IMPROVED UPPER BOUNDS FOR NEARLY ANTIPODAL CHROMATIC NUMBER OF PATHS

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Abstract

For paths \(P_n\), G. Chartrand, L. Nebeský and P. Zhang showed that 
\(ac'(P_n) \leq \binom{n-2}{2} + 2\) for every positive integer \(n\), where \(ac'(P_n)\) denotes 
the nearly antipodal chromatic number of \(P_n\). In this paper we show 
that \(ac'(P_n) \leq \binom{n-2}{2} - \frac{n}{2} - \left\lfloor \frac{10}{n} \right\rfloor + 7\) if \(n\) is even positive integer and 
\(n \geq 10\), and \(ac'(P_n) \leq \binom{n-2}{2} - \frac{n+1}{2} - \left\lfloor \frac{13}{n} \right\rfloor + 8\) if \(n\) is odd positive integer and \(n \geq 13\). For all even positive integers \(n \geq 10\) and all odd positive integers \(n \geq 13\), these results improve the upper bounds for 
neearly antipodal chromatic number of \(P_n\).

Keywords: radio colorings, nearly antipodal chromatic number, paths.

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1. Introduction

Radio \( k \)-colorings are generalizations of ordinary colorings of graphs, which were inspired by (FM Radio) Channel Assignments Problem (see [5, 7]) and introduced by G. Chartrand, D. Erwan, F. Harary and P. Zhang [1]. For a connected graph \( G \) of order \( n \) and diameter \( d \) and an integer \( k \) with \( 1 \leq k \leq d \), a radio \( k \)-coloring of \( G \) is a function \( c : V(G) \to \mathbb{N} \), such that \( d(u, v) + |c(u) - c(v)| \geq k + 1 \) for every pair \( u \) and \( v \) of distinct vertices of \( G \), where \( d(u, v) \) denotes the distance between \( u \) and \( v \) (the length of a shortest \( u-v \) path) in \( G \). Clearly, radio 1-colorings and ordinary colorings are synonymous. The value \( rck(c) \) of a radio \( k \)-coloring \( c \) of \( G \) is the maximum color assigned to a vertex of \( G \); while the radio \( k \)-chromatic number \( rck(G) \) of \( G \) is \( \min\{rck(c)\} \) taken over all \( k \)-coloring \( c \) of \( G \). In particular, radio \( d \)-colorings are referred to as radio labelings and the radio \( d \)-chromatic number is called the radio number. Radio \((d-1)\)-colorings are referred to as radio antipodal coloring or, more simply, as an antipodal coloring, and the radio \((d-1)\)-chromatic number is called the antipodal chromatic number, denoted by \( ac(G) \). Radio \( k \)-coloring and radio labeling of graphs were studied in [1, 2]. Radio antipodal coloring of paths were studied in [3, 4, 6].

Furthermore, G. Chartrand, L. Nebeský and P. Zhang gave the concepts of nearly antipodal colorings in [4]. For a connected graph \( G \) of diameter \( d \), a nearly antipodal coloring of \( G \) is a function \( c : V(G) \to \mathbb{N} \), such that \( d(u, v) + |c(u) - c(v)| \geq d - 1 \) for every two distinct vertices \( u \) and \( v \) of \( G \). The value \( ac'(c) \) of a nearly antipodal coloring \( c \) of \( G \) is the maximum color assigned to a vertex of \( G \). The nearly antipodal chromatic number \( ac'(G) \) of \( G \) is \( \min\{ac'(c)\} \) taken over all nearly antipodal colorings of \( G \) (In fact, for \( d \geq 3 \), a nearly antipodal coloring is a radio \((d-2)\)-coloring).

Clearly, if \( G \) is a connected graph of diameter 1 or 2, then \( ac'(G) = 1 \); while if \( \text{diam}(G) = 3 \), then \( ac'(G) \) is the chromatic number of \( G \). Thus nearly antipodal colorings are most interesting for connected graphs of diameter 4 or more. For this reason, the nearly antipodal chromatic number of paths \( P_n \) were investigated in [4] by G. Chartrand, L. Nebeský and P. Zhang. And they showed that \( ac'(P_5) = 5 \), \( ac'(P_6) = 7 \), \( ac'(P_7) = 11 \) and \( ac'(P_8) = 16 \). Moreover, they presented an upper bound for the nearly antipodal chromatic number of paths \( P_n \) for every positive integer \( n \) as follows.

**Theorem 1.1 ([4]).** If \( n \) is a path of order \( n \geq 1 \), then \( ac'(P_n) \leq \left(\frac{n-2}{2}\right) + 2 \).
2. Our Results and the Idea of the Proof

In this paper we will provide an improved version for Theorem 1.1. We will show that

Theorem 2.1.

1. If $P_n$ is even and $n \geq 10$, then $ac'(P_n) \leq \left(\frac{n-2}{2}\right) - \frac{n}{2} - \left\lfloor \frac{10}{n} \right\rfloor + 7$;
2. If $n$ is odd and $n \geq 13$, then $ac'(P_n) \leq \left(\frac{n-2}{2}\right) - \frac{n-1}{2} - \left\lfloor \frac{13}{n} \right\rfloor + 8$.

