GENERALISED IRREDUNDANCE IN GRAPHS: NORDHAUS-GADDUM BOUNDS

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Abstract

For each vertex \( s \) of the vertex subset \( S \) of a simple graph \( G \), we define Boolean variables \( p = p(s, S), q = q(s, S) \) and \( r = r(s, S) \) which measure existence of three kinds of \( S \)-private neighbours (\( S \)-pns) of \( s \). A 3-variable Boolean function \( f = f(p, q, r) \) may be considered as a compound existence property of \( S \)-pns. The subset \( S \) is called an \( f \)-set of \( G \) if \( f = 1 \) for all \( s \in S \) and the class of \( f \)-sets of \( G \) is denoted by \( \Omega_f(G) \). Only 64 Boolean functions \( f \) can produce different classes \( \Omega_f(G) \), special cases of which include the independent sets, irredundant sets, open irredundant sets and CO-irredundant sets of \( G \).

Let \( Q_f(G) \) be the maximum cardinality of an \( f \)-set of \( G \). For each of the 64 functions \( f \), we establish sharp upper bounds for the sum \( Q_f(G) + Q_f(\overline{G}) \) and the product \( Q_f(G)Q_f(\overline{G}) \) in terms of \( n \), the order of \( G \).

Keywords: graph, generalised irredundance, Nordhaus-Gaddum.

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1. Introduction

Generalised irredundant sets were defined in [2]. We repeat the definition here for completeness but omit motivation which may be found in [2]. The open (closed) neighbourhood of the vertex subset \( S \) of a simple graph \( G = (V, E) \) is denoted by \( N(S) \) (\( N[S] \)) and as usual, for \( s \in V, N(\{s\}) \) and \( N[\{s\}] \) are abbreviated to \( N(s) \) and \( N[s] \).
The basic ingredients of the definition of generalised irredundant sets are three properties which make a vertex \( s \) (informally) important in a vertex subset \( S \) of a graph \( G \). It will also help the intuition to replace the word “important” by “essential” or “non-redundant.” Each property depends on the existence of one of the three types of \( S \)-private neighbour (\( S \)-pn) \( t \) for \( s \), which we now formally define.

For \( s \in S \), vertex \( t \) is an:

(i) \( S \)-self private neighbour (\( S \)-spn) of \( s \) if \( t = s \) and \( s \) is an isolated vertex of \( G \mid S \),

(ii) \( S \)-internal private neighbour (\( S \)-ipn) of \( s \) if \( t \in S - \{s\} \) and \( N(t) \cap S = \{s\} \),

(iii) \( S \)-external private neighbour (\( S \)-epn) of \( s \) if \( t \in V - S \) and \( N(t) \cap S = \{s\} \).

Observe that each such \( t \) is an element of \( N[s] - N(S - \{s\}) \) and that no \( s \in S \) may have \( S \)-pns of both type (i) and type (ii).

For \( s \in S \) let \( p(s, S) \), \( q(s, S) \), \( r(s, S) \) be Boolean Variables which take the value 1 if and only if \( s \) has an \( S \)-pn of type (i), (ii), (iii) respectively. Whenever possible we use the abbreviations \( p \), \( q \), \( r \) for these variables. Further let \( S(s) = (p(s, S), q(s, S), r(s, S)) \). Observe that for all \( s \) and \( S \), \( p(s, S) \cap q(s, S) = 0 \), i.e., the three Boolean variables are not independent and \( S(s) \) is never \((1, 1, 0) \) or \((1, 1, 1) \).

**Example 1.** Consider the vertex subset \( S = \{a, b, c, d\} \) of the graph \( G \) depicted in Figure 1. The \( S \)-pns of vertices of \( S \) are tabulated in Table 1 and we observe

\[
S(a) = (0, 1, 1), \quad S(b) = (0, 0, 0), \quad S(c) = (0, 0, 1), \quad S(d) = (1, 0, 1).
\]

We are now ready to define generalised irredundant sets. Let \( f \) be a Boolean function of the three variables \( p(s, S) \), \( q(s, S) \), \( r(s, S) \).

