INFINITE INDEPENDENT SYSTEMS OF IDENTITIES OF ALTERNATIVE COMMUTATIVE ALGEBRA OVER A FIELD OF CHARACTERISTIC THREE

NICOLAE ION SANDU

Tiraspol State University
The author’s home address:
Deleanu str 1, Apartment 60
Kishinev MD-2071, Moldova
e-mail: sandumn@yahoo.com

Abstract

Let $\mathfrak{A}_3$ denote the variety of alternative commutative (Jordan) algebras defined by the identity $x^3 = 0$, and let $\mathcal{S}_2$ be the subvariety of the variety $\mathfrak{A}_3$ of solvable algebras of solvability index 2. We present an infinite independent system of identities in the variety $\mathfrak{A}_3 \cap \mathcal{S}_2$. Therefore we infer that $\mathfrak{A}_3 \cap \mathcal{S}_2$ contains a continuum of infinite based subvarieties and that there exist algebras with an unsolvable words problem in $\mathfrak{A}_3 \cap \mathcal{S}_2$.

It is worth mentioning that these results were announced in 1999 in works of the international conference “Loops’99” (Prague).

Keywords: infinite independent system of identities, alternative commutative algebra, solvable algebra, commutative Moufang loop.

2000 Mathematics Subject Classification: 17D05, 20N05.

In [8] A.M. Slin’ko has formulated the question (Problem 1.129): if any variety of solvable alternative (Jordan) algebras would be finitely based. U.U. Umirbaev has got an affirmative answer to this question for alternative algebras over a field of characteristic $\neq 2, 3$ (see [14]), and Yu.A. Medvedev [7] has given a negative answer for characteristic 2. The main topic of this work is the construction of an example of an alternative commutative (Jordan) algebra also in the case of characteristic three*, which, together with

*Another example was constructed (independently) by A.V. Badeev, see Added in proof on the end of this paper.
the former results, completes the settlement of Slin’ko’s problem for solvable alternative algebras.

Let \((u, v, w) = uv \cdot w − u \cdot vw\) mean the associator in a considered algebra, let \((u_1, \ldots, u_{2i−1}, u_{2i}, u_{2i+1}) = ((u_1, \ldots, u_{2i−1}), u_{2i}, u_{2i+1})\) and let \(F\) be an infinite field of characteristic 3. Let \(\mathfrak{A}_3\) be the variety of alternative commutative (or Jordan) \(F\)-algebras, determined by the identities

\[(1) \quad x^3 = 0,\]

\[(2) \quad (x_1, x_2, x_3, x_4, x_5), (x_6, x_7, x_8, x_9, x_{10}),\]

\[(x_{11}, x_{12}, x_{13}, x_{14}, x_{15}), (x_{16}, x_{17}, x_{18}), x_{19}) = 0.\]

We denote by \(\mathfrak{S}_2\) the variety of alternative commutative (Jordan) \(F\)-algebras being solvable of index 2. We also write

\[\mu_k = ((x_1, x_2, x_3, x_4, x_5), (y_1, y_2, y_3), \ldots, (y_{12k−2}, y_{12k−1}, y_{12k}),\]

\[((x_1, x_2, x_3, x_4, x_5), (y_{12k+1}, y_{12k+2}, y_{12k+3}), \ldots\]

\[\ldots (y_{24k−2}, y_{24k−1}, y_{24k})), (x_1, x_2, x_3, x_4, x_5)).\]

In this work it is proved that the system of identities \(\{\mu_k = 0 \mid k = 1, 2, \ldots\}\) is independent in the variety \(\mathfrak{A}_3 \cap \mathfrak{S}_2 \mathfrak{S}_2\), i.e. no identity of this system follows from other identities of the system. From (1), it follows that the variety \(\mathfrak{A}_3 \cap \mathfrak{S}_2 \mathfrak{S}_2\) is locally nilpotent [15]. Consequently, it is easy to show that any nilpotent variety of algebras, not necessary alternative or Jordan, has a finite basis of identities. We also note that in [6] it is shown that a lot of classic algebras being solvable of index 2, alternative and Jordan among them, have a finite basis of identities.