Clearly, it holds that $-\frac{n}{2} - \left\lfloor \frac{10}{n} \right\rfloor + 7 \leq 1$ for all even integers $n \geq 10$, and $-\frac{n-1}{2} - \left\lfloor \frac{13}{n} \right\rfloor + 8 \leq 1$ for all odd integers $n \geq 13$. Thus, for all even integers $n \geq 10$ and all odd integers $n \geq 13$, Theorem 2.1 improves the upper bounds of $ac'(P_n)$.

We will prove Theorem 2.1 in Section 3, and the proof will virtually provide a nearly antipodal coloring $c$ for paths $P_n$ with $ac'(c)$ that is equal to the bound presented in Theorem 2.1. The idea of performing the coloring $c$ is based on pseudo greedy algorithm: Let $V(P_n) = \{p_1, p_2, \ldots, p_n\}$. At first, we use the color $c_1 = 1$ to color some vertex $p_{n_1} \in \{p_1, p_2, \ldots, p_n\}$, where $p_{n_1}$ is the (a) central vertex of $P_n$. Suppose that for $1 \leq i \leq n-1$ the vertices in $\{p_{n_1}, p_{n_2}, \ldots, p_{n_i}\} \subset \{p_1, p_2, \ldots, p_n\}$ have been colored with $c(p_{n_j}) = c_j$ for all $1 \leq j \leq i$, then we choose a color $c_{i+1} \in \mathbb{N}$ as small as possible to color one vertex $p_{n_{i+1}} \in V(P_n) \setminus \{p_{n_1}, p_{n_2}, \ldots, p_{n_i}\}$, such that $d(p_{n_{i+1}}, p_{n_j}) + |c(p_{n_{i+1}}) - c(p_{n_j})| \geq d - 1$ for all $1 \leq j \leq i$. And if there are two vertices can be chosen for $p_{n_{i+1}}$, then we take $p_{n_{i+1}}$ close to central vertices of $P_n$ as near as possible. Finally, we obtain that $ac'(c) = c(p_{n_i})$ and hence $ac'(P_n) \leq ac'(c)$. In Section 4 we will give some examples which present the nearly antipodal coloring $c$ for some paths $P_n$ with $ac'(c)$ showed in Theorem 2.1 by our methods.

3. Proof of Theorem 2.1

Proof. 1. $n$ is even and $n \geq 10$. Firstly, we let $n \geq 12$, note that $-\left\lfloor \frac{10}{n} \right\rfloor = 0$, it suffices to show that $ac'(P_n) \leq \left(\frac{n-2}{2}\right) - \frac{n}{2} + 7$. Write $n = 2k = 10 + 2(4p + q)$, where $p \in \{0, 1, 2, \ldots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 5 + (4p + q)$ and $d - 1 = \text{diam}(P_n) - 1 = 2k - 2$.

We denote the vertices of $P_n$ by $x_1', x_2', x_3'; v_1', v_2', \ldots, v_2'_{p-1}, v_2'; w_1, w_2, \ldots, w_q; v_{2p}, v_{2p-1}, \ldots, v_2, v_1; x_2, x_1; y_1, y_2; u_1, u_2, \ldots, u_{2p-1}, u_2; \ldots$.
Step 1. Color the vertices in $V_1$ (see Figure 1). And we write

$$V_1 = \{x_1, x_2; y_1, y_2; x'_1, x'_2, x'_3; y'_1, y'_2, y'_3\},$$

$$V_2 = \{v_1, u_2, v_3, u_4, \ldots, v_{2p-1}, u_{2p}, v'_1, v'_2, \ldots, v'_{2p-1}, v'_2, u'_1, u'_2, \ldots, u'_{2p-1}, u'_2\},$$

$$V_3 = \{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q; v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}.$$

In the following we will present a coloring $c$ for $P_n$ by three steps, such that

$$d(u, v) + |c(u) - c(v)| \geq d - 1 = 2k - 2$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $ac'(c) = \binom{n-2}{2} - \frac{n}{2} + 7$ (note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

**Step 1.** Color the vertices in $V_1$ (see Figure 1).

Let

$$c(x_1) = 1 \text{ (} x_1 \text{ is a central vertex of } P_n \text{); }$$

$$c(y'_1) = c(x_1) + (k - 2) = k - 1, \quad c(x'_1) = c(x_1) + (k - 1) = k;$$

$$c(y_1) = c(x'_1) + (k - 2) = 2k - 2;$$

$$c(x'_2) = c(y_1) + k - 1 = 3k - 3, \quad c(y'_2) = c(x'_2) + 1 = 3k - 2;$$

$$c(x_2) = c(x'_2) + (k + 1) = 4k - 2;$$

$$c(y'_3) = c(x_2) + (k - 1) = 5k - 3, \quad c(x'_3) = c(y'_3) + 3 = 5k;$$

$$c(y_2) = c(x'_3) + (k - 1) = 6k - 1.$$

Then by the definition of $c$ and the value of $d(u, v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.