**Definition.** The vertex subset \( S \) of \( G \) is an \( f \)-set of \( G \) if for each \( s \in S \)

\[
f(S(s)) = f(p(s, S), q(s, S), r(s, S)) = 1.
\]

The function \( f \) may be viewed as a compound existence/non-existence property of the three types of \( S \)-pn. The class of all \( f \)-sets of \( G \) will be denoted by \( \Omega_f(G) \) (abbreviated to \( \Omega_f \) whenever possible).
Figure 1. Graph $G$ for Example 1

Table 1. $S$-pns of vertices of $S$ for graph $G$ for Example 1.

<table>
<thead>
<tr>
<th></th>
<th>type(i)</th>
<th>type(ii)</th>
<th>type(iii)</th>
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<tbody>
<tr>
<td>$a$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$b$</td>
<td></td>
<td>$b,c$</td>
<td></td>
</tr>
<tr>
<td>$c$</td>
<td></td>
<td></td>
<td>$e$</td>
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<tr>
<td>$d$</td>
<td></td>
<td></td>
<td>$g$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$h,i$</td>
</tr>
</tbody>
</table>

The rows of the truth table of $f$ will be labelled $0, \ldots, 7$, so that the entry in row $i$ is $f(p,q,r)$, where $pqr$ is the binary representation of the integer $i$ (e.g., $f(1,0,1)$ is the fifth entry in the table). Recall that for each $s \in S$, $S(s)$ is never equal to $(1,1,0)$ or $(1,1,1)$. We deduce:

(a) If the only 1’s in the truth table for $f$ occur in rows 6 or 7, then $\Omega_f = \emptyset$.

(b) If $f'$ is formed from $f$ by replacing the values in rows 6 and 7 by 0’s, then $\Omega_f' = \Omega_f$.

Thus we will only be concerned with the set $F$ of 64 functions with 0’s in rows 6 and 7. Two of these are in fact rather uninteresting since $f = 0$ gives $\Omega_f = \emptyset$ and the function $g$ with 1’s in all rows 0, 1, \ldots, 5 has $\Omega_g$ equal to the class of all subsets of $V$.

The functions of $F$ will be numbered (as in [4]) as follows. Let $a_0a_1a_2a_3a_4a_5$ be the binary representation of $i$. Then $f_i$ is defined to be the
function with entries $a_0a_1a_2a_3a_4a_5$ in rows 0 through 5, respectively. Note that $F = \{f_0, \ldots, f_{63}\}$.

We now list four special classes of $f$-sets. Additional examples may be found in [2, 4].

**Example 2.**

(i) The function $p$
The truth table column is 0, 0, 0, 1, 1, 0, 0. Since 3 (decimal) = 00011 (binary), $p = f_3$. The subset $S$ of $V(G)$ is an $f$-set of $G$ if and only if each $s \in S$ is isolated in $G[S]$, i.e., $S$ is independent in $G$. Thus $\Omega_p = \Omega_{f_3}$ is precisely the class of independent sets of $G$.

(ii) The function $p \lor r$
The truth table column is 0, 1, 0, 1, 1, 0, 0. Since 010111 (binary) = 23 (decimal), $p \lor r = f_{23}$. Then $S \subseteq V(G)$ is an $f_{23}$-set of $G$ if and only if each $s \in S$ is isolated in $G[S]$ or has an $S$-epn, i.e., $S$ is an irredundant set of $G$ (originally defined in [7]). Hence $\Omega_{f_{23}}$ is precisely the class of irredundant sets of $G$. See [18] for a bibliography of over 100 papers concerning irredundance.

(iii) The function $p \lor q \lor r$
The truth table column is 0, 1, 1, 1, 1, 0, 0. So that $p \lor q \lor r = f_{31}$. Each vertex of an $f_{31}$-set $S$ has at least one $S$-epn, i.e., $\Omega_{f_{31}}$ is the class of CO-irredundant sets which are defined in [14] and studied in [8, 9, 12, 21].

(iv) The function $r$
The truth table column is 0, 1, 0, 1, 0, 1, 0, 0. Since (010101) binary = 21 (decimal), $r = f_{21}$. The subset $S$ is an $f_{21}$-set if each $s \in S$ has an $S$-epn. Such sets (called open irredundant) were introduced in [14] and applied to broadcast networks. They are also known as $OC$-irredundant sets and have been studied in [1, 2, 3, 5, 13, 15, 16, 17, 19].

In view of Example 2, we regard each $\Omega_f$ as a class of generalised irredundant sets.

