A NOTE ON THE RAMSEY NUMBER AND THE PLANAR RAMSEY NUMBER FOR $C_4$ AND COMPLETE GRAPHS

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Abstract

We give a lower bound for the Ramsey number and the planar Ramsey number for $C_4$ and complete graphs. We prove that the Ramsey number for $C_4$ and $K_7$ is 21 or 22. Moreover we prove that the planar Ramsey number for $C_4$ and $K_6$ is equal to 17.

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1 Introduction

Let $F, G, H$ be simple graphs with at least two vertices. The Ramsey number $R(G, H)$ is the smallest integer $n$ such that in arbitrary two-colouring (say red and blue) of $K_n$ a red copy of $G$ or a blue copy of $H$ is contained (as subgraphs).

Let the planar Ramsey number $PR(G, H)$ be the smallest integer $n$ such that any planar graph on $n$ vertices contains a copy of $G$ or its complement contains a copy of $H$.

So we have an immediate inequality between planar and ordinary Ramsey number, i.e., $PR(G, H) \leq R(G, H)$.

Walker in [9] and Steinberg and Tovey in [8] studied the planar Ramsey number but only in the case when both graphs are complete.

In this paper we will only consider the case when $G$ is a cycle $C_4$ of order 4 and $H$ is a complete graph $K_t$ of order $t$. In that case one can say that the Ramsey number is the smallest integer $n$ such that any graph on $n$ vertices contains a copy of $C_4$ or an independent set of cardinality $t$. The
problem for the case when \( G \), i.e., the first graph of the pair, is a cycle has been studied by J.A. Bondy, P. Erdős in [3] and by P. Erdős, R.J. Faudree, C.C. Rousseau, R.H. Schelp in [6]. We give a lower bound for the Ramsey number and the planar Ramsey number for \( C_4 \) and complete graphs. We prove that the Ramsey number for \( C_4 \) and \( K_7 \) is 21 or 22.

Moreover in Theorem 6 we prove that \( PR(C_4, K_6) = 17 \).

A graph \( F \) is said to be a \((G, K_t)\)-Ramsey-free graph if it does not contain any copy of \( G \) and any independent set of cardinality \( t \). For graphs \( G, H \) the symbol \( G \cup H \) denotes a disjoint union of graphs, \( tG \) a disjoint union of \( t \) copies of the graph \( G \), \( \overline{G} \) a complement of \( G \), \( G - S \) a subgraph of \( G \) induced by a subset \( V(G) - S \) of the vertices of \( G \) where \( S \subseteq V(G) \), and \( G \supset H \) express the fact that a graph \( H \) is a subgraph of \( G \). Then \( \deg_G(x) \) denotes the degree of the vertex \( x \) in the graph \( G \), and \( \delta(G) \) is the minimum vertex degree over all vertices of \( G \). Moreover \( N(x) \) is the set of vertices adjacent to \( x \), and \( N[x] \) is the closed neighbourhood, i.e., \( N[x] = N(x) \cup \{x\} \).

The following theorems summarises the results for ordinary and planar Ramsey numbers known so far referring to the cases when the first graph is a cycle of order 4 and the second one is a complete graph.

**Theorem 1** [4], [5], [7].
(i) \( R(C_4, K_3) = 7; \)
(ii) \( R(C_4, K_4) = 10; \)
(iii) \( R(C_4, K_5) = 14; \)
(iv) \( R(C_4, K_6) = 18. \)

**Theorem 2** [1].
(i) \( PR(C_4, K_3) = 7; \)
(ii) \( PR(C_4, K_4) = 10; \)
(iii) \( PR(C_4, K_5) = 13. \)

## 2 Main Results

We use the following lemma to prove some further results for the Ramsey and the planar Ramsey number of pair of graphs.

**Lemma 3** [2]. Let \( G \) be a graph of order 17 with independence number less than 6 and without \( C_4 \). Then \( G \) is isomorphic to one of the graphs presented in Figure 1.

Therefore we have the following simple general observation.
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Figure 1. Graphs of order 17 without $C_4$ and with $\alpha(G) < 6.$
Proposition 4. For each integer $t \geq 6$, $R(C_4, K_{t+1}) \geq 3t + 2[\frac{t}{3}] + 1$.