It follows from the main result of the work that the variety \(\mathfrak{A}_3 \cap \mathfrak{S}_2 \mathfrak{S}_2\) contains a continuum of infinite based subvarieties and there are algebras with an unsolvable words problem in \(\mathfrak{A}_3 \cap \mathfrak{S}_2 \mathfrak{S}_2\).

Now, we recall some notions and results from the theory of commutative Moufang loops (CML’s), which can be found, e.g. in [1] (with some modifications). Any commutative Moufang loop \((Q; \cdot)\) (CML \(Q\), for short) is characterized by the identity \(x^2 \cdot yz = xy \cdot xz\). The inner mapping group \(I(Q)\) of a CML \(Q\) is the group generated by all the inner mappings \(L(x, y) = L(xy)^{−1}L(x)L(y)\), where \(L(x)y = xy\), of the CML \(Q\). A subloop
$H$ of a CML $Q$ is normal in $Q$, if it is invariant under the group $I(Q)$. The associator (of multiplicity 1) $[x_1, x_2, x_3]$ of elements $x_1, x_2, x_3 \in Q$ is determined by the equality $x_1 x_2 \cdot x_3 = (x_1 \cdot x_2) x_3$. The associators of multiplicity $i$ are determined by induction: $[x_1, \ldots, x_{2i-1}, x_{2i}, x_{2i+1}] = [[x_1, \ldots, x_{2i-1}], x_{2i}, x_{2i+1}]$. We denote by $Q_i$ the CML $Q$ generated by all the associators of multiplicity $i$. A CML $Q$ is centrally nilpotent of class $n$ if its lower central series has the form $Q = Q_0 \supset Q_1 \supset \ldots \supset Q_{n-1} \supset Q_n = \{1\}$. Let $f(Q)$ be the Frattini subloop of $Q$. If $Q$ is a centrally nilpotent loop, then $Q_1 \subseteq f(Q)$. Hence a set $\{a_i \mid a_i \in Q, i \in I\}$ generates $Q$ if and only if the set $\{a_i Q_1 \mid i \in I\}$ generates the abelian group $Q/Q_1$.

We recall (see [1], Chapter VIII, and [12]):

**Lemma 1** (Bruck-Slaby’s Theorem). Any finitely generated CML is centrally nilpotent.

Every CML satisfies the following identities:

(3) $[x, y, z] = [y, z, x] = [y, x, z]^{-1}$;

(4) $[[x, y, z], u, v] = [[x, u, v], y, z][x, [y, u, v], z][x, y, [z, u, v]]$;

(5) $[xy, u, v] = [x, u, v][[x, u, v], x, y] \cdot [y, u, v][[y, u, v], y, x]$;

and the relation

(6) $[Q_i, Q_j, Q_k] \subseteq Q_{i+j+k+1}$.

Let $Q$ be an arbitrary CML and let $FQ$ be its loop algebra. We remind [2] that $FQ$ is a free $F$-module with the basis $\{g \mid g \in Q\}$ and the product of elements of this basis is determined as their product in CML $Q$. We denote by $\omega H$ the ideal of algebra $FQ$, generated by all the elements $1-h$ ($h \in H$), for a normal subloop $H$ of the CML $Q$. If $H = Q$, then $\omega Q$ is called the augmentation ideal of algebra $FQ$. Let $J$ denote the ideal of algebra $FQ$, generated by all the expressions $(u, v, w)+(v, u, w), u, v, w \in Q$. The Moufang identities hold in CML (see [1]), however these identities do not always hold in $FQ$, i.e. the algebra $FQ$ is not always alternative. (An algebra is called alternative if the identities $(x, x, y) = 0$ and $(y, x, x) = 0$ hold in it). It is shown in [13] that if $Q$ is a relatively free CML, then the
quotient algebra \( FQ/J \) is alternative and the CML \( Q \) can be embedded in the multiplication groupoid of algebra \( FQ/J \). Now let \( Q \) be a finite generated CML. By Lemma 1, \( Q \) is centrally nilpotent. Then \( F(Q/Q_1) \) is a non-trivial associative algebra. Moreover an alternative algebra \( FQ/J \) is non-trivial. CML \( Q \) contains a minimal set of generators. Then, as in [12], we introduce for elements in \( Q \) the notion of normal reduced word. Repeating the proof of Theorem 1 from [13] almost word for word, we prove that any finite generated CML \( Q \) can be embedded in the multiplication groupoid of \( FQ/J \). We identify CML \( Q \) with this isomorphic image. In [13] the algebra \( FQ/J \) is called a “loop algebra” of the CML \( Q \) and \( \omega Q/J \) (always \( J \subseteq \omega Q \)) an “augmentation ideal” (now we use these phrases in quotation marks) and are denoted by the same symbols \( FQ \) and \( \omega Q \), respectively.

