \leq l}(x_i) = \emptyset \) and \( C_D^{d \leq l+1}(x_i) \neq \emptyset \).

We will show that \( J_i \) is a 1-element set containing an \( l \)-dominating vertex of \( D_i^c \). Using Theorem 6(a) we obtain that \( J_i \) is an \( l \)-dominating set of \( D_i^c \). Because \( J_i \) contains exactly one vertex, so \( J_i = \{ (x_i, y_i^j) \} \), where \( (x_i, y_i^j) \) is an \( l \)-dominating vertex of \( D_i^c \).

II. Let \( J \subset V(D) \) be a \((k, l)\)-kernel of the digraph \( D \). Let \( I = \{ i : x_i \in J \} \) and \( J_i \) be as in the statements of the theorem. We shall prove that \( J^* = \bigcup_{i \in I} J_i \) is a \((k, l)\)-kernel of \( \sigma(\alpha, D) \). Firstly we will prove that \( J^* \) is a \( k \)-independent set of the \( D \)-join \( \sigma(\alpha, D) \). Let \( (x_i, y_i^p), (x_j, y_i^q) \in J^* \) be two different vertices. Consider the following cases:

II.1. \((x_i, y_i^p) \in J_i \) and \((x_i, y_i^j) \in J_j \), where \( i \neq j \).

Evidently, \( x_i, x_j \in J \) and because \( J \) is \( k \)-independent so by Theorem 4 we have that \( d_D(x_i, x_j) = d_{\sigma(\alpha, D)}((x_i, y_i^p), (x_j, y_i^q)) \geq k \).

II.2. \((x_i, y_i^p), (x_i, y_i^q) \in J_i \) for some \( i \in I \).

Since \( J_i \) contains at least two vertices, so by assumption \( J_i \) is a \((k, l)\)-kernel of \( D_i^c \). Hence \( d_D((x_i, y_i^p), (x_i, y_i^q)) \geq k \). Assume on the contrary that \( d_{\sigma(\alpha, D)}((x_i, y_i^p), (x_i, y_i^q)) < k \). If \( k = 2 \), then there is a contradiction with the independence of \( J_i \) in \( D_i^c \). Let \( k \geq 3 \). This means that there exists a path \( (x_i, y_i^p) \ldots (x_i, y_i^q) \in \sigma(\alpha, D) \) of length less than \( k \) such that at least one inner vertex of this path does not belong to \( V(D_i^c) \). Hence there exists in \( D \) a circuit containing the vertex \( x_i \) of length less than \( k \) and by Theorem 5(b) the set \( J_i \) contains exactly one vertex from \( V(D_i^c) \), a contradiction to \( (x_i, y_i^p), (x_i, y_i^q) \in J_i \).

Taking the two above cases into consideration we obtain that for distinct \( (x_i, y_i^p), (x_j, y_i^q) \in J^* \) there holds \( d_{\sigma(\alpha, D)}((x_i, y_i^p), (x_j, y_i^q)) \geq k \). Hence \( J^* \) is a \( k \)-independent set of \( \sigma(\alpha, D) \).
Now we shall prove that \( J^* = \bigcup_{i \in I} J_i \), where \( I = \{i : x_i \in J\} \), is an \( l \)-dominating set of the \( D \)-join \( \sigma(\alpha, D) \). Consider the possible cases:

II.3. Let \((x_j, y_j^p) \not\in J^* \) and \( j \not\in I \).

Then by the definition of the set \( J \) we have that \( x_j \not\in J \). Since \( J \) is \( l \)-dominating in \( D \), so there exists \( i \in I \) such that \( x_i \in J \) and \( d_D(x_j, x_i) \leq l \).

Consequently, there exists \( 1 \leq q \leq p_i \) such that \((x_i, y_q^p) \in J^* \) and by Theorem 4 we obtain that \( d_{\sigma(\alpha, D)}((x_j, y_q^p), (x_i, y_q^p)) \leq l \).

II.4. Let \((x_j, y_j^p) \not\in J^* \) and \( j \in I \).