**Claim 3.1.** For all distinct vertices $u, v \in V_1$, the inequality (1) holds. At the same time, $\max_{v \in V_1} c(v) = c(y_2) = 6k - 1$ and $\max_{v \in V_1 \backslash \{y_2\}} c(v) = c(x'_3) = 5k$.

**Step 2.** Color the vertices in $V_2$ (see Figure 1).

For $i = 1, 2, \ldots, p$, let
Claim 3.2. Then we have the following claim. Improved Upper Bounds for Nearly Antipodal ... 163

At the same time, it holds that max \( d \in V \) \( c(v) = 6k - 1 + 2p(2k + 2p + 3) \) and max \( v \in V \) \( c(v) = c(v_2p) = 5k + 2p(2k + 2p + 3) \).

In fact, note \( d - 1 = 2k - 2 \). Since that \( d(y_2, u_1) = k - 2 \), \( d(y_2, u_1) = k - 5 \), \( d(y_2, u_1) = 2k - 7 \), \( c(v_1) = c(y_2) + k \) and \( c(v_1') = c(y_2) + k + 5 \), then for all distinct vertices \( u, v \in \{ y_2, v_1', u_1 \} \), the inequality (1) holds. As \( \max_{v \in V \setminus \{ y_2 \}} c(v) = c(x_1) \) by Claim 3.1, \( c(v_1') = c(y_2) + k = c(x_1) + 2k - 1 \) and \( c(u_1') > c(v_1') \), we have that \( c(v_1') - c(x_1) \geq d - 1 \) and \( c(u_1') - c(x_1) \geq d - 1 \). Therefore for all distinct vertices \( u, v \in V \setminus \{ v_1', u_1 \} \), the inequality (1) holds.

Since that \( d(u_1, v_1) = k - 1 \), \( d(v_1, v_1') = k - 6 \), and \( c(v_1) = c(v_1') + (k - 1) = c(v_1') + 5 + (k - 1) \), then for all distinct vertices \( u, v \in \{ v_1, v_1', u_1 \} \), the inequality (1) holds. As \( \max_{v \in V \setminus \{ y_2 \}} c(v) = c(y_2) \) by Claim 3.1, \( c(v_1) = c(y_2) + k + 5 + (k - 1) \), we have that \( c(v_1) - c(y_2) \geq d - 1 \). Therefore for all distinct vertices \( u, v \in V \setminus \{ v_1', u_1 \} \), the inequality (1) holds.

Note the fact that \( d(v_1, u_2') = k - 2, d(v_1, u_2') = k - 5 - 2, d(u_2, u_2') = 2k - 7 - 2, c(u_2') = c(v_1) + k, c(v_2') = c(v_1) + k + 5 + 2 \); and \( d(v_2, u_2) = k - 1, d(u_2, u_2') = k - 6 - 2, c(u_2) = c(v_2) + (k - 1) = c(v_2') + 5 + 2 + (k - 1) \). Similar to the above discussion we can obtain that for all distinct vertices \( u, v \in V \setminus \{ v_1', v_1 \} \cup \{ u_2, u_2' \} \cup \{ v_2', v_2', u_2 \} \cup \{ v_2', v_2', u_2 \} \cup V_1 \cup V_2 \), the inequality (1) holds.

Continue the above discussion we can conclude that for all distinct vertices \( u, v \in V \setminus \{ v_1', v_1 \} \cup \{ u_2', u_2', v_2 \} \cup \ldots \cup \{ v_2p, v_2p', u_2p \} \cup V_1 \cup V_2 \), the inequality (1) holds.
By the definition of $c$, it is easy to verify that $\max_{v \in V_1 \cup V_2} c(v) = c(w_{2p}) = 6k - 1 + 2p(2k + 2p + 3)$ and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v_{2p}^\prime) = 5k + 2p(2k + 2p + 3)$.

Figure 1: A nearly antipodal coloring for $P_n$ ($n = 2k \geq 10$).

**Step 3.** Color the vertices in $V_3$ (see Figure 1).

**Step 3.1.** Color the vertices in $\{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q\}$.

According the value of $q$, there are four cases.

**Case 1.** $q = 1$. Let

- $c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3)$,
- $c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5)$.

**Case 2.** $q = 2$. Let

- $c(u_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3)$,
- $c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5)$,
- $c(u_2) = c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5)$,
- $c(z_2) = c(w_2) + 3 + 2(2p + 2) = 8k + 10 + 2p(2k + 2p + 7)$.