Proof. Let $H$ be a graph of order 17 presented in Figure 1. Note that $H$ does not contain any subgraph $C_4$ and $\alpha(H) = 5$. Therefore $[\frac{t}{5}]H \cup (t - [\frac{t}{5}]5)K_3$, $t \geq 5$, shows that $R(C_4, K_{t+1}) \geq 3t + 2[\frac{t}{3}] + 1.$

Theorem 5. $21 \leq R(C_4, K_7) \leq 22.$

Proof. Immediately by Proposition 4 we get $21 \leq R(C_4, K_7)$. Suppose for the contrary that $R(C_4, K_7) > 22$. Let $G$ be a $(C_4, K_7)$-Ramsey-free graph of order 22. Note that $\delta(G) < 5$, else a $C_4$ should be a subgraph of $G$.

Let $m$ be an arbitrary vertex of $G$ of the minimum degree $\delta(G)$.

Suppose that $\delta(G) \leq 3$. Then deleting a 3-degree vertex $m$ and all its neighbours we get a graph $F$ of the order at least 18. By Theorem 1(iv) the graph $F$ contains an independent set $S$ of cardinality 6. Thus $S \cup \{m\}$ is an independent set of cardinality 7, a contradiction.

Therefore $\delta(G) = 4$. Let $m_i$, $i = 1, 2, 3, 4$ be the neighbours of $m$ in $G$. Let us consider the graph $F$ obtained from $G$ by deleting the vertex $m$ and all its neighbours. Since $G$ does not contain any $C_4$ then by degree condition each $m_i$, $i = 1, 2, 3, 4$ has at least two neighbours in $F$. Evidently the order of $F$ equals 17 and $F$ must be isomorphic to one of the $(C_4, K_6)$-Ramsey-free graphs presented in Figure 1 (else we get a contradiction as before).

Suppose that $F$ is isomorphic to $H_1$ or $H_2$. Since the vertex $w$ has degree 3 in $F$ then it must be adjacent to one of the neighbours of $m$, say $m_1$. Let us consider the graph $Y = G - N[w]$. Note that the vertex $m$ has degree 3 in $Y$. Hence $Y$ must be one of the $(C_4, K_6)$-Ramsey-free graphs $H_1$ or $H_2$ presented in Figure 1. Evidently $m$ is not adjacent to any vertex of the set $\{d, v, b, h\}$. Therefore each of the four vertices must be adjacent to a vertex of the set $\{m_2, m_3, m_4\}$. It is impossible without creating $C_4$ because each two vertices of the set $\{d, v, b, h\}$ are at distance 2. A contradiction. Therefore we can assume that $F$ is not isomorphic to $H_i$, $i = 1, 2$.

Suppose that $F$ is isomorphic to $B_1$. Let the vertex $x$ be adjacent to $m_1$. Then $1m_1 \in E(G)$, else $\deg(m_1) < 4$. Moreover without loss of generality $m_1m_2 \in E(G)$. Note that $\deg(m_1) = 4$. So we consider the graph $Y = G - N[m_1]$. Since $Y$ cannot be isomorphic to $H_i$, $i = 1, 2$ then each of the vertices of the set $\{2, 3, 4, 5\}$ must be adjacent to $m_3$ or $m_4$ and we get $C_4$, a contradiction. Therefore $xm_1 \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry, $ym_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Suppose that $f$ is adjacent to $m_1$. Since $C_4$ cannot be a subgraph then $b, v, e$ or 4 is not adjacent to $m_i$, $i = 2, 3, 4$. Therefore $\deg(m_1) > 4$, else
the graph $G - N[m_1]$ has a 3-degree vertex, so it should be isomorphic to $H_i$, $i = 1, 2$ and we get a case above. Then $m_1$ should be adjacent to 3 and $h$, and without loss of generality $m_1m_2 \in E(G)$. Note that $m_2$ can be adjacent to one of the vertices $d, 1$ or $u$. So $\deg(m_2) < 4$ or a $C_4$ exists, a contradiction.

Hence $fm_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $bm_i \notin E(G)$, for $i = 1, 2, 3, 4$.

If the vertex 2 is adjacent to $m_1$ then $\deg(m_1) > 4$, else $G - N[m_1]$ has a 3-degree vertex, and we get a case above. Then $m_1$ must be adjacent to $e$ and to one of $g, u$. Moreover without loss of generality $m_1m_2 \in E(G)$. Note that $\deg(m_2) < 4$ or a $C_4$ exists, a contradiction.

Hence $2m_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $cm_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Similar arguments give that 5 and $h$ cannot be adjacent to $m_i$, $i = 1, 2, 3, 4$.

Now without loss of generality we can assume that $m_1, m_2$ and $u$ create an independent set. Therefore $\{m_1, m_2, 2, 5, y, f, u\}$ is an independent set.