Lemma 2 ([13]). Let \( Q \) be a relatively free (or finite generated) CML and let \( \phi \) be the homomorphism of “loop algebra” \( FQ \). Then, by the homomorphism \( \phi \), the image of CML \( Q \) is CML. ■

Lemma 3 ([13]). Let \( H \) be a normal subloop of relatively free (or finite generated) CML \( Q \) and let \( FQ, \omega Q \) be its “loop algebra” and “augmentation ideal”, respectively. Then

(i) \( \omega Q = \{ \sum_{q \in Q} \lambda_q q | \sum_{q \in Q} \lambda_q = 0 \} \);

(ii) \( FQ/\omega H \cong F(Q/H) \) and \( \omega Q/\omega H \cong \omega(Q/H) \);

(iii) the “augmentation ideal” is generated as \( F \)-module by the elements of the form \( 1-q \) \((q \in Q)\). ■

Lemma 4. The relatively free (or finite generated) CML \( Q \) satisfies the identity

(7) \[ x^3 = 1 \]

if and only if the “augmentation ideal” \( \omega Q \) of the “loop algebra” \( FQ \) satisfies the identity (1).

Proof. Let the CML \( Q \) satisfy the identity (7). By (iii) of Lemma 3, any element \( h \) in \( \omega Q \) has the form \( h = \lambda_1 q_1 + \ldots + \lambda_n q_n \), where \( \lambda_i \in F, q_i = 1-g_i, g_i \in Q \). Since \( F \) is a field of characteristic 3, the equality \( g^3 = 0 \) follows from the equality \( g^3 = 1 \). Suppose that \( h_{n-1}^3 = 0 \), where \( h_{n-1} = \lambda_1 q_1 + \ldots + \lambda_{n-1} q_{n-1} \)
Lemma 5. Let \( A \) be an alternative commutative algebra with identity 1 and \( B \) its subalgebra, satisfying (1). Then \( 1 - B = \{ 1 - b \mid b \in B \} \) is CML and \((1 - b)^{-1} = 1 + b + b^2\).

**Proof.** We put \( 1 - u = a, 1 - v = b \) and \( 1 - w = c \). Then we have
\[
[1 - u, 1 - v, 1 - w] = ([a \cdot (bc)^{-1}](ab - c) = ([a \cdot (bc)^{-1}](ab - c) - (a \cdot (bc)^{-1})(a \cdot (bc)^{+1}) = 1 + (a \cdot (bc)^{-1}(a, b, c)) = 1 + ((1 - w)^{-1} \cdot (1 - v)^{-1})w)(1 - u, 1 - v, 1 - w) = 1 - ((1 - w)^{-1} \cdot (1 - v)^{-1})w)(1 - u, 1 - v, 1 - w) = 1 - ((1 - w)^{-1} \cdot (1 - v)^{-1})(u, v, w) = 1 - ((1 + w + w^2)(1 + v + v^2) \cdot (1 + u + u^2))(u, v, w). \]
This completes the proof of Lemma 5. \( \blacksquare \)

Let now \( A \) be an alternative commutative \( F \)-algebra with identity 1 and \( B \) a subalgebra of \( A \), satisfying (1). Then \( 1 - B = \{ 1 - b \mid b \in B \} \) is CML and \((1 - b)^{-1} = 1 + b + b^2\).