**Case 3.** $q = 3$. Let

- $c(w_1) = c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3)$,
- $c(z_1) = c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5)$,
- $c(w_3) = c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5)$,
- $c(z_2) = c(w_3) + k = 9k + 3 + 2p(2k + 2p + 5)$,
- $c(w_2) = c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7)$,
- $c(z_3) = c(w_2) + k = 10k + 10 + 2p(2k + 2p + 7)$. 
Case 4. $q = 4$. Let
\[
\begin{align*}
c(w_1) &= c(u_{2p}) + k = 7k - 1 + 2p(2k + 2p + 3), \\
c(z_1) &= c(w_1) + 3 + 2(2p + 1) = 7k + 4 + 2p(2k + 2p + 5), \\
c(w_4) &= c(z_1) + (k - 1) = 8k + 3 + 2p(2k + 2p + 5), \\
c(z_2) &= c(w_4) + k = 9k + 3 + 2p(2k + 2p + 5), \\
c(w_2) &= c(z_2) + 3 + 2(2p + 2) = 9k + 10 + 2p(2k + 2p + 7), \\
c(z_3) &= c(w_2) + (k - 1) = 10k + 9 + 2p(2k + 2p + 7), \\
c(w_3) &= c(z_3) + 3 + 2(2p + 3) = 10k + 18 + 2p(2k + 2p + 9), \\
c(z_4) &= c(w_3) + (k + 1) = 11k + 19 + 2p(2k + 2p + 9).
\end{align*}
\]

Claim 4.3. Color the vertices in \{\(v_{2p}, u_{2p-1}, \ldots, v_4, v_2, u_1\}\). For any case above (\(q = 1, 2, 3, 4\)), we let
\[
\begin{align*}
c(v_{2p}) &= c(z_q) + [(k + q) - 1], \\
c(u_{2p-1}) &= c(v_{2p}) + [(k + q - 1) + 2], \\
c(v_{2p-2}) &= c(u_{2p-1}) + [(k + q - 1) + 2 \cdot 2], \\
c(u_{2p-3}) &= c(v_{2p-2}) + [(k + q - 1) + 2 \cdot 3], \\
\cdots \\
c(v_2) &= c(u_3) + [(k + q - 1) + 2(2p - 2)], \\
c(u_1) &= c(v_2) + [(k + q - 1) + 2(2p - 1)] \\
&= c(z_q) + 2p(k + q - 1) + 2 \cdot \frac{2p(2p-1)}{2} \\
&= c(z_q) + 2p(k + q + 2p - 2).
\end{align*}
\]

Then by a similar method to prove Claim 3.2, we can obtain the following claim.

Claim 3.3. For all distinct vertices \(u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)\), the inequality (1) holds. And \(
\max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2).
\)

By Claim 3.3, we have shown that for all even integers \(n \geq 12\), \(c\) is a nearly antipodal coloring for \(P_n\). Therefore \(ac'(P_n) \leq ac'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 2)\). To finish the proof of Theorem 2.1 for all even integers \(n \geq 12\), it suffices to prove the following claim.

Claim 3.4. For any \(p \in \{0, 1, 2, \ldots\}\) and any \(q \in \{1, 2, 3, 4\}\), it holds that
\[
c(u_1) = c(z_q) + 2p(k + q + 2p - 2) = \binom{n}{2} - \frac{n}{2} + 7, \text{ where } n = 2k = 2(5 + 4p + q).
\]
In fact, if \( q = 1 \), then \( k = 4p + 6, 2p = \frac{k-6}{2} \). Thus
\[
c(u_1) = c(z_1) + 2p(k + q + 2p - 2) = 7k + 4 + 2p(2k + 2p + 5) + 2p(k + 2p - 1) = 2k^2 - 6k + 10 = \frac{n^2}{2} - 3n + 10 = \left(\frac{n - 2}{2}\right) - \frac{n}{2} + 7.
\]

If \( q = 2 \), then \( k = 4p + 7, 2p = \frac{k-7}{2} \). Thus
\[
c(u_1) = c(z_2) + 2p(k + q + 2p - 2) = 8k + 10 + 2p(2k + 2p + 7) + 2p(k + 2p) = 8k + 10 + 2p(3k + 4p + 7) = \frac{n^2}{2} - 3n + 10 = \left(\frac{n - 2}{2}\right) - \frac{n}{2} + 7.
\]

If \( q = 3 \), then \( k = 4p + 8, 2p = \frac{k-8}{2} \). Thus
\[
c(u_1) = c(z_3) + 2p(k + q + 2p - 2) = 10k + 10 + 2p(2k + 2p + 7) + 2p(k + 2p + 1) = 10k + 10 + 2p(3k + 4p + 8) = \frac{n^2}{2} - 3n + 10 = \left(\frac{n - 2}{2}\right) - \frac{n}{2} + 7.
\]

If \( q = 4 \), then \( k = 4p + 9, 2p = \frac{k-9}{2} \). Thus
\[
c(u_1) = c(z_4) + 2p(k + q + 2p - 2) = 11k + 19 + 2p(2k + 2p + 9) + 2p(k + 2p + 2) = 11k + 19 + 2p(3k + 4p + 11) = \frac{n^2}{2} - 3n + 10 = \left(\frac{n - 2}{2}\right) - \frac{n}{2} + 7.
\]

Thus Claim 3.4 holds and hence \( ac'(P_n) \leq ac'(c) = \left(\frac{n-2}{2}\right) - \frac{n}{2} + 7 \) for all even integers \( n \geq 12 \).

