SOME RESULTS CONCERNING THE ENDS OF MINIMAL CUTS OF SIMPLE GRAPHS

Xiaofeng Jia

Department of Mathematics
Taiyuan University of Technology (West Campus)
Taiyuan, Shanxi, P.R. China 030024

Abstract

Let $S$ be a cut of a simple connected graph $G$. If $S$ has no proper subset that is a cut, we say $S$ is a minimal cut of $G$. To a minimal cut $S$, a connected component of $G - S$ is called a fragment. And a fragment with no proper subset that is a fragment is called an end.

In the paper ends are characterized and it is proved that to a connected graph $G = (V, E)$, the number of its ends $\Sigma \leq |V(G)|$.

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In this paper $G = (V, E)$ will always denote finite non-complete connected graph. The notations not mentioned are the same with those in reference [1]. For $A \subset V(G)$ we use $\Gamma(A)$ to denote the adjacent set of $A$, that is, $\Gamma(A) = \{v|uv \in E(G), u \in A\}$, and we put $N(A) = \Gamma(A) - A$. $\langle A \rangle$ is a subgraph induced by $A \subset V(G)$, that is, $\langle A \rangle = G[A]$. A set of vertices $S \subset V(G)$ is called a cut of $G$ if there are at least two connected components in $G - S$. A minimal cut is a cut without a proper subset that is a cut. If $S$ is a minimal cut, then a connected component of $G - S$ is called a fragment. And a fragment with no proper subset that is a fragment is called an end.

It is obvious that the graphs we discuss all have cuts and furthermore, minimal cuts. Thus a graph has at least two fragments. Since all fragments have ends, a graph has at least two ends.

It is obvious that:
Proposition P. Let $S$ be a cut of $G$. Then $S$ is a minimal cut if and only if for any $u \in S$ and a connected component of $G - S$, say $\langle A \rangle$, $N(u) \cap A \neq \emptyset$.

Definition 1. If $S_1$ and $S_2$ are two minimal cuts of $G$ and there are at least two connected components of $G - S_1$ which contain vertices of $S_2$, then $S_1$ interferes with $S_2$.

Theorem 2. Let $S_1$ and $S_2$ be two minimal cuts of $G$ and $S_1$ interferes with $S_2$, then there exist vertices of $S_1$ in every fragment of $G - S_2$.

Proof. Suppose $\langle A \rangle$ is a fragment of $G - S_2$ and $A \cap S_1 = \emptyset$, then because $\langle A \rangle$ is connected, all the vertices in $A$ must belong to a fragment of $G - S_1$. By Proposition P, for each $v \in S_2 - S_1$, $N(v) \cap A \neq \emptyset$, then $\langle A \cup (S_2 - S_1) \rangle$ is connected. Thus the vertices which belong to $S_2 - S_1$ can only be in one fragment of $G - S_1$, and it contradicts the fact that $S_1$ interferes with $S_2$.

By Theorem 2, $S_1$ interferes with $S_2$ and $S_2$ interferes with $S_1$ are equivalent assertions.

Theorem 3. Let $\langle A \rangle$ be an end of $G$, then for any $u \in A$, every minimal cut of $G$ that contains $u$ interferes with $N(A)$.

Proof. Suppose $S$ is a minimal cut of $G$ that contains $u$ and does not interfere with $N(A)$, then by applying Theorem 2 and Definition 1, $S - N(A) \subset A$. And since $N(u) \subset (A \cup N(A))$, from Proposition P, every fragment in $G - S$ contains a vertex in $A$ or $N(A)$. Since $S$ does not interfere with $N(A)$, there is at least one of such fragments that does not contain any vertex in $N(A)$. Thus it contains vertices in $A$ and only in $A$. But this fragment does not contain $u$, then it is a proper subset of $A$, contradicting the fact that $\langle A \rangle$ is an end.

Corollary 4. Let $S$ be a minimal cut of $G$. If $G$ does not contain any minimal cuts that interfere with $S$, then a vertex in $S$ cannot belong to any end of $G$.

Theorem 5. Suppose $\langle A \rangle$ is a fragment of $G$, then $\langle A \rangle$ is also an end of $G$ if and only if $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$.

Proof. (a) From Theorem 3, if $\langle A \rangle$ is an end of $G$, then $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$. 
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(b) For every non-empty proper subset of $A$, say $A'$, since $N(A') \subset A \cup N(A)$, and $N(A)$ is the only minimal cut that is contained by $A \cup N(A)$, then $N(A')$ is not a minimal cut, thus $\langle A' \rangle$ is not a fragment and $\langle A \rangle$ is an end.

**Theorem 6.** Let $\langle A \rangle$ be a fragment of $G$, then $\langle A \rangle$ is an end if and only if for any $u \in A$ and $v \in N(A)$, $uv \in E(G)$.

**Proof.** (a) Suppose $A$ is an end. There are $u \in A$, $v \in N(A)$ and $uv \notin E$. Consider all the $u-v$ paths in $G$. There must exist vertices of $(A \cup N(A)) - \{u, v\}$ in every such path. Delete all the vertices in $(A \cup N(A)) - \{u, v\}$ from each path, then we obtain a disconnected graph with no $u-v$ paths, thus the vertices we deleted are a cut of $G$. But this cut is contained by $A \cup N(A)$ and it is not $N(A)$, contradicting Theorem 5.

(b) Let $A'$ be a proper subset of $A$. It is obvious that $N(A') \subset (A \cup N(A))$ and $N(A') \cap (A - A') \neq \emptyset$ under the conditions of the theorem, then $N(A')$ is not a minimal cut. Thus $\langle A' \rangle$ is not a fragment, then $\langle A \rangle$ is an end.

**Corollary 7.** Let $\langle A \rangle$ be an end of $G$, then all the minimal cuts that interfere with $N(A)$ contain $A$.

**Proof.** Let $S$ be a minimal cut that interferes with $N(A)$. If there exist $v \in A$ and $v \notin S$, then there must exist a fragment of $G - S$ that contains at least one vertex $u \in N(A)$ and does not contain vertex $v$, which contradicts Theorem 6.

**Theorem 8.** Let $\langle A \rangle$ be an end of $G$ and $\langle B \rangle$ be a fragment of $G$ that does not contain $A$, then $A \cap B = \emptyset$.

**Proof.** Under the conditions of the theorem, if $A \cap B \neq \emptyset$, then since $A - B \neq \emptyset$ and $\langle A \rangle$ is connected, we have $N(B) \cap A \neq \emptyset$.

Thus, if $N(B)$ interferes with $N(A)$, by applying Corollary 7, $A \subset N(B)$. It contradicts $A \cap B \neq \emptyset$. If $N(B)$ does not interfere with $N(A)$, from $N(B) \cap A \neq \emptyset$ we have $N(B) \subset (A \cup N(A))$. If $N(B) \neq N(A)$, it will contradict Theorem 5.

From Theorem 8 we know, for any two distinct ends of $G$, $\langle A \rangle$ and $\langle B \rangle$, there must be $A \cap B = \emptyset$. Thus by denoting the number of distinct ends of $G$ as $\Sigma$, there is

**Corollary 9.** $\Sigma \leq |V(G)|$. 
References


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