We write \( \sum x = 1 + x + x^2, \{ x, y, z \} = \{ \sum x \cdot \sum y \cdot \sum z \}(x, y, z) \) and \( \{ x_1, \ldots, x_{2i-1}, x_{2i}, x_{2i+1} \} = \{ \{ x_1, \ldots, x_{2i-1} \}, x_{2i}, x_{2i+1} \} \). If a CML \( Q \) satisfies the identity (7), then from Lemmas 4 and 5 it follows that for \( u, v, w \in \omega Q \) \( [1 - u, 1 - v, 1 - w] = 1 - \{ u, v, w \} \), and consequently by induction, we get
\[
[1 - u_1, 1 - u_2, \ldots, 1 - u_{2i+1}] = 1 - \{ u_1, u_2, \ldots, u_{2i+1} \}. \tag{8} \]

In an arbitrary algebra \( A \), we define by induction:
\[
A^1 = A, \quad A^n = \sum_{i+j=n} A^i \cdot A^j, \quad A^{(1)} = A^2, \quad A^{(n)} = (A^{(n-1)})^2. \]
We remind that algebra \( A \) is called nilpotent (respectively solvable) if there is an \( n \), such that \( A^n = 0 \) (respectively \( A^{(n)} = 0 \)). The least \( n \) is called the nilpotent (respectively solvable) index. Let \( f(x_1, x_2, \ldots, x_i) \) be a polynomial of free algebras. We say that \( f(x_1, x_2, \ldots, x_i) = 0 \) is a partial identity of the algebra \( A \) with the generating set \( B \) if \( f(b_1, b_2, \ldots, b_i) = 0 \) for any \( b_1, b_2, \ldots, b_i \) in \( B \).
Lemma 6. Let $A$ be an alternative commutative $F$-algebra and $I$ be an ideal of $A$. Then $I^{(n)}$, $n = 1, 2, \ldots$, is also an ideal of $A$.

Proof. As $F$ is a field of characteristic 3, we have
\begin{align*}
(u, v, w) + (v, u, w) &= 0, \\
 uv \cdot w - u \cdot vw + vu \cdot w - v \cdot uw &= 0, \\
 2uv \cdot w - u \cdot vw - v \cdot uw &= 0, \\
 -uw \cdot w - u \cdot vw - v \cdot uw &= 0, \\
 uv \cdot w &= -u \cdot vw - v \cdot uw
\end{align*}
for all $u, v, w \in A$.

We will prove the statement by induction on $n$. Let $x \in A, u, v \in I^{(n)}$ and assume that $I^{(n)}$ is an ideal of $A$. Then $x \cdot uv = -xu \cdot v - u \cdot xv$. But $xu, xv \in I^{(n)}$. Therefore $xI^{(n+1)} \subseteq I^{(n+1)}$. Consequently, $I^{(n+1)}$ is an ideal of $A$. The statement is proved by analogy for $n = 1$. This completes the proof of Lemma 6.

Let $L$ be a free CML that satisfies the identity (7), with a set of free generators $Y = \{y_1, y_2, \ldots\}$, where the cardinal number $|Y| \geq 5$. Let $\omega L$ be the “augmentation ideal” of the “loop algebra” $FL$. Let us consider the homomorphism $\alpha : FL \rightarrow FL/(\omega L)^{(2)}$. Then $H = \{h \in L \mid 1 - h \in (\omega L)^{(2)}\}$ is the kernel of the homomorphism $\overline{\alpha}$ of the CML $L$, induced on $L$ by the homomorphism $\alpha$. By Lemma 2, the quotient loop $L/H = \overline{L}$ is a CML.

Lemma 7. Let $L$ be a free CML and $\overline{\alpha} : L \rightarrow \overline{L}$ be the homomorphism of CML defined above. Then the inequality $[\overline{y}_1, \overline{y}_2, \overline{y}_3, \overline{y}_4, \overline{y}_5] \neq 1$, where $\overline{y}_i \in \overline{Y} = \{\overline{y}_i \mid y_i \in Y\}$, holds in the CML $\overline{L}$.

Proof. First we construct an alternative commutative solvable $F$-algebra of index 2, in which identity (1) holds and the following partial identity does not hold:
\begin{equation}
\{x_1, x_2, x_3, x_4, x_5\} = 0.
\end{equation}