Suppose that $F$ is isomorphic to $B_2$. Let $g$ be adjacent to $m_1$. Then $m_1$ must be adjacent to 3 and $y$, and without loss of generality $m_1m_2 \in E(G)$, else the graph $G - N[m_1]$ has a 3-degree vertex, so it should be isomorphic to $H_i$, $i = 1, 2$ and we get a case above. So $m_2$ must be adjacent to 4 and $e$, and it has degree four. Therefore the vertices 5, $b, u, f$ must be adjacent to $m_3$ or $m_4$, else we get a 3-degree vertex in $G - N[m_2]$. Without loss of generality we can assume that the vertex $m_3$ is adjacent to $b, f$, and the vertex $m_4$ is adjacent to 5, $u$. Note that $m_4$ has only these two neighbours in $B_2$. Hence $m_4$ must be adjacent to $m_3$ and $\deg(m_4) = 4$. Since $h$ cannot be adjacent to $m_i$, $i = 1, 2, 3, 4$ the graph $G - N[m_4]$ has a 3-degree vertex and we get a case above.

Hence $gm_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $ym_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Let 2 be adjacent to $m_1$. Then $m_1$ should be adjacent to one of the vertices $u, a, b$. So the graph $G - N[m_1]$ contains a 3-degree vertex $g$ or $y$, and we get a case above. Hence $2m_i \notin E(G)$, for $i = 1, 2, 3, 4$. By symmetry $cm_i \notin E(G)$, for $i = 1, 2, 3, 4$.

Now without loss of generality we can assume that $m_1, m_2$ and 4 create an independent set. Therefore $\{m_1, m_2, 4, c, 2, g, y\}$ is an independent set.

Suppose that $F$ is isomorphic to $B_3$. Let $d$ be adjacent to $m_1$. Then $m_1$ must be adjacent to one of the vertices 3, $b, g, h$, and without loss of generality
$m_1m_2 \in E(G)$. Since $\text{deg}(m_1) = 4$ and $m_2$ cannot be adjacent to $3,h,f,u$, then the graph $G - N[m_1]$ has a 3-degree vertex, and we get a case above.

Hence $dm_i \notin E(G)$, for $i = 1,2,3,4$. By symmetry $gm_i \notin E(G)$, for $i = 1,2,3,4$.

Let $a$ be adjacent to $m_1$. Then $m_1$ must be adjacent to one of the vertices $u,f,4$. As before $\text{deg}(m_1) = 4$. Note that one of the vertices $2,b,h,y$ has 3-degree in $G - N[m_1]$, and we get a case above.

Hence $a$ and $4$ (by symmetry) cannot be adjacent to $m_i$, $i = 1,2,3,4$. Now without loss of generality we can assume that $m_1, m_2$ and $1$ create an independent set. Therefore $\{m_1, m_2, 1, 4, a, d, g\}$ is an independent set.

All cases lead to a contradiction.

For the planar case we get the following theorem.

**Theorem 6.** $PR(C_4, K_6) = 17$.

**Proof.** Since by Lemma 3 each $(C_4, K_6)$-Ramsey-free graph of order 17 is not planar and $R(C_4, K_6) = 18$ we get $PR(C_4, K_6) \leq 17$. The graph presented in Figure 2 is $(C_4, K_6)$-Ramsey-free planar graph. So $PR(C_4, K_6) > 16$.

![Figure 2](image-url)

Figure 2. A planar graph of order 16 with independence number less than 6 and without $C_4$.

**Proposition 7.** For each integer $t \geq 5$, $PR(C_4, K_{t+1}) \geq 3t + \left\lfloor \frac{t}{3} \right\rfloor + 1$. 
**Proof.** Let \( H \) be a graph of order 16 presented in Figure 2. Note that \( H \) does not contain any subgraph \( C_4 \) and \( \alpha(H) = 5 \). Therefore \( \left\lceil \frac{t}{5} \right\rceil H \cup (t - \left\lfloor \frac{t}{5} \right\rfloor)K_3, \ t \geq 6, \) shows that \( PR(C_4, K_{t+1}) \geq 3t + \left\lceil \frac{t}{5} \right\rceil + 1. \) □

**Added in Proof.** The result cited in Lemma 3 can be also find in: C.J. Jayawardene, C.C. Rousseau, An upper bound for Ramsey number of a quadrilateral versus a complete graph on seven vertices, Congressus Numerantium 130 (1998) 175–188.

**References**


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