If \( J_j \) is a \((k, l)\)-kernel of \( D_j^c \), then \( J_j \) is an \( l \)-dominating set of \( D_j^c \), so \( d_{D_j^c}((x_j, y_j^p), J_j) \leq l \). Hence \( d_{\sigma(\alpha, D)}((x_j, y_j^p), J^*) \leq l \). If \( J_j \) contains exactly one arbitrary vertex of \( V(D_j^c) \), then by assumption of the theorem there exists in \( D \) a circuit containing the vertex \( x_j \) of length less than or equal to \( l \). So there exists in \( \sigma(\alpha, D) \) a path from \((x_j, y_j^p)\) to \( J_j \) and \( d_{\sigma(\alpha, D)}((x_j, y_j^p), J_j) \leq l \).

Hence \( d_{\sigma(\alpha, D)}((x_j, y_j^p), J^*) \leq l \). If \( J_j \) is a \( l \)-dominating vertex of \( D_j^c \), then by the definition of the \( l \)-dominating vertex \( d_{D_j^c}((x_j, y_j^p), J_j) \leq l \). Hence \( d_{\sigma(\alpha, D)}((x_j, y_j^p), J^*) \leq l \).

Thus it follows that \( J^* \) is an \( l \)-dominating set of \( \sigma(\alpha, D) \).

Taking the above cases into consideration we obtain that \( J^* \) is a \((k, l)\)-kernel of \( \sigma(\alpha, D) \).

Thus the theorem is proved.

If the digraph \( D \) is without circuits of length less than \( k \), then we obtain Theorem 3.

**Theorem 8.** Let \( k \geq 2 \), \( l \geq k \) be integers. The subset \( J^* \subset V(\sigma(\alpha, D)) \) is a \((k, l)\)-kernel of the \( D \)-join \( \sigma(\alpha, D) \) if and only if there exists a \((k, l)\)-kernel \( J \subset V(D) \) such that \( J^* = \bigcup_{i \in I} J_i \), where \( I = \{i : x_i \in J\} \), \( J_i \subseteq V(D_i^c) \) and for every \( i \in I \)

\( (a) \) \( J_i \) is a \((k, l)\)-kernel of \( D_i^c \) if \( C_d^{\leq l}(x_i) = \emptyset \) and for each \( j \in I \) and \( j \neq i \) there holds \( d_D(x_i, x_j) > l \) or

\( (b) \) \( J_i \) is a 1-element set containing an arbitrary vertex from \( V(D_i^c) \) if \( C_d^{\leq k-1}(x_i) \neq \emptyset \) or

\( (c) \) \( J_i \) is an arbitrary nonempty \( k \)-independent set of \( D_i^c \), otherwise.

**Proof.** I. Let \( k \geq 2 \), \( l \geq k \) be integers. Let \( J^* \) be a \((k, l)\)-kernel of the \( D \)-join \( \sigma(\alpha, D) \). Denote \( J = \{x_i \in V(D) : J^* \cap V(D_i^c) \neq \emptyset\} \). Proving analogously as in Theorem 7 we obtain that \( J \) is a \((k, l)\)-kernel of the
digraph $D$. Of course, the definition of the set $J$ implies that $J^* = \bigcup_{i \in \mathcal{I}} J_i$, where $\mathcal{I} = \{i : x_i \in J\}$. Consider the following cases:

I. Let $C_d^{d \leq l}(x_i) = \emptyset$.
Since $l \geq k$, so there does not exist in $D$ a circuit containing the vertex $x_i$ of length less than $k$. Then from Theorem 5(a) we obtain that $J_i$ is a $k$-independent set of $D_i^c$. By our assumption $l \geq k$, so to establish sets $J_i$ we consider the following possibilities:

I.1. There exists $j \in \mathcal{I}$ and $j \neq i$ such that $d_D(x_i, x_j) \leq l$.
By Theorem 6 (b) an arbitrary nonempty subset of $V(D_i^c)$ is $l$-dominating in $D_i^c$, so $J_i$ is an arbitrary $k$-independent set of $D_i^c$.

I.1.1. For each $j \in \mathcal{I}$ and $j \neq i$ there holds $d_D(x_i, x_j) > l$.
Then by Theorem 6(a) we obtain that $J_i$ is an $l$-dominating set of $D_i^c$. Consequently, $J_i$ is a $(k, l)$-kernel in this case.