In [2, 4] the hereditary classes among the $\Omega_f$’s were determined and Ramsey properties of the classes were investigated.

Let $Q_i(G)$ be the maximum cardinality of an $f_i$-set of $G$. Wherever possible we abbreviate $Q_i(G)$, $Q_i(G)$ to $Q_i$, $\overline{Q}_i$ respectively. In this paper we determine Nordhaus-Gaddum type bounds (see [20]) for these parameters.
More specifically for each $i = 1, \ldots, 63$ we find upper bounds for 
\[ \max_G (Q_i + \overline{Q}_i) \quad \text{and} \quad \max_G (Q_i \overline{Q}_i) \]
where the maximum is taken over all $n$ vertex graphs $G$. The bounds are 
attained for an infinite number of values of $n$.

2. The Bounds

The Nordhaus-Gaddum bounds for the 63 non-zero values of $i$, will be given 
in Theorems 3, 5 and 11. We first state an obvious Lemma.

**Lemma 1.** If $f_i \implies f_j$, then for any graph $G$, $Q_i \leq Q_j$.

**Theorem 1.** If $i \geq 32$ and $n \geq 5$, then 
\[ \max_G (Q_i + \overline{Q}_i) = 2n \quad \text{and} \quad \max_G (Q_i \overline{Q}_i) = n^2. \]

**Proof.** If $i \geq 32$, then $f_{32} \implies f_i$, so that for all $G$ (using Lemma 1) 
$Q_{32} \leq Q_i \leq n$ and $\overline{Q}_{32} \leq \overline{Q}_i \leq n$. Hence 
\[ Q_{32} + \overline{Q}_{32} \leq Q_i + \overline{Q}_i \leq 2n \]
and 
\[ Q_{32} \overline{Q}_{32} \leq Q_i \overline{Q}_i \leq n^2. \]

However for $n \geq 5$, $Q_{32}(C_n) = Q_{32}(\overline{C}_n) = n$ and the result follows.

We next use the Nordhaus-Gaddum bounds for standard irredundant (i.e., 
f$_{23}$-) sets obtained by Cockayne and Mynhardt [10] to deduce the same 
bounds for other values of $i$.

**Theorem 2 ([10]).** If $n \geq 3$, then for any graph $G$ 
\[ Q_{23} + \overline{Q}_{23} \leq n + 1 \quad \text{and} \quad Q_{23} \overline{Q}_{23} \leq \left\lceil \frac{n^2 + 2n}{4} \right\rceil. \]

**Theorem 3.** If $n \geq 5$ and $i \in \{2, 3, 6, 7, 18, 19, 22, 23\}$, then 
\[ \max_G (Q_i + \overline{Q}_i) = n + 1 \quad \text{and} \quad \max_G (Q_i \overline{Q}_i) = \left\lceil \frac{n^2 + 2n}{4} \right\rceil. \]
Proof. If \( i \in \{2, 3, 6, 7, 18, 19, 22, 23\} \), then \( f_2 \implies f_i \implies f_{23} \) hence by Lemma 1 and Theorem 3
\[
Q_2 + Q_2 \leq Q_1 + Q_1 \leq Q_{23} + Q_{23} \leq n + 1
\]
and
\[
Q_2 Q_2 \leq Q_1 Q_1 \leq Q_{23} Q_{23} \leq \left\lceil \frac{n^2 + 2n}{4} \right\rceil.
\]
Consider the graph \( H \) which consists of a set \( X \) of \( \lfloor \frac{n+1}{2} \rfloor \) vertices, a set \( Y \) of \( \lceil \frac{n+1}{2} \rceil \) vertices (where \( X \cap Y = \{x\} \)), the edges to make \( H[Y] \) complete and a matching joining the vertices of \( X - \{x\} \) to \( Y - \{x\} \). In the case where \( n \) is even, an edge is added between the vertex of \( Y \) which was not previously matched and any vertex of \( X - \{x\} \).

Since each vertex of an \( f_2 \)-set \( S \) is a \( S \)-spn and has no \( S \)-epn, it is easily seen that \( X, Y \) are \( f_2 \)-sets of \( H \) respectively and so \( Q_2(H) \geq |X| \) and \( Q_2(H) \geq |Y| \). Hence for \( H \) all of the above inequalities are equalities and the result follows.