Secondly, for \( n = 10 \), in the above proof we take \( p = 0 \) and \( q = 0 \). Namely, \( V_2 = V_3 = \emptyset \). \( V(P_{10}) = V_1 = \{x_1', x_2', x_3'; x_2, x_1; y_1, y_2; y_6', y_7'; y_1', y_3', y_4'\} \) (also see Figure 1 and let \( p = q = 0 \)). Then coloring \( c|_{v \in V_1}(v) \) is a nearly antipodal coloring for \( P_{10} \). Thus by Claim 3.1, \( ac'(P_{10}) \leq ac'(c|_{v \in V_1}) = \max_{v \in V_1} c(v) = c(y_2) = (6k - 1)|_{k=5} = 29 = \left(\binom{10}{2} - 2\right) + 1 \). Since \( -\left\lfloor\frac{10}{10}\right\rfloor = -1 \) for \( n = 10 \), it follows that \( ac'(P_{10}) \leq ac'(c|_{v \in V_1}) = \left(\binom{10}{2} - 2\right) + 1 = \left(\binom{10}{2} - 1\right) + 7 \).
Thus we complete the proof of assertion 1 in Theorem 2.1.

2. $n$ is odd and $n \geq 13$. Firstly, we let $n \geq 15$, note that $-\lfloor \frac{13}{n} \rfloor = 0$, it suffices to show that $ac'(P_n) \leq \left(\begin{array}{c}n-2 \\ 2 \end{array}\right) - \frac{n}{2} + 8$. Write $n = 2k + 1 = 13 + 2(4p + q)$, where $p \in \{0, 1, 2, \ldots\}$ and $q \in \{1, 2, 3, 4\}$. Then we have that $k = 6 + (4p + q)$ and $d - 1 = \text{diam}(P_n) - 1 = 2k - 1$.

We denote the vertices of $P_n$ by $x_1', x_2', x_3', x_4'$; $v_1', v_2', \ldots, v_{2p-1}', v_{2p}$; $w_1, w_2, \ldots, w_q; v_{2p}, v_{2p-1}, \ldots, v_2, v_1; x_1', x_2', x_0; y_1, y_2; u_1, u_2, \ldots, u_{2p-1}, u_{2p}; z_1, z_2, z_3; u_1', v_1', u_2', u_2', \ldots, u_{2p-1}', u_{2p}'$ (see Figure 2). And we write

$V_1 = \{x_0; x_1, x_2; y_1, y_2; x_1', x_2', x_0'; y_1', y_2', y_3', y_4\}$,

$V_2 = \{v_1, u_2, v_3, u_4, \ldots, v_{2p-1}, u_{2p}; v_1', v_2', \ldots, v_{2p-1}', v_{2p}'; u_1', u_2', \ldots, u_{2p-1}', u_{2p}'\}$,

$V_3 = \{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q; v_{2p}, v_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}$.

Similar to the method of proof assertion 1, we will present a coloring $c$ for $P_n$ by three steps, such that

$$d(u, v) + |c(u) - c(v)| \geq d - 1 = 2k - 1$$

holds for all distinct vertices $u, v \in V_1 \cup V_2 \cup V_3 = V(P_n)$, and $ac'(c) = \left(\begin{array}{c}n-2 \\ 2 \end{array}\right) - \frac{n}{2} + 8$(note that $V_2 = \emptyset$ if $p = 0$, and it is easy to see that the following proof is also suitable for $V_2 = \emptyset$).

**Step 1.** Color the vertices in $V_1$ (see Figure 2).

Let

$c(x_0) = 1$ ($x_0$ is the central vertex of $P_n$);
$c(x_1') = c(x_0) + (k - 1) = k$,
$c(x_1) = c(x_1') + k = 2k$;
$c(y_1') = c(x_1) + (k - 1) = 3k - 1$,
$c(y_1) = c(y_2') + (k + 1) = 4k$;
$c(x_3') = c(y_1) + k = 5k$,
$c(y_3') = c(y_3') + 3 = 5k + 3$;
$c(x_2) = c(x_3') + (k + 3) = 6k + 3$;
$c(y_4') = c(x_2) + k = 7k + 3$,
$c(x_4') = c(y_4') + 5 = 7k + 8$;
$c(y_2) = c(x_4') + k = 8k + 8$.

Then by the definition of $c$ and the value of $d(u, v)$ for $u, v \in V_1$, it is easy to verify that the following claim holds.
Claim 3.5. For all distinct vertices \( u, v \in V_1 \), the inequality (2) holds. At the same time, \( \max_{v \in V_1} c(v) = c(y_2) = 8k + 8 \) and \( \max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x_4') = 7k + 8 \).

Step 2. Color the vertices in \( V_2 \) (see Figure 2).