Let $M$ be a free $F$-module with a set of generators $X$ and let $N$ be the “exterior” algebra of module $M$, satisfying the identity $3uvw = 0$. We add a new symbol $b \notin N$ to the generators $X$ and assume that $B$ is an $F$-algebra generated by the set $X \cup \{b\}$ which besides the relations of the “exterior” algebra $N$ also satisfies the relations $bu \cdot v = b \cdot uv, bu = -ub$, for all $u, v \in X$. Let $E$ denote the $F$-submodule of module $B$ with the basis
Let us show that the algebra \( R \) becomes a commutative \( F \)-algebra and it satisfies the identity (1). We define the product (\( \cdot \)) on the set \( R \):

\[
\bar{u} \cdot \bar{v} = \begin{pmatrix}
bu_2v_3 + bv_2u_3 \\
bv_3v_1 + bv_3u_1 \\
bv_1v_2 + bv_1u_2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
= \begin{pmatrix}
bu_2v_3 + bv_2u_3 \\
bv_3v_1 + bv_3u_1 \\
bv_1v_2 + bv_1u_2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
= \begin{pmatrix}
bv_3w_3 + bv_2w_3 \\
bv_2w_1 + bv_3w_1 \\
bv_1w_2 + bv_1w_2
\end{pmatrix}
\]

We also define the sum (+) as the componentwise addition. Then, obviously, \((R, +, \cdot)\) becomes a commutative \( F \)-algebra and it satisfies the identity (1). Let us show that the algebra \( R \) is alternative. Let \( \bar{u}, \bar{v}, \bar{w} \in R \). Using (10), we obtain

\[
(\bar{u}, \bar{v}, \bar{w}) = \overline{uv} - \overline{uw} =
\]

\[
\left( b(u_2v_3 + bv_2u_3) \right) \left( b(u_3v_1 + bv_3u_1) \right) \left( b(u_1v_2 + bv_1u_2) \right) - \left( b(u_2v_3 + bv_2u_3) \right) \left( b(u_3v_1 + bv_3u_1) \right) \left( b(u_1v_2 + bv_1u_2) \right) =
\]

\[
= \begin{pmatrix}
bv_3w_3 + bv_2w_3 \\
bv_2w_1 + bv_3w_1 \\
bv_1w_2 + bv_1w_2
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix}
= \begin{pmatrix}
bv_3w_3v_1 + bv_2w_3v_1 \\
bv_2w_1v_2 + bv_3w_1v_2 \\
bv_1w_2v_3 + bv_1w_2v_3
\end{pmatrix}
\]

consisting of the monomials of odd degree from \( B \), except the monomials of the form \( b^{2k+1} \). Let \( u, v \) be monomials from \( E \). There is an odd number of generators from \( X \cup \{ b \} \) in the composition of \( u, v \), because \( uv = -vu \). Moreover, as there are necessary generators from \( X \) in the composition of \( u \), we have \( uu = 0 \). Consequently, it follows easily that for the polynomials \( s, t \) from \( E \) the equalities \( st = -ts \) and \( ss = 0 \) hold. Let

\[
\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \text{where } u_i \in E,
\]

denote the elements of the direct product \( R = E \times E \times E \).
If $\overline{u} = \overline{v}$, then the first component of the associator gets the form
\[
\begin{align*}
&b^2u_3w_3u_1 + b^2u_3u_3w_1 + b^2w_3u_3u_1 + b^2u_2w_2u_1 + b^2u_2w_2u_1 + \\
&\quad + b^2w_2u_2u_1 = b^2u_3w_3u_1 + b^2w_3u_3u_1 + b^2u_2w_2u_1 + b^2w_2u_2u_1 = \\
&= b^2u_3w_3u_1 - b^2u_3w_3u_1 + b^2u_2w_2u_1 - b^2u_2w_2u_1 = 0.
\end{align*}
\]

It is shown by analogy that all the other components are equal to zero. Therefore $(\overline{u}, \overline{u}, \overline{w}) = 0$, i.e. the algebra $R$ is alternative.

