KERNELS IN EDGE COLOURED LINE DIGRAPH

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Abstract
We call the digraph $D$ an $m$-coloured digraph if the arcs of $D$ are coloured with $m$ colours. A directed path (or a directed cycle) is called monochromatic if all of its arcs are coloured alike. A set $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if it satisfies the two following conditions ($i$) for every pair of different vertices $u, v \in N$ there is no monochromatic directed path between them and ($ii$) for every vertex $x \in V(D) \setminus N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic directed path.

Let $D$ be an $m$-coloured digraph and $L(D)$ its line digraph. The inner $m$-coloration of $L(D)$ is the edge coloration of $L(D)$ defined as follows: If $h$ is an arc of $D$ of colour $c$, then any arc of the form $(x, h)$ in $L(D)$ also has colour $c$.

In this paper it is proved that if $D$ is an $m$-coloured digraph without monochromatic directed cycles, then the number of kernels by monochromatic paths in $D$ is equal to the number of kernels by monochromatic paths in the inner edge coloration of $L(D)$.

Keywords: kernel, kernel by monochromatic paths, line digraph, edge coloured digraph.

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1. Introduction

For general concepts we refer the reader to [1]. The existence of kernels by monochromatic paths in edge coloured digraphs was studied primarily by Sauer, Sands and Woodrow in [4]; they proved that any 2-coloured digraph
has a kernel by monochromatic paths; sufficient conditions for the existence of kernels by monochromatic paths in $m$-coloured digraphs have been studied in [2], [3], [4], [5].

**Definition 1.1.** The line digraph of $D = (X,U)$ is the digraph $L(D) = (U,W)$ (we also denote $U = V(L(D)))$ and $W = A(L(D))$ with a set of vertices as the set of arcs of $D$, and for any $h,k \in U$ there is $(h,k) \in W$ if and only if the corresponding arcs $h,k$ induce a directed path in $D$; i.e., the terminal endpoint of $h$ is the initial endpoint of $k$.

In what follows, we denote the arc $h = (u,v) \in U$ and the vertex $h$ in $L(D)$ by the same symbol.

If $H$ is a subset of arcs in $D$ it is also a subset of vertices of $L(D)$. When we want to emphasize our interest in $H$ as a set of vertices of $L(D)$, we use the symbol $H_L$ instead of $H$.

**Definition 1.2.** Let $D$ be an $m$-coloured digraph and $L(D)$ its line digraph; the inner $m$-coloration of $L(D)$ is the edge coloration of $L(D)$ defined as follows: If $h$ is an arc of $D$ with colour $c$ then any arc of the form $(x,h)$ in $L(D)$ also has colour $c$.

**Definition 1.3.** A subset $N \subseteq V(D)$ is said to be independent by monochromatic paths if for every pair of different vertices $u,v \in N$ there is no $uv$-monochromatic directed path. The subset $N \subseteq V(D)$ is absorbant by monochromatic paths if for every vertex $x \in V(D) - N$ there is a vertex $y \in N$ such that there is an $xy$-monochromatic directed path. And a subset $N \subseteq V(D)$ is said to be a kernel by monochromatic paths if $N$ is both independent and absorbant by monochromatic paths.

**Definition 1.4.** A sequence of vertices $x_1,x_2,\ldots,x_n$ such that $(x_i,x_{i+1}) \in U$ for $1 \leq i \leq n - 1$ is called a directed walk; when $x_i \neq x_j$ for $i \neq j$, $1 \leq i,j \leq n$ will be called a directed path.

2. Kernels in Edge Coloured Line Digraph

**Lemma 2.1.** Let $D$ be an $m$-coloured digraph, $x_0,x_n \in V(D)$, $T = (x_0,x_1,\ldots,x_{n-1},x_n)$ a monochromatic directed path in $D$ and $a_0 = (x,x_0)$ be an arc of $D$ whose terminal endpoint is $x_0$. There exists an $a_0a_n$-monochromatic directed path in the inner $m$-coloration of $L(D)$, where $a_n = (x_{n-1},x_n)$. 
Kernels in Edge Coloured Line Digraph

Proof. Denote by $a_i = (x_{i-1}, x_i)$; for $i = 1, 2, \ldots, n$. Since $T$ is a directed path in $D$, it follows from Definition 2.1 that $(a_1, a_2, \ldots, a_n)$ is a directed path in $L(D)$; in fact, the choice of $a_0$ and Definition 2.1 imply $(a_0, a_1, \ldots, a_n)$ is a directed path in $L(D)$.