For \( i = 1, 2, \ldots, p \), let

\[
\begin{align*}
c(v'_{2i-1}) &= c(y_2) + (2i - 1)(k + 1) + 5(2i - 2) + 2[1 + 2 + \ldots + (2i - 2)] + (2i - 2)k, \\
c(u'_{2i-1}) &= c(y_2) + (2i - 1)(k + 1) + 5(2i - 1) + 2[1 + 2 + \ldots + (2i - 1)] + (2i - 2)k; \\
c(v_{2i-1}) &= c(y_2) + (2i - 1)(k + 1) + 5(2i - 1) + 2[1 + 2 + \ldots + (2i - 1)] + (2i - 1)k; \\
c(u_{2i}) &= c(y_2) + (2i)(k + 1) + 5(2i) + 2[1 + 2 + \ldots + (2i)] + (2i - 1)k; \\
c(u_{2i}) &= c(y_2) + (2i)(k + 1) + 5(2i) + 2[1 + 2 + \ldots + (2i)] + (2i)k.
\end{align*}
\]

Then we have the following claim.

Claim 3.6. For all distinct vertices \( u, v \in V_1 \cup V_2 \), the inequality (2) holds. At the same time, it holds that \( \max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 8k + 8 + 2p(2k + 2p + 7) \) and \( \max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v'_{2p}) = 7k + 8 + 2p(2k + 2p + 7) \).

In fact, note \( d - 1 = 2k - 1 \). Since that \( d(y_2, v'_1) = k - 2 \), \( d(y_2, u'_1) = k - 6 \), \( d(v'_1, u'_1) = 2k - 8 \), \( c(v'_1) = c(y_2) + (k + 1) \) and \( c(u'_1) = c(y_2) + (k + 1) + 7 \), then for all distinct vertices \( u, v \in \{y_2, v'_1, u'_1\} \), the inequality (2) holds. As \( \max_{v \in V_1 \setminus \{y_2\}} c(v) = c(x_4') \) by Claim 3.5, \( c(v'_1) = c(y_2) + (k + 1) = c(x_4') + 2k + 1 \) and \( c(u'_1) > c(v'_1) \), we have that \( c(v'_1) - c(x_4') \geq d - 1 \) and \( c(u'_1) - c(x_4') \geq d - 1 \). Therefore for all distinct vertices \( u, v \in V_1 \cup \{v'_1, u'_1\} \), the inequality (2) holds.

Since that \( d(u'_1, v_1) = k - 1 \), \( d(v_1, v'_1) = k - 7 \), and \( c(v_1) = c(v'_1) + k = c(v'_1) + 7 + k \), then for all distinct vertices \( u, v \in \{v_1, v'_1, u'_1\} \), the inequality (2) holds. As \( \max_{v \in V_1} c(v) = c(y_2) \) by Claim 3.5, and \( c(v_1) = c(y_2) + (k + 1) + 7 + k \), we have that \( c(v_1) - c(y_2) \geq d - 1 \). Therefore for all distinct vertices \( u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \), the inequality (2) holds.

Note the fact that \( d(v_1, u'_2) = k - 2 \), \( d(v_1, v'_2) = k - 6 - 2 \), \( d(u'_2, v'_2) = 2k - 8 - 2 \), \( c(u'_2) = c(v_1) + (k + 1) \), \( c(v'_2) = c(v_1) + (k + 1) + 7 + 2 \); and
$d(v'_2, u_2) = k - 1$, $d(u_2, u'_2) = k - 7 - 2$, $c(u_2) = c(v'_2) + k = c(u'_2) + 7 + 2 + k$. Similar to the above discussion we can obtain that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\}$, the inequality (2) holds.

Continue the above discussion we can conclude that for all distinct vertices $u, v \in V_1 \cup \{v'_1, u'_1, v_1\} \cup \{u'_2, v'_2, u_2\} \cup \ldots \cup \{v'_{2p-1}, u'_{2p-1}, v_{2p-1}\} \cup \{u'_{2p}, v'_{2p}, u_{2p}\} = V_1 \cup V_2$, the inequality (2) holds. By the definition of $c$, it is easy to see that $\max_{v \in V_1 \cup V_2} c(v) = c(u_{2p}) = 8k + 8 + 2p(2k + 2p + 7)$, and $\max_{v \in (V_1 \cup V_2) \setminus \{u_{2p}\}} c(v) = c(v'_{2p}) = 7k + 8 + 2p(2k + 2p + 7)$.

![Figure 2. A nearly antipodal coloring for $P_n$ (n = 2k + 1 ≥ 13).](image)

**Step 3.** Color the vertices in $V_3$ (see Figure 2).

**Step 3.1.** Color the vertices in $\{w_1, w_2, \ldots, w_q; z_1, z_2, \ldots, z_q\}$.

According the value of $q$, there are four cases.