Let $A$ denote the $F$-subalgebra of the algebra $R$, generated by the elements of the form $u = (u_1, u_2, 0)$, i.e. by the elements that have the third component equal to zero. Using (10) it is easy to see that the algebra $A$ is solvable of index 2. Then $\{u, v, w\} = ((1 + u + u^2) \cdot (1 + v + v^2)(1 + w + w^2))(u, v, w) = (1 + u + v + w)^2$. Let us write it in detail. We put $b^2u_2w_2v_1 + b^2v_2u_2w_1 + b^2w_2v_2u_1 = c$ and $b^2u_1w_1v_2 + b^2v_1u_1w_2 + b^2w_1v_1u_2 = d$. We have
\[
\begin{align*}
\{u, v, w\} = (1 + u + v + w)(u, v, w) &= \begin{pmatrix} c & u_1 + v_1 + w_1 \\ d & u_2 + v_2 + w_2 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \\
&= \begin{pmatrix} c & u_1 + v_1 + w_1 \\ d & c(u_2 + v_2 + w_2) + (u_1 + v_1 + w_1)d \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix},
\end{align*}
\]
where $a = b^3w_2v_2u_1u_2 + b^3u_2w_2v_1v_2 + b^3v_2u_2w_1v_2 + b^3u_1w_1v_1u_2 + b^3v_1u_1w_1v_2 + b^3v_1w_1u_1v_2$. Consequently, we can choose generating elements $u, v, w, y, z$ in $A$ such that $\{\{u, v, w\}, y, z\} \neq 0$. Therefore, (9) is not a partial identity of the algebra $A$.

As we have shown before in Lemma 5, the set $C = 1 - A$ is a CML. Since $F$ is a field of characteristic 3, we have $(1 - a)^3 = 1 - 3a + 3a^2 - a^3 = 1$ for $a \in A$, i.e. the CML $C$ satisfies the identity (7). It is obvious that $A = \omega C$. We suppose that the CML $C$ has the same number of generating elements as the free CML $L$ and let $C = L/H$. The loop homomorphism
L → C induces the algebra homomorphisms ωL → ωC, ωL/(ωL)(2) → ωC/(ωC)(2) = A/A(2). Since the algebra A is solvable of index 2, then A(2) = (0) and we have the algebra homomorphism ωL/(ωL)(2) → A. As the partial identity (9) does not hold also in the algebra A, the same holds in algebra ωL/(ωL)(2). But FL/(ωL)(2) ∼= F(L/H). Then it follows from (8) that the inequality [y_1, y_2, y_3, y_4, y_5] ≠ 1 holds in the CML L = L/H. This completes the proof of Lemma 7.

Let Q_n denote the CML, constructed in the proof of Theorem of [11]. It is the semi-direct product of B_n and G_n, where G_n is the free centrally nilpotent CML of class 2 with free generators g_1, g_2, ..., g_{24n}, B_n is the centrally nilpotent CML of class 2, generated by the set \{ad \mid α ∈ I(G_n)\}, where d /∈ G_n, I(G_n) is the inner mapping group of CML G_n. It follows from the definition of the CML B_n and G_n that they satisfy the identity

\[ [[x_1, x_2, x_3], x_4, x_5] = 1. \]

Since the CML Q_n is finite, it is centrally nilpotent by Lemma 1. In the CML Q_n the identities (7) and

\[ \lambda = \lambda(x_1, x_2, \ldots, x_{19}) = [[x_1, x_2, x_3, x_4, x_5], [x_6, x_7, x_8, x_9, x_{10}], \]
\[ [x_{11}, x_{12}, x_{13}, x_{14}, x_{15}], [x_{16}, x_{17}, x_{18}], x_{19}] = 1 \]

hold, and for k ≠ n, k ≠ 2n also the following identities are satisfied:

\[
\tau_k = \tau_k(y_1, y_2, y_3, y_4, y_5; x_1, x_2, \ldots, x_{24k}) = \\
= [[y_1, y_2, y_3, y_4, y_5], [x_1, x_2, x_3], \ldots, [x_{12k-2}, x_{12k-1}, x_{12k}], \\
[ [y_1, y_2, y_3, y_4, y_5], [x_{12k+1}, x_{12k+2}, x_{12k+3}], \ldots \\
\ldots, [x_{24k-2}, x_{24k-1}, x_{24k}]], [y_1, y_2, y_3, y_4, y_5] = 1,
\]
and the following inequality holds:

\[
\nu_k = \nu_k(y_2, y_3, y_5; x_1, x_2, \ldots, x_{24k}; u_1, u_2, \ldots, u_{12}) = \\
= [(u_1, y_2, y_3, u_2, y_5), [x_1, x_2, x_3], \ldots, [x_{12k-2}, x_{12k-1}, x_{12k}], \\
[u_3, y_2, y_3, u_4, y_5], [x_{12k+1}, x_{12k+2}, x_{12k+3}], \ldots \\
\ldots, [x_{24k-2}, x_{24k-1}, x_{24k}], [u_5, y_2, y_3, u_6, y_5]] \\
\times [u_7, y_2, y_3, u_8, y_5], [x_1, x_2, x_3], \ldots, [x_{12k-2}, x_{12k-1}, x_{12k}], \\
[u_9, y_2, y_3, u_{10}, y_5], [x_{12k+1}, x_{12k+2}, x_{12k+3}], \ldots \\
\ldots, [x_{24k-2}, x_{24k-1}, x_{24k}], [u_{11}, y_2, y_3, u_{12}, y_5]] = 1,
\]