We now proceed in a similar manner using the bounds for CO-irredundant (i.e., \( f_{31} \)-) sets established by Cockayne, McCrea and Mynhardt [9].

**Theorem 4** ([9]). For any graph \( G \),
\[
Q_{31} + Q_{31} \leq n + 2 \quad \text{and} \quad Q_{31} Q_{31} \leq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor.
\]

**Theorem 5.** If \( 8 \leq i \leq 15 \) or \( 24 \leq i \leq 31 \), then
\[
\max_G (Q_i + Q_i) \leq n + 2, \quad \max_G (Q_i Q_i) \leq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor
\]
and these bounds are attained for \( n \equiv 2 \pmod{4}, n \geq 6 \).

**Proof.** For any \( i \) satisfying \( 8 \leq i \leq 15 \) or \( 24 \leq i \leq 31 \), \( f_8 \implies f_i \implies f_{31} \). Thus, by Lemma 1, for any \( G \),
\[
Q_8 Q_8 \leq Q_i Q_i \leq Q_{31} Q_{31} \leq \left\lfloor \frac{(n+2)^2}{4} \right\rfloor
\]
and
\[
Q_8 + Q_8 \leq Q_i + Q_i \leq Q_{31} + Q_{31} \leq n + 2.
\]
Thus the bounds of the theorem are established. Now let \( n \equiv 2 \pmod{4} \) and \( n \geq 6 \). Let the graph \( H \) consist of vertex sets \( X \) and \( Y \) where \( |X| = |Y| = (n + 2)/2 \) and \( |X \cap Y| = 2 \). Add edges so that \( H[X] \) and \( H[Y] \) are both isomorphic to \((\frac{n+2}{2})K_2\) and add a matching from \( X - Y \) to \( Y - X \).

Since a subset \( S \) is an \( f_8 \)-set if each vertex has an \( S \)-ipn and no \( S \)-epn, it is easily seen that \( X, Y \) are \( f_8 \)-sets of \( H \), respectively. Therefore \( H \) attains the bounds.

In order to find the bounds for the remaining values of \( i \), it will be necessary to improve the following result of Cockayne [3] concerning open irredundant (i.e., \( f_21 \)-) sets. A set \( S \) is an \( f_21 \)-set if each \( s \in S \) has an \( S \)-epn.

**Theorem 6 ([3]).** For any graph \( G \) with \( n \geq 16 \),

\[
Q_{21} + \overline{Q}_{21} \leq \left\lfloor \frac{3n}{4} \right\rfloor .
\]

Further if \( n \geq 17 \), then

\[
Q_{21} \overline{Q}_{21} < \frac{9n^2}{64}.
\]

We show that for larger \( n \), the second bound of Theorem 6 can be improved to \( n^2/8 \). This will be accomplished by more detailed analysis of the various cases used in the proof of Theorem 6 given in [3]. Some of the details of our proof may be found in [3] but must be repeated here for completeness.

Let \( X(Y) \) be open irredundant sets of \( G(\overline{G}) \), \( |X| = x \) and \( |Y| = y \). Each \( u \in X(v \in Y) \) has an at least one \( X \)-epn in \( G \) (\( Y \)-epn in \( \overline{G} \)). Let \( u_r(v_b) \) be any \( X \)-epn of \( u \) in \( G \) (\( Y \)-epn of \( v \) in \( \overline{G} \)). The edges of \( G \) (resp. \( \overline{G} \)) will be coloured red (blue). Occasionally \( u_r(v_b) \) will be called a red epn of \( u \) (blue epn of \( v \)). Let \( X' = \{u_r|u \in X\} \). Then each edge of \( \{u_u|u \in X\} \) is red while all other edges joining \( X \) to \( X' \) are blue. Hence the set \( \{u_u|u \in X\} \) induces a matching in \( G \). Similarly, it can be seen that, the set \( \{v_v|v \in Y\} \) induces a matching in \( \overline{G} \). Note that the set \( X' \) is also an open irredundant set of \( G \) and \( u \) is an \( X' \)-epn of \( u_r \) in \( G \). Let \( Z = V - (X \cup X') \).

The principal result will follow immediately from three propositions which are broken down into cases depending on the distribution of vertices of \( Y \) and blue epns among the three sets \( X, X', Z \).