1. **Case 1.** $q = 1$. Let

   \[
   c(w_1) = c(u_{2p}) + (k + 1) = 9k + 9 + 2p(2k + 2p + 7),
   
   c(z_1) = c(w_1) + 5 + 2(2p + 1) = 9k + 16 + 2p(2k + 2p + 9).
   \]

2. **Case 2.** $q = 2$. Let

   \[
   c(w_1) = c(u_{2p}) + (k + 1) = 9k + 9 + 2p(2k + 2p + 7),
   
   c(z_1) = c(w_1) + 5 + 2(2p + 1) = 9k + 16 + 2p(2k + 2p + 9),
   
   c(w_2) = c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9),
   
   c(z_2) = c(w_2) + 5 + 2(2p + 2) = 10k + 25 + 2p(2k + 2p + 11).
   \]
Case 3. $q = 3$. Let
\[
\begin{align*}
    c(w_1) &= c(u_{2p}) + (k + 1) = 9k + 9 + 2p(2k + 2p + 7), \\
    c(z_1) &= c(w_1) + 5 + 2(2p + 1) = 9k + 16 + 2p(2k + 2p + 9), \\
    c(w_3) &= c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9), \\
    c(z_2) &= c(w_3) + (k + 1) = 11k + 17 + 2p(2k + 2p + 9), \\
    c(w_2) &= c(z_2) + 5 + 2(2p + 2) = 11k + 26 + 2p(2k + 2p + 11), \\
    c(z_3) &= c(w_2) + (k + 1) = 12k + 27 + 2p(2k + 2p + 11).
\end{align*}
\]

Case 4. $q = 4$. Let
\[
\begin{align*}
    c(w_1) &= c(u_{2p}) + (k + 1) = 9k + 9 + 2p(2k + 2p + 7), \\
    c(z_1) &= c(w_1) + 5 + 2(2p + 1) = 9k + 16 + 2p(2k + 2p + 9), \\
    c(w_4) &= c(z_1) + k = 10k + 16 + 2p(2k + 2p + 9), \\
    c(z_2) &= c(w_4) + (k + 1) = 11k + 17 + 2p(2k + 2p + 9), \\
    c(w_2) &= c(z_2) + 5 + 2(2p + 2) = 11k + 26 + 2p(2k + 2p + 11), \\
    c(z_3) &= c(w_2) + k = 12k + 26 + 2p(2k + 2p + 11), \\
    c(w_3) &= c(z_3) + 5 + 2(2p + 3) = 12k + 37 + 2p(2k + 2p + 13), \\
    c(z_4) &= c(w_3) + (k + 2) = 13k + 39 + 2p(2k + 2p + 13).
\end{align*}
\]

Step 3.2. Color the vertices in \{v_{2p}, u_{2p-1}, \ldots, v_4, u_3, v_2, u_1\}.

For each case above ($q = 1, 2, 3, 4$), we let
\[
\begin{align*}
    c(v_{2p}) &= c(z_q) + (k + q), \\
    c(u_{2p-1}) &= c(v_{2p}) + [(k + q) + 2], \\
    c(v_{2p-2}) &= c(u_{2p-1}) + [(k + q) + 2 \cdot 2], \\
    c(u_{2p-3}) &= c(v_{2p-2}) + [(k + q) + 2 \cdot 3], \\
    c(v_2) &= c(u_3) + [(k + q) + 2(2p - 2)], \\
    c(u_1) &= c(v_2) + [(k + q) + 2(2p - 1)] \\
    &= c(z_q) + 2p(k + q) + 2 \cdot \frac{2p(2p-1)}{2} \\
    &= c(z_q) + 2p(k + q + 2p - 1).
\end{align*}
\]

Then by a similar method to prove Claim 3.6, we can obtain the following claim.
Claim 3.7. For all distinct vertices \( u, v \in V_1 \cup V_2 \cup V_3 = V(P_n) \), the inequality (2) holds. And \( \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 1) \).

By Claim 3.7, we have shown that for all odd integers \( n \geq 15 \), \( c \) is a nearly antipodal coloring for \( P_n \). Therefore \( ac'(P_n) \leq ac'(c) = \max_{v \in V(P_n)} c(v) = c(u_1) = c(z_q) + 2p(k + q + 2p - 1) \). To finish the proof of Theorem 2.1 for all odd integers \( n \geq 15 \), it suffices to prove the following claim.

Claim 3.8. For any \( p \in \{0, 1, 2, \ldots\} \) and any \( q \in \{1, 2, 3, 4\} \), it holds that
\[
c(u_1) = c(z_q) + 2p(k + q + 2p - 1) = \binom{n - 2}{2} - \frac{n - 1}{2} + 8,
\]
where \( n = 2k + 1 = 13 + 2(4p + q) \).