where either

1) \( u_1 = y_s, u_3 = y_t, u_7 = y_t, u_{11} = y_s, u_2 = u_4 = u_6 = u_8 = u_{10} = u_{12} = y_4, \) and \( u_5 = u_9 = y_1 \)

or 2) \( u_2 = y_s, u_4 = y_t, u_8 = y_t, u_{12} = y_s, u_1 = u_3 = u_5 = u_7 = u_9 = u_{11} = y_1, \) and \( u_6 = u_{10} = y_4 \)

or 3) \( u_1 = y_s, u_5 = y_t, u_7 = y_t, u_9 = y_s, u_2 = u_4 = u_6 = u_8 = u_{10} = u_{12} = y_4, \) and \( u_3 = u_{11} = y_1 \)

or 4) \( u_2 = y_s, u_6 = y_t, u_{10} = y_s, u_1 = u_3 = u_5 = u_7 = u_9 = u_{11} = y_1, \) and \( u_4 = u_{12} = y_4 \)

or 5) \( u_3 = y_s, u_5 = y_t, u_{11} = y_t, u_9 = y_s, u_2 = u_4 = u_6 = u_8 = u_{10} = u_{12} = y_4, \) and \( u_1 = u_7 = y_1; \)

or 6) \( u_4 = y_s, u_6 = y_t, u_{12} = y_t, u_{10} = y_s, u_1 = u_3 = u_5 = u_7 = u_9 = u_{11} = y_1, \) and \( u_2 = u_8 = y_4 \)

and the following inequality holds:
The identities (1), (12), (13) and the inequality (15) are proved in [11]. The identity (14) is proved as the proof of the identity \( w_k = 1 \) of the Theorem of [11], using Lemma 9 of [11].

By construction, the CML \( Q_n \) is a semi-direct product of the CML’s \( B_n \) and \( G_n \). Then, by (ii) of Lemma 3, \( F Q_n / \omega B_n \cong F G_n \), \( \omega Q_n / \omega B_n \cong \omega G_n \), where \( F B_n \) and \( F G_n \) are subalgebras of “loop algebra” \( F Q_n \), and \( \omega G_n \) is the “augmentation ideal” of “loop algebra” \( F G_n \). We will consider the homomorphism \( \eta : F Q_n \rightarrow F Q_n / (\omega B_n)^{(2)} \). By Lemma 2, \( \eta \) induces the homomorphism \( \overline{\eta} \) of the CML \( Q_n \).

We will show that \( \overline{\eta} \) is the isomorphism of the CML \( Q_n \). Indeed, let \( \alpha \) and \( \beta \) be the homomorphisms of CML \( Q_n \) and \( \overline{\eta} Q_n \), respectively, which, by Lemma 2, are induced by the homomorphisms of algebras \( F Q_n \rightarrow F Q_n / \omega B_n \) and \( F Q_n / (\omega B_n)^{(2)} \rightarrow (F Q_n / (\omega B_n)^{(2)}) / (\omega B_n / (\omega B_n)^{(2)}) \).

By (i) of Lemma 3, \( G_n \cap \omega B_n = \emptyset \). Then it follows from the relations \( F G_n \cong F Q_n / \omega B_n \cong (F Q_n / (\omega B_n)^{(2)}) / (\omega B_n / (\omega B_n)^{(2)}) \) that \( G_n \cong \alpha G_n \) and \( \alpha G_n \cong \beta(\overline{\eta} G_n) \). Therefore, \( |G_n| = |\beta(\overline{\eta} G_n)| \). We suppose that the homomorphism \( \overline{\eta} \) of CML \( G_n \) is not an isomorphism. By construction, the CML \( G_n \) is finite. Then \( |\overline{\eta} G_n| < |G_n| \) and \( |\beta(\overline{\eta} G_n)| < |G_n| \). We have obtained a contradiction. Hence \( \overline{\eta} \) is an isomorphism of CML \( G_n \).