Suppose without loss of generality that $T$ is monochromatic of colour $c$. Since $a_{i+1}$ has colour $c$ for $0 \leq i \leq n - 1$ it follows from Definition 1.2 that $(a_i, a_{i+1})$ has colour $c$ for $0 \leq i \leq n - 1$, hence $(a_0, a_1, \ldots, a_n)$ is a monochromatic directed path of colour $c$.

Lemma 2.2. Let $D$ be an $m$-coloured digraph without monochromatic directed cycles, $a_0, a_n \in V(L(D))$. If there exists an $a_0, a_n$-monochromatic directed path in the inner $m$-coloration of $L(D)$, then the terminal endpoint of $a_0$ is different from the terminal endpoint of $a_n$ and there exists a monochromatic directed path from the terminal endpoint of $a_0$ to the terminal endpoint of $a_n$ in $D$.

Proof. Let $(a_0, a_1, \ldots, a_n)$ be a monochromatic directed path of colour $c$ in the inner $m$-coloration of $L(D)$ and $a_i = (x_i, x_{i+1}), 0 \leq i \leq n$. It follows from Definition 2.1 that $(x_1, \ldots, x_{n+1})$ is a directed walk in $D$; since $(a_i, a_{i+1})$ has colour $c, 0 \leq i \leq n - 1$ it follows from Definition 1.2 that $a_{i+1}$ has colour $c$ in $D, 0 \leq i \leq n - 1$. Hence $(x_1, x_2, \ldots, x_n, x_{n+1})$ is a monochromatic directed walk of colour $c$ in $D$. Since $D$ has no monochromatic directed cycles it follows that $x_i \neq x_j \forall i \neq j, 1 \leq i \leq n + 1, 1 \leq j \leq n + 1$; in particular $x_1 \neq x_{n+1}$ (Notice that any monochromatic closed directed walk contains a monochromatic directed cycle) and $(x_1, \ldots, x_{n+1})$ is a monochromatic directed path.

Definition 2.1. Let $D = (X, U)$ be a digraph. We denote by $\mathcal{P}(X)$ the set of all the subsets of the set $X$ and $f: \mathcal{P}(X) \to \mathcal{P}(U)$ will denote the function defined as follows: for each $Z \subseteq X, f(Z) = \{(u, x) \in U | x \in Z\}$.

Lemma 2.3. Let $D$ be an $m$-coloured digraph without monochromatic directed cycles; if $Z \subseteq V(D)$ is independent by monochromatic paths in $D$, then $f(Z)_L$ is independent by monochromatic paths in the inner $m$-coloration of $L(D)$.

Proof. We proceed by contradiction. Let $D$ be an $m$-coloured digraph and $Z \subseteq V(D)$ independent by monochromatic paths. Suppose (by contradiction) that $f(Z)_L$ is not independent by monochromatic paths in the
inner \(m\)-coloration of \(L(D)\). Then there exists \(h, k \in f(Z)_L\) and an \(hk\)-monochromatic directed path in the inner \(m\)-coloration of \(L(D)\). It follows from Lemma 2.2 that the terminal endpoint of \(h\) is different from the terminal endpoint of \(k\) and there exists a monochromatic directed path from the terminal endpoint of \(h\) to the terminal endpoint of \(k\). Since \(h \in f(Z)_L\) (resp. \(k \in f(Z)_L\)) we have from Definition 2.1 that the terminal endpoint of \(h\) (resp. of \(k\)) is in \(Z\); so we have a monochromatic directed path between two vertices of \(Z\), a contradiction.

\[\text{Theorem 2.1. Let } D = (X, U) \text{ be an } m\text{-coloured digraph without monochromatic directed cycles. The number of kernels by monochromatic paths of } D \text{ is equal to the number of kernels by monochromatic paths in the inner } m\text{-coloration of } L(D).\]

\[\text{Proof. Denote by } K \text{ the set of all the kernels by monochromatic paths of } D \text{ and by } K^* \text{ the set of all the kernels by monochromatic paths in the inner } m\text{-coloration of } L(D).\]

(1) If \(Z \in K\), then \(f(Z)_L \in K^*\). Since \(Z \in K\), we have that \(Z\) is independent by monochromatic paths and Lemma 2.3 implies that \(f(Z)_L\) is independent by monochromatic paths. Now we will prove that \(f(Z)_L\) is absorbant by monochromatic paths. Let \(k = (u, v)\) be a vertex of \(L(D)\) such that \(k \in (V(L(D)) - f(Z)_L)\), it follows from Definition 2.1 that \(v \in (V(D) - Z)\).