I.2. Let $C_d^{d \leq l}(x_i) \neq \emptyset$.
Because $l \geq k$, we consider the following possibilities:

I.2.1. $C_d^{k \leq d \leq l}(x_i) \neq \emptyset$ and $C_d^{d \leq k-1}(x_i) = \emptyset$.
Then by Theorem 5(a) and Theorem 6(b) we obtain that the set $J_i$ is an arbitrary $k$-independent set of $V(D_i^c)$.

I.2.2. $C_d^{d \leq k-1}(x_i) \neq \emptyset$.
We shall prove that $J_i$ is a 1-element set containing an arbitrary vertex from $V(D_i^c)$. By Theorem 5(b) the set $J_i$ contains exactly one vertex from $V(D_i^c)$. Because $l \geq k$, so there exists in $D$ a circuit containing the vertex $x_i$ of length less than or equal to $l$. Hence by Theorem 6(b) we obtain that $J_i$ contains exactly one arbitrary vertex from $V(D_i^c)$.

II. Let $J \subset V(D)$ be a $(k, l)$-kernel of the digraph $D$ and let $\mathcal{I} = \{i : x_i \in J\}$. Proving analogously as in Theorem 7 we can show that $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a $(k, l)$-kernel of the $D$-join $\sigma(\alpha, D)$ where $J_i$, $i \in \mathcal{I}$ satisfy the assumption of the theorem.

Thus the theorem is proved.

3. On $(k, l)$-Kernel Perfectness of the $D$-Join

From the definition of the $\sigma(\alpha, D)$ it follows immediately:
Proposition 1. Every induced subdigraph of $\sigma(\alpha, D)$ is
(a) a digraph of the form $\sigma(\tilde{\alpha}, \tilde{D})$, where $\tilde{D}$ is an induced subdigraph of $D$ with $V(\tilde{D}) = \{x_t : t \in \tilde{T}\}$, $|\tilde{T}| > 1$, $\tilde{T} \subseteq \{1, \ldots, n\}$ and $\tilde{\alpha}$ is a family of induced subdigraphs of $D_t$, where $t \in \tilde{T}$ or
(b) an induced subdigraph of $D_i$ for some $1 \leqslant i \leqslant n$ or
(c) the union of the digraphs from (a) and (b).

From the definition of the $(k,l)$-kernel perfect digraph and by Proposition 1 it follows immediately:

Proposition 2. If $\sigma(\alpha, D)$ is $(k,l)$-kernel perfect, then $D$ and $D_i$, $i = 1, \ldots, n$ are $(k,l)$-kernel perfect digraphs.

In [5] it has been proved:

Theorem 9 [5]. Let $D$ be a digraph without circuits of length less than $k$ and let $\alpha = (D_i)_{i \in \{1, \ldots, n\}}$ be a sequence of vertex disjoint digraphs. The $D$-join $\sigma(\alpha, D)$ is a $(k,l)$-kernel perfect digraph if and only if the digraph $D$ and the digraphs $D_i$, $i = 1, \ldots, n$ are $(k,l)$-kernel perfect digraphs.

In this section, we generalize this result for an arbitrary digraph $D$.

Theorem 10. Let $D$ be a $(k,l)$-kernel perfect digraph. Let $D_i$, $i = 1, \ldots, n$ be a $(k,l)$-kernel perfect digraph if $C_d^{d \leq k-1}(x_i) = \emptyset$ or every subdigraph of $D_i$ has an $l$-dominating vertex, otherwise. Then $\sigma(\alpha, D)$ is a $(k,l)$-kernel perfect digraph.

Proof. Assume that $D$ and $D_i$, $i = 1, \ldots, n$ are as in the statements of the theorem. We shall show that $\sigma(\alpha, D)$ is a $(k,l)$-kernel perfect digraph. From Proposition 1 it follows that we need only to prove that $\sigma(\alpha, D)$ has a $(k,l)$-kernel. By Theorem 7, Theorem 8 and from our assumptions there exists a $(k,l)$-kernel $J \subset V(D)$ such that $J^* = \bigcup_{i \in \mathcal{I}} J_i$ is a $(k,l)$-kernel of the $D$-join, where $\mathcal{I} = \{i : x_i \in J\}$, $J_i \subseteq V(D_i^c)$ and $J_i$ is a $(k,l)$-kernel of $D_i^c$ if $C_d^{d \leq k-1}(x_i) = \emptyset$ or $J_i$ is a 1-element set containing an $l$-dominating vertex of $D_i^c$.