In fact, if \( q = 1 \), then \( k = 4p + 7 \), \( 4p = k - 7 \), \( 2p = \frac{k - 7}{2} \). Thus
\[
c(u_1) = c(z_1) + 2p(k + q + 2p - 1) = 9k + 16 + 2p(2k + 2p + 9) + 2p(k + 2p)
= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n - 2}{2} - \frac{n - 1}{2} + 8.
\]
If \( q = 2 \), then \( k = 4p + 8 \), \( 4p = k - 8 \), \( p = \frac{k - 8}{2} \). Thus
\[
c(u_1) = c(z_2) + 2p(k + q + 2p - 1)
= 10k + 25 + 2p(2k + 2p + 11) + 2p(k + 2p + 1)
= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n - 2}{2} - \frac{n - 1}{2} + 8.
\]
If \( q = 3 \), then \( k = 4p + 9 \), \( 4p = k - 9 \), \( p = \frac{k - 9}{2} \). Thus
\[
c(u_1) = c(z_3) + 2p(k + q + 2p - 1)
= 12k + 27 + 2p(2k + 2p + 11) + 2p(k + 2p + 2)
= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n - 2}{2} - \frac{n - 1}{2} + 8.
\]
If \( q = 4 \), then \( k = 4p + 10 \), \( 4p = k - 10 \), \( 2p = \frac{k - 10}{2} \). Thus
\[
c(u_1) = c(z_4) + 2p(k + q + 2p - 1)
= 13k + 39 + 2p(2k + 2p + 13) + 2p(k + 2p + 3)
= 2k^2 - 4k + 9 = \frac{n^2}{2} - 3n + \frac{23}{2} = \binom{n - 2}{2} - \frac{n - 1}{2} + 8.
\]
Thus Claim 3.8 holds and hence \( ac'(P_n) \leq ac'(c) = \left(\frac{n-2}{2}\right) - \frac{n-1}{2} + 8 \) for all odd integers \( n \geq 15 \).

Secondly, for \( n = 13 \), in the above proof we take \( p = 0 \) and \( q = 0 \). Namely, \( V_2 = V_3 = \emptyset \), \( V(P_{13}) = V_1 = \{x_1', x_2', x_3', x_4'; x_2, x_1; x_0; y_1, y_2; y_4', y_3', y_2', y_1'\} \) (also see Figure 2 and let \( p = q = 0 \)). Then coloring \( c_{|v\in V_1}(v) \) is a nearly antipodal coloring for \( P_{13} \). Thus by Claim 3.5, \( ac'(P_{13}) \leq ac'(c_{|v\in V_1}) = \max_{v\in V_1} c(v) = c(y_2) = (8k + 8)|_{k=6} = \left(\frac{13-2}{2}\right) + 1. \) Since \(-\left\lfloor \frac{13}{13} \right\rfloor = -1\) for \( n = 13 \), it follows that \( ac'(P_{13}) \leq ac'(c_{|v\in V_1}) = \left(\frac{13-2}{2}\right) + 1 = \left(\frac{13}{2}\right) - \frac{13-1}{2} = \frac{13}{2} + 8 \).

Thus the assertion 2 in Theorem 2.1 holds.

4. Examples

In this section we give some examples which present the nearly antipodal coloring \( c \) for some \( P_n \) with \( ac'(c) \) presented in Theorem 2.1 by our methods.

Example 4.1. A nearly antipodal coloring \( c \) for \( P_{10} \) with \( ac'(c) = \left(\frac{10-2}{2}\right) - \frac{10}{2} - \left\lfloor \frac{10}{10} \right\rfloor + 7 = \left(\frac{10-2}{2}\right) + 1 = 29 \) (see Figure 3).

\[
\begin{array}{cccccccc}
5 & 12 & 25 & 18 & 1 & 8 & 22 & 13 \\
x_1' & x_2' & x_3' & x_2 & x_1 & y_1 & y_2 & y_3'
\end{array}
\]

Figure 3. A nearly antipodal coloring for \( P_{10} \).

Example 4.2. A nearly antipodal coloring \( c \) for \( P_{13} \) with \( ac'(c) = \left(\frac{13-2}{2}\right) - \frac{13-1}{2} - \left\lfloor \frac{13}{13} \right\rfloor + 8 = \left(\frac{13-2}{2}\right) + 1 = 56 \) (see Figure 4).

\[
\begin{array}{cccccccc}
6 & 19 & 30 & 50 & 39 & 12 & 1 & 24 & 56 \\
x_1' & x_2' & x_3' & x_4 & x_2 & x_1 & x_0 & y_1 & y_2 & y_3' & y_2' & y_1
\end{array}
\]

Figure 4. A nearly antipodal coloring for \( P_{13} \).

Example 4.3 A nearly antipodal coloring \( c \) for \( P_{32} \) with \( ac'(c) = \left(\frac{32-2}{2}\right) - \frac{32}{2} + 7 = \left(\frac{32-2}{2}\right) - 9 = 426 \) (see Figure 5).
Here $n = 2k + 1 = 13 + 2(4p + q) = 33$, then $k = 16$, $p = 2$ and $q = 2$.

**Example 4.4.** A nearly antipodal coloring $c$ for $P_{33}$ with $ac'(c) = \frac{(33-2)}{2} - \frac{33-1}{2} + 8 = \frac{(33-2)}{2} - 8 = 457$ (see Figure 6).

Here $n = 2k + 1 = 13 + 2(4p + q) = 33$, then $k = 16$, $p = 2$ and $q = 2$.

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**References**


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