Since \(Z\) is a kernel by monochromatic paths of \(D\), it follows from Definition 1.3 that there exists \(z \in Z\) and a monochromatic directed path from \(v\) to \(z\) in \(D\), say \((v = x_0, x_1, \ldots, x_{n-1}, x_n = z)\). Then it follows from Lemma 2.1 that there exists an \((u, v)(x_{n-1}, x_n)\)-monochromatic directed path in the inner \(m\)-coloration of \(L(D)\) and since \(z \in Z\), we have from Definition 2.1 that \((x_{n-1}, x_n = z) \in f(Z)_L\).

(2) The function \(f' : K \rightarrow K^*\), where \(f'\) is the restriction of \(f\) to \(K\) is an injective function. Let \(Z_1, Z_2 \in K\) and \(Z_1 \neq Z_2\). Let us suppose, e.g., that \(Z_1 - Z_2 \neq \emptyset\). Let \(v \in (Z_1 - Z_2)\), since \(Z_2\) is a kernel by monochromatic paths of \(D\), it follows from Definition 1.3 that there exists \(u \in Z_2\) and a \(vu\)-monochromatic directed path, let \(h = (x_n, u)\) be the last arc of such a path. It follows from Definition 2.1 that \(h \in f(Z_2)_L\). Finally, notice that since \(v \in Z_1\), the subset \(Z_1\) is independent by monochromatic paths and there exists a \(vu\)-monochromatic directed path, we have that \(u \notin Z_1\) and then \(h \notin f(Z_1)_L\). Hence \(h \in (f(Z_2)_L - f(Z_1)_L)\) and so \(f(Z_1)_L \neq f(Z_2)_L\).

Define a function \(g : P(U) \rightarrow P(X)\) as follows:
If $H \subseteq U$, then $g(H) = C(H) \cup D(H)$, where $C(H) = \{x \in X \mid$ there exists $(z, x) \in H\}$ (the set of all the terminal endpoints of arcs of $H$).
$D(H) = \{x \in X \mid \delta_D(x) = 0 \text{ and there is no monochromatic directed path from } x \text{ to } C(H)\}$. (Where $\delta_D(x) = \{y \in V(D) \mid (y, x) \in U\}$).

(3) If $H_L \in \mathcal{K}^*$, then $g(H_L) \in \mathcal{K}$.

(3.1) If $H_L \in \mathcal{K}^*$, then $g(H_L)$ is independent by monochromatic paths. Suppose that $H_L \in \mathcal{K}^*$, and let $u, v \in g(H_L)$, $u \neq v$; we will prove that there is no $uv$-monochromatic directed path in $D$. We will analyze several cases:

Case 1. $u, v \in C(H_L)$.

In this case we proceed by contradiction. Suppose (by contradiction) that there exists an $uv$-monochromatic directed path $T = (u = x_0, x_1, \ldots, x_n = v)$ in $D$. Since $u, v \in C(H_L)$, $u$ is the terminal endpoint of an arc $h \in H_L$ and $v$ is the terminal endpoint of an arc $k \in H_L$.

When $k = (x_{n-1}, x_n = v)$ we have from Lemma 2.1 that there exists an $hk$-monochromatic directed path, a contradiction (because $H_L$ is independent by monochromatic paths and $h, k \in H_L$).

Otherwise if $k \neq (x_{n-1}, x_n = v)$, we have $(x_{n-1}, x_n = v) \notin H_L$ (because if $(x_{n-1}, x_n = v) \in H_L$ we would have the monochromatic directed path $(h, a_0, a_1, \ldots, a_{n-1})$ where $a_i = (x_i, x_{i+1}), 0 \leq i \leq n - 1$; from $h$ to $(x_{n-1}, x_n = v) = a_{n-1}$ with $h, a_{n-1} \in H_L$, a contradiction). Since $H_L$ is absorbant by monochromatic paths and $a_{n-1} = (x_{n-1}, x_n = v) \notin H_L$, there exists $b \in H_L$ and an $a_{n-1}b$-monochromatic directed path in the inner $m$-coloration of $L(D)$; let $(a_{n-1} = b_0, b_1, \ldots, b_m = b)$ be such a path. Since the terminal endpoint of $k$ is $v$ (the same as $a_{n-1} = b_0$) we have from Definitions 1.1 and 1.2 that also $(k, b_1, b_2, \ldots, b_m = b)$ is a monochromatic directed path in the inner $m$-coloration of $L(D)$ with $k, b \in H_L$, a contradiction.

Case 2. $u \in C(H_L), v \in D(H_L)$.

Since $v \in D(H_L)$, we have $\delta_D(v) = 0$, so there is no $uv$-monochromatic directed path in $D$.

Case 3. $u \in D(H_L), v \in C(H_L)$.