Thus the theorem is proved. ■
4. The Total Number of \((k,l)\)-Kernels of the \(D\)-Join

In this section, we calculate the number of all \(k\)-independent sets, \(l\)-dominating sets and \((k,l)\)-kernels of the \(D\)-join \(\sigma(\alpha,D)\).

**Theorem 11.** Let \(k \geq 2\), \(n \geq 2\) be integers. Let \(\sigma(\alpha,D)\) be a \(D\)-join of the digraph \(D\) on \(n\) vertices and \(\alpha\) be a sequence of vertex disjoint digraphs \((D_i)_{i \in \{1,\ldots,n\}}\) on \(p_i\) vertices, \(p_i \geq 1\). Let \(S = \{S_1,\ldots,S_j\}\), \(j \geq 1\) be a family of all nonempty \(k\)-independent sets of the digraph \(D\) and let \(S \ni S_r = \{x_i : i \in I_r\}\), where \(I_r \subseteq \{1,\ldots,n\}\). Then \(NkI(\sigma(\alpha,D)) = 1 + \sum_{r=1}^{j} \prod_{i \in I_r} \varphi(D_i)\), where

\[
\varphi(D_i) = \begin{cases} 
NkI(D_i) - 1 & \text{if } C_D^{l \leq k-1}(x_i) = \emptyset, \\
p_i & \text{otherwise}.
\end{cases}
\]

**Proof.** Let \(D\) be a given digraph on \(n\)-vertices, \(n \geq 2\). Theorem 4 implies that to obtain a \(k\)-independent set of \(\sigma(\alpha,D)\) first we have to choose a \(k\)-independent set of \(D\). Let \(S = \{S_1,\ldots,S_j\}\), \(j \geq 1\) be a family of all nonempty \(k\)-independent sets of the digraph \(D\). Assume that \(S \ni S_r = \{x_i : i \in I_r\}\), where \(I_r \subseteq \{1,\ldots,n\}\). Next by Theorem 5 in each of the \(D_i^r\), \(i \in I_r\) we have to choose an nonempty \(k\)-independent set if there does not exist in \(D\) a circuit containing the vertex \(x_i\) of length less than \(k\) or we choose an arbitrary vertex from \(V(D_i^r)^c\), otherwise. Evidently we can do it on \(NkI(D_i^r) - 1\) or \(p_i\) ways, respectively. Hence from the fundamental combinatorial statement we have \(\sum_{r=1}^{j} \prod_{i \in I_r} \varphi(D_i)\) sets being \(k\)-independent, where

\[
\varphi(D_i) = \begin{cases} 
NkI(D_i) - 1 & \text{if } C_D^{l \leq k-1}(x_i) = \emptyset, \\
p_i & \text{otherwise}.
\end{cases}
\]

Moreover, the empty set also is a \(k\)-independent set of \(\sigma(\alpha,D)\). Consequently, \(NkI(\sigma(\alpha,D)) = 1 + \sum_{r=1}^{j} \prod_{i \in I_r} \varphi(D_i)\).

Thus the theorem is proved. \(\blacksquare\)

**Theorem 12.** Let \(l \geq 1\), \(n \geq 2\) be integers. Let \(\sigma(\alpha,D)\) be a \(D\)-join of the digraph \(D\) on \(n\) vertices and \(\alpha\) be a sequence of vertex disjoint digraphs \((D_i)_{i \in \{1,\ldots,n\}}\) on \(p_i\) vertices, \(p_i \geq 1\). Let \(Q = \{Q_1,\ldots,Q_j\}\), \(j \geq 1\) be a family of all \(l\)-dominating sets of the digraph \(D\) and let \(Q \ni Q_r = \{x_i : i \in I_r\}\), where \(I_r \subseteq \{1,\ldots,n\}\). Then \(NI\sigma(\alpha,D)) = \sum_{r=1}^{j} \prod_{i \in I_r} \psi(D_i)\), where
ψ(D_i) = \begin{cases} 
NlD(D_i) & \text{if } C_D^{d \leq l}(x_i) = \emptyset \text{ and for each } \mathcal{I} \ni j \neq i \\
2^{p_i} - 1 & \text{otherwise.} 
\end{cases}