The open irredundance property implies that both \( x \) and \( y \) are at most \( n/2 \). From this we deduce that \( xy \leq \frac{n^2}{4} \) if \( x \) (or \( y \)) \( \leq \frac{n}{4} \). Hence it is sufficient to establish the propositions under the assumption \( x, y > \frac{n}{4} \) and we use this
hypothesis in the proofs without further emphasis. We also repeatedly use
the following obvious fact.

**Lemma 2.** Let $A$ be an open irredundant set in a graph $F$ and $B \subseteq V(F)$. If each $u \in A \cap B$ has $A$-epn in $B$, then $|A \cap B| \leq |B|/2$.

**Proposition 7.** If $n \geq 32$ and $|Y \cap X| \geq 3$, then $xy \leq n^2/8$.

**Proof.** Since $|Y \cap X| \geq 3$, for each $u \in Y \cap X$, $u \notin X'$. Hence $u_b \in X \cup Z$. Define

$$
X_1 = \{u \in Y \cap X | u_b \in X\},
X_2 = \{u \in Y \cap X | u_b \in Z\},
X_3 = X - (X_1 \cup X_2)
$$

and for $i = 1, 2, 3$, let $|X_i| = x_i$.

For $w \in Y \cap Z$, $w_b \notin X_1 \cup X_2 \cup X'$, hence $w_b \in X_3 \cup Z$.

**Case 1.** $Y \cap X' = \emptyset$.
Let $t = |\{w \in Y \cap Z | w_b \in X_3\}|$. Then by Lemma 2

$$(1) \quad |\{w \in Y \cap Z | w_b \in Z\} \leq (n - 2x - x_2 - t)/2.
$$

We will now give more detailed justification for (1). Similar explanations will be omitted in future cases of the propositions. Define

$$
B = Z - (\{w \in Y \cap Z | w_b \in X_3\} \cup \{w_b \in Z | w \in X_2\})
$$

(disjoint Union).

Note that $|B| = (n - 2x - x_2 - t)$ and

$$
\{w \in Y \cap Z | w_b \in Z\} = \{w \in Y \cap B | w_b \in B\}.
$$

Then (1) follows by applying Lemma 2 with $A = Y$. 
Now

\[ x + y = x + |Y \cap X| + |Y \cap Z| \]

(2)

\[ \leq x + (x_1 + x_2) + t + \left(\frac{n - 2x - x_2 - t}{2}\right) \]

\[ = x_1 + \frac{x_2}{2} + \frac{t}{2} + \frac{n}{2}. \]

The blue epns in \( X_3 \) are distinct and so \( x_3 \geq t + x_1 \), i.e.,

(3)

\[ \frac{t}{2} \leq \frac{x_3}{2} - \frac{x_1}{2}. \]

From (2) and (3) we obtain

\[ x + y \leq \left(\frac{x_1 + x_2 + x_3}{2}\right) + \frac{n}{2} = \frac{x}{2} + \frac{n}{2}. \]

Therefore \( y \leq \frac{n}{2} - \frac{x}{2} \) and \( xy \leq \frac{nx}{2} - \frac{x^2}{2} \). By elementary calculus, \( xy \) attains its maximum \( \frac{n^2}{8} \) when \( x = \frac{n}{2} \).

**Case 2.** \(|Y \cap X'| \geq 2.\)

In this case \( x_1 = 0 \), each \( w \in Y \cap Z \) has \( w_b \in Z \) and for each \( w \in Y \cap X' \), \( w_b \notin X' \) i.e., \( w_b \in X_3 \cup Z \).

**Subcase 2(a).** \( w \in Y \cap X' \) has \( w_b \in X_3 \).

This implies \(|Y \cap X'| = 2.\) Let \( Y \cap X' = \{w, v\} \). Now

\[ x + y = x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \]

\[ \leq x + x_2 + 2 + \frac{(n - 2x - x_2 - \lambda)}{2} \]

where \( \lambda = 1 \) (resp. 0) if \( v_b \in Z(X_3) \). Hence

(4)

\[ x + y \leq \frac{n}{2} + \frac{x_2}{2} - \frac{\lambda}{2} + 2. \]