Since $u \in D(H_L)$, we have that there is no monochromatic directed path from $u$ to $C(H_L)$, in particular there is no $uv$-monochromatic directed path.

Case 4. $u, v \in D(H_L)$.

Since $v \in D(H_L)$, we have $\delta_D(v) = 0$ and clearly, there is no $uv$-monochromatic directed path in $D$.

(3.2) If $H_L \in \mathcal{K}^*$, then $g(H_L)$ is absorbant by monochromatic paths.
Let \( u \in X - g(H_L) = X - (C(H_L) \cup D(H_L)) \). Since \( u \notin (C(H_L) \cup D(H_L)) \), we have that there is no arc in \( H \) whose terminal endpoint is \( u \), and at least one of the two following conditions holds: \( \delta_D^-(u) > 0 \) or there exists a monochromatic directed path from \( u \) to \( C(H_L) \).

We will analyze the two possible cases.

**Case 1.** There is no arc in \( H_L \) whose terminal endpoint is \( u \) and \( \delta_D^-(u) > 0 \). The hypothesis in this case implies that there exists an arc \( (t,u) \in U - H_L \). Since \( H_L \in K^* \), we have that \( H_L \) is absorbant by monochromatic paths; hence there exists \( p = (s,m) \in H_L \) and a monochromatic directed path from \( (t,u) \) to \( p \). Now it follows from Lemma 2.2 that \( u \) is different from \( m \) and there exists a monochromatic directed path from \( u \) to \( m \). Finally, notice that since \( (s,m) \in H_L \), we have \( m \in g(H_L) \). So there exists a monochromatic directed path from \( u \) to \( m \) with \( m \in g(H_L) \).

**Case 2.** There is no arc in \( H_L \) whose terminal endpoint is \( u \) and there exists a monochromatic directed path from \( u \) to \( g(H_L) = C(H_L) \cup D(H_L) \).

(4) The function \( g':K^* \rightarrow K \), where \( g' \) is the restriction of \( g \) to \( K \) is an injective function. Let \( N_L, P_L \in K^* \), such that \( N_L \neq P_L \). Let us suppose, e.g., that \( N_L - P_L \neq \emptyset \). Let \( h \in N_L - P_L \) and \( u \) the terminal endpoint of \( h \). Since \( u \) is the terminal endpoint of an arc in \( N_L \), we have that \( u \in g(N_L) \).

Now we will prove that \( u \notin g(P_L) \). Since \( P_L \) is absorbant by monochromatic paths and \( h \notin P_L \), we have that there exists \( k \in P_L \) and an \( hk \)-monochromatic directed path in the inner \( m \)-coloration of \( L(D) \).

Let \( v \) be the terminal endpoint of \( k \); hence \( v \in g(P_L) \) and it follows from Lemma 2.2 that \( u \) is different from \( v \) and there exists an \( uv \)-monochromatic directed path in \( D \). Since \( g(P_L) \) is independent by monochromatic paths (This follows directly from (3) and Definition 1.3), we have that \( u \notin g(P_L) \).

We conclude \( u \in g(N_L) - g(P_L) \) and so \( g(N_L) \neq g(P_L) \). Finally, notice that it follows from (2) and (4) that: Card \( K \leq \text{Card } K^* \leq \text{Card } K \) and hence Card \( K = \text{Card } K^* \).

**Note 2.1.** Let \( D \) be an \( m \)-coloured digraph and \( L(D) \) its line digraph; similarly as in Definition 1.2 we can define the outer \( m \)-coloration of \( L(D) \) as follows: If \( h \) is arc of \( D \) with colour \( c \), then any arc of the form \( (h,x) \) in \( L(D) \) also has colour \( c \). However, Theorem 2.1 does not hold if we change inner \( m \)-coloration of \( L(D) \) by outer \( m \)-coloration of \( L(D) \). In Figure 1, we show a digraph \( D \) without monochromatic directed cycles with one kernel.
by monochromatic paths such that the outer $m$-coloration of its line digraph (Figure 2) has no kernel by monochromatic paths.
Note 2.2. Theorem 2.1 does not hold if we drop the hypothesis that \( D \) has no monochromatic directed cycles. In Figure 3, we show a digraph \( D \) with monochromatic directed cycles which has two kernels by monochromatic paths such that the inner \( m \)-coloration of its line digraph (Figure 4) has just one kernel by monochromatic paths. And in Figure 5, we show a digraph with monochromatic directed cycles without a kernel by monochromatic paths and its line digraph has two kernels by monochromatic paths (see Figure 6).
Kernels in Edge Coloured Line Digraph

Figure 6

REFERENCES


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