**Proof.** Let \( D \) be a given digraph on \( n \) vertices, \( n \geq 2 \). By Theorem 4 we have that to obtain an \( l \)-dominating set of \( \sigma(\alpha, D) \) first we have to choose an \( l \)-dominating set of \( D \). Let \( Q = \{ Q_1, \ldots, Q_j \} \), \( j \geq 1 \) be a family of all \( l \)-dominating sets of the digraph \( D \). Assume that \( Q \ni Q_r = \{ x_i : i \in \mathcal{I}_r \} \) and \( \mathcal{I}_r \subseteq \{1, \ldots, n\} \). Next by Theorem 6 in each of the \( D_{c_i}^r \), \( i \in \mathcal{I}_r \) we have to choose an \( l \)-dominating set if for each \( j \in \mathcal{I}_r \) and \( j \neq i \) there does not exist a path \( x_i \ldots x_j \) of length less than or equal to \( l \) and there does not exist in \( D \) a circuit containing the vertex \( x_i \) of length less than or equal to \( l \) or we have to choose in \( D_{c_i}^r \) an arbitrary nonempty subset of \( V(D_{c_i}^r) \). Evidently, we can do it on \( NlD(D_i) \) or \( 2^{p_i} - 1 \) ways, respectively. Hence from the fundamental combinatorial statement we have \( \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \psi(D_i) \) sets being \( l \)-dominating sets of \( \sigma(\alpha, D) \), where

\( \psi(D_i) = \begin{cases} 
NlD(D_i) & \text{if } C_D^{d \leq l}(x_i) = \emptyset \text{ and for each } \mathcal{I} \ni j \neq i \\
2^{p_i} - 1 & \text{otherwise.} 
\end{cases} \)

Thus the theorem is proved.  

Using the same method as in Theorems 11 and 12 we can prove:

**Theorem 13.** Let \( k \geq 2 \), \( 1 \leq l \leq k - 1 \), \( n \geq 2 \) be integers. Let \( \sigma(\alpha, D) \) be a \( D \)-join of the digraph \( D \) on \( n \) vertices and \( \alpha \) be a sequence of vertex disjoint digraphs \( (D_i)_{i \in \{1, \ldots, n\}} \) on \( p_i \) vertices, \( p_i \geq 1 \). Let \( J = \{ J_1, \ldots, J_j \} \), \( j \geq 1 \) be a family of all \((k,l)\)-kernels of the digraph \( D \) and let \( J \ni J_r = \{ x_i : i \in \mathcal{I}_r \} \), where \( \mathcal{I}_r \subseteq \{1, \ldots, n\} \). Then \( NklK(\sigma(\alpha, D)) = \sum_{r=1}^j \prod_{i \in \mathcal{I}_r} \mu(D_i) \), where

\( \mu(D_i) = \begin{cases} 
NklK(D_i) & \text{if } C^{d \leq k-1}_D(x_i) = \emptyset \\
p_i & \text{if } C^{d \leq l}_D(x_i) \neq \emptyset \\
Nld(D_i) & \text{otherwise.} 
\end{cases} \)

**Theorem 14.** Let \( k \geq 2 \), \( l \geq k \), \( n \geq 2 \) be integers. Let \( \sigma(\alpha, D) \) be a \( D \)-join of the digraph \( D \) on \( n \) vertices and \( \alpha \) be a sequence of vertex disjoint digraphs
Let \((D_i)_{i \in \{1, \ldots, n\}}\) on \(p_i\) vertices, \(p_i \geq 1\). Let \(\mathcal{J} = \{J_1, \ldots, J_j\}, j \geq 1\) be a family of all \((k, l)\)-kernels of the digraph \(D\) and let \(\mathcal{J} \ni J_r = \{x_i : i \in I_r\}\), where \(I_r \subseteq \{1, \ldots, n\}\). Then \(N_kI_k(D) = \sum_{r=1}^{J} \prod_{i \in I_r} \eta(D_i),\) where \(\eta(D_i) = \begin{cases} N_kI_k(D) & \text{if } C^d_{\leq 1}(x_i) = \emptyset \text{ and for each } I \ni j \neq i \text{ holds } d_D(x_i, x_j) > l, \\ p_i & \text{if } C^{d_{\leq k-1}}_{D}(x_i) \neq \emptyset, \\ N_kI_k(D) - 1 & \text{otherwise}. \end{cases}\)

References


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