By counting blue epns in \( X_3 \), we obtain \( x_3 \geq 2 - \lambda \) and since \( |Z| \geq x_2 \), we deduce \( x_2 \leq n - 2x \). Use of these gives

\[ x_2 \leq n - 2(x_1 + x_2 + x_3) = n - 2(x_2 + x_3). \]
Therefore

\[ x_2 \leq \frac{n - 2x_3}{3} \leq \frac{n - 4 - 2\lambda}{3}. \]

From (4) and (5)

\[ x + y \leq \frac{2n + 4}{3} - \frac{5\lambda}{6} \leq \frac{2n + 4}{3}, \]

so that \( xy \leq x \left( \frac{2n+4}{3} - x \right) \). Calculus shows that \( xy \leq \left\lfloor \left( \frac{n+2}{3} \right)^2 \right\rfloor \leq \frac{n^2}{8} \) \((n \geq 32)\).

**Subcase 2(b).** Each \( w \in Y \cap X' \) has \( w_b \in Z \).
In this situation every \( v \in Y \) has \( v_b \in Z \). Therefore \( y \leq |Z| = n - 2x \) and \( xy \leq nx - 2x^2 \). The maximum of this for \( x \in \left[ \frac{n}{4}, \frac{n}{2} \right] \) is \( \frac{n^2}{8} \).

**Case 3.** \( |Y \cap X'| = \{ v \} \).
Define \( \lambda \) as in subcase 2(a) and let \( \mu (= 0 \text{ or } 1) \) be the number of vertices in \( Y \cap Z \) with blue epns in \( X_3 \).

The set \( Z \) contains \( \lambda + x_2 \) blue epns of vertices in \( Y \cap (X \cup X') \) and \( \mu \) vertices of \( Y \cap Z \) have blue epns in \( X_3 \). Hence using Lemma 2 we obtain

\[ x + y = x + |Y \cap X| + |Y \cap X'| + |Y \cap Z| \]

\[ \leq x + (x_1 + x_2) + 1 + \mu + \left( \frac{n - 2x - \mu - x_2 - \lambda}{2} \right) \]

\[ = \frac{n}{2} + x_1 + \frac{x_2}{2} + \frac{\mu - \lambda}{2} + 1. \]

By counting blue epns in \( X_3 \) we obtain \( x_3 \geq (1 - \lambda) + x_1 + \mu \) and since \( |Z| \geq x_2 \) we have \( x_2 \leq n - 2x \). Use of these gives

\[ x_2 \leq n - 2 \left( x_1 + x_2 + x_3 \right). \]

Hence

\[ x_2 \leq \frac{n - 2(x_1 + x_3)}{3} \]

\[ \leq \frac{n - 2x_1 - 2[(1 - \lambda) + x_1 + \mu]}{3} \]

\[ = \frac{n - 4x_1 - 2 - 2(\mu - \lambda)}{3}. \]
Combining (6) and (7) we obtain
\[ x + y \leq \frac{2n + 2}{3} + \frac{x_1}{3} + \frac{\mu - \lambda}{6}. \]

However hypothesis and the private neighbour property imply that \( x_1 + \mu \leq 1 \). Hence from (8) we deduce
\[ x + y \leq \frac{2n + 3}{3} - \left( \frac{\lambda + \mu}{6} \right) \leq \frac{2n + 3}{3}. \]

Calculus shows that \( xy \leq \left( \frac{2n + 3}{6} \right)^2 \leq \frac{n^2}{8} \) \((n \geq 32)\). This completes the proof of Proposition 7.

**Proposition 8.** If \( n \geq 32 \) and \(|Y \cap X| \leq 2\), then \( xy \leq n^2/8\).

**Proof.** Define \( Y' = \{ v \in Y \mid v \notin X \} \). If \(|Y \cap X'| \) \((|Y' \cap X| \) or \(|Y' \cap X'|) > 2\), then we may apply Proposition 7 to the open irredundant sets \( Y, X' \) \((Y', X \) or \(Y', X')\) of \( \overline{G}, G\) and infer the result. Thus we assume that \(|Y \cap X'| \)|\(|Y' \cap X|\) and \(|Y' \cap X'|\) are at most two. Then
\[ n \geq |X| + |X'| + |Y' - Y \cap X| - |X' \cap Y'|. \]
\[ \geq 2x + 2y - 2 - 2 - 2. \]

Hence \( x + y \leq \frac{n + 8}{2} \) and therefore by elementary calculus \( xy \leq \left( \frac{n + 8}{4} \right)^2 \leq \frac{n^2}{8} \) \((n \geq 32)\).

The preceding propositions have established a bound for \( Q_{21} \).

**Theorem 9.** If \( n \geq 32 \), then \( Q_{21} \leq n^2/8 \).

**Proof.** Immediate from Propositions 7 and 8.

We now use Theorems 6 and 9 to determine exact Nordhaus-Gaddum bounds for the remaining values of \( i \).

**Theorem 10.** If \( n \geq 32 \) and \( i \in \{1, 4, 5, 16, 17, 20, 21\} \), then \( \max_G (Q_i + \overline{Q_i}) \leq 3n/4 \), \( \max_G (Q_i \overline{Q_i}) \leq n^2/8 \) and these bounds are attained for infinitely many values of \( n \).
Proof. For any \( i \in \{1, 4, 5, 16, 17, 20, 21\} \),
\[
f_1 \implies f_i \implies f_{21},
\]
\[
f_4 \implies f_i \implies f_{21}
\]
or
\[
f_{16} \implies f_i \implies f_{21}.
\]
Hence by Lemma 1, Theorems 6 and 9, for any \( G \)
\[
Q_j + \overline{Q}_j \leq Q_i + \overline{Q}_i \leq Q_{21} + \overline{Q}_{21} \leq \frac{3n}{4}
\]
and
\[
Q_j \overline{Q}_j \leq Q_i \overline{Q}_i \leq Q_{21} \overline{Q}_{21} \leq \frac{n^2}{8},
\]
where \( j \in \{1, 4, 16\} \). Thus the bounds of the theorem are established. To show that they are attained it is sufficient to exhibit for each \( j \in \{1, 4, 16\} \) graphs satisfying
\[
Q_j + \overline{Q}_j \geq \frac{3n}{4} \quad \text{and} \quad Q_j \overline{Q}_j \geq \frac{n^2}{8}.
\]
In order to describe the three examples we need the following definition. Let \( A, B \) be disjoint \( m \)-vertex subsets of a graph \( L \). We say there is an induced matching from \( A \) to \( B \) in \( L \) if the bipartite subgraph of \( L \) defined by \( A, B \) is isomorphic to \( nK_2 \).

We form the graph \( H \) as follows. Let \( V(H) = X \cup Y \cup Y' \) (disjoint union) where \( |X| = \frac{n}{2} \) where \( n \equiv 0 \pmod{4}, n \geq 32, |Y| = |Y'| = \frac{n}{4} \) and \( X' = Y \cup Y' \). Add edges so that there are induced matchings from \( X \) to \( X' \) in \( H \) and from \( Y \) to \( Y' \) in \( H \).

Each of the three examples will be formed by adding edges to \( H \). For each of the three values of \( j \) it is easily checked that \( X \) and \( Y \) are \( f_j \)-sets of the constructed graph \( H^* \) and \( \overline{H^*} \) respectively, so that \( H^* \) satisfies (9). In each case we remind the reader of the \( f_j \)-set definition.

\( j = 1 \) : Subset \( S \) is an \( f_1 \)-set if each \( s \in S \) is a \( S-spm \) and has an \( S-epm \). Form \( H^* \) from \( H \) by adding edges so that \( H^*[Y] \) is complete.

\( j = 4 \) : Subset \( S \) is an \( f_4 \)-set if each \( s \in S \) has both an \( S-ipm \) and an \( S-epm \). In this case we require \( n \equiv 0 \pmod{8} \). Form \( H^* \) from \( H \) by adding edges so that \( H^*[X] \) and \( \overline{H^*}[Y] \) are isomorphic to \( \frac{n}{2}K_2 \) and \( \frac{n}{8}K_2 \), respectively.
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\[ j = 16 : \text{Subset } S \text{ is an } f_{16}\text{-set if each } s \in S \text{ has an } S\text{-epn, has no } S\text{-imn} \]
\[ \text{and is not an } S\text{-spn. Form } H^* \text{ from } H \text{ by adding edges so that } H^*[X] \text{ and } H^*[Y] \text{ are isomorphic to } C_2^n \text{ and } C_4^n \text{ respectively.} \]

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References


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