By construction, the CML \( B_n \) is generated by the set \( \{ \varphi b \mid \varphi \in I(G_n) \} \), where \( b \notin G_n \), and \( I(G_n) \) is the inner mapping group of CML \( G_n \). It is determined by the identities (7) and (11) and by the relations of the form \( [\varphi_1 b, \varphi_2 b, \varphi_3 b] = 1 \) or \( [\varphi_1 b, \varphi_2 b, \varphi_3 b] = t(\varphi_1, \varphi_2, \varphi_3) \neq 1 \), where \( \varphi_1, \varphi_2, \varphi_3 \in I(G_n) \). By Lemma 4 and (8), the system of identities (7), (11) is equivalent to the system consisting of the identity (1) and the partial identity (9) of the algebra \( \omega B_n \). The meaning of the elements \( t(\varphi_1, \varphi_2, \varphi_3) \) depends only on the inner mappings \( \varphi_1, \varphi_2, \varphi_3 \) of the CML \( G_n \). Then, as \( G_n \cong \overline{\eta} G_n \), the generators \( \varphi b \) and the elements \( t(\varphi_1, \varphi_2, \varphi_3) \) are not mapped on the unit of the CML \( \overline{\eta} G_n \) under the homomorphism \( \eta \). We suppose that \( B_n = L / H \), where \( L \) is a free CML, as in Lemma 7. By (ii) of Lemma 3,
\(FB_n \cong FL/\omega H\). We will consider the homomorphism \(FL \rightarrow FL/(\omega L)^{(2)}\), taking into account Lemma 6. By Lemma 2, a normal subloop \(K\) of the CML \(L\) corresponds to this homomorphism. We have \(FL + (\omega H + (\omega L)^{(2)}) = FL + \omega H + (\omega H + (\omega L)^{(2)}) = FL + \omega H + (\omega B_n)^{(2)} = FB_n + (\omega B_n)^{(2)}\). It means that the ideal \(\omega H + (\omega L)^{(2)}\) is the kernel of the homomorphism \(FL \rightarrow FB_n/\omega B_n)^{(2)}\). The normal subloop \(HK\) will be the kernel of the homomorphism \(L \rightarrow \bar{\eta}B_n\), which by Lemma 2, is induced by the homomorphism \(FL \rightarrow FB_n/\omega B_n)^{(2)}\). We suppose that the determining relation \([\varphi_1 b, \varphi_2 b, \varphi_3 b] \neq 1\) of CML \(B_n\) corresponds to the associator \([y_1, y_2, y_3]\) of the CML \(L\) under the homomorphism \(L \rightarrow B_n\). Then \([y_1, y_2, y_3] \notin H\) and, by Lemma 7, \([y_1, y_2, y_3] \notin K\). Therefore, \([y_1, y_2, y_3] \notin HK\). It means that the determining relation \([\varphi_1 b, \varphi_2 b, \varphi_3 b]\) of the CML \(B_n\) is not mapped on the unit of the CML \(\bar{\eta}B_n\) under the homomorphism \(B_n \rightarrow \bar{\eta}B_n\). Therefore the CMLs \(B_n\) and \(\eta B_n\), which have the same determining relations, are isomorphic. Consequently, \(\bar{\eta}Q_n/\eta B_n \cong \bar{\eta}G_n\) and \(\bar{\eta}\) is the isomorphism of the CML \(Q_n\). Moreover, \(\eta(FQ_n)/\eta(\omega B_n) \cong \eta(FG_n)\). As the homomorphism \(\eta\) keeps the sum of the coefficients of the polynomials, by (i) of Lemma 3, \(\eta(\omega Q_n)/\eta(\omega B_n) \cong \eta(\omega G_n)\).

Taking into account Lemma 6, we can consider the homomorphism \(\xi : \eta(FQ_n) \rightarrow \eta(FG_n)/\eta(FG_n)^{(2)}\). Let \(\bar{\xi}\) be the homomorphism of the CML \(\eta Q_n\) which, by Lemma 2, is induced by the homomorphism \(\xi\). By Lemma 3, \(\eta(\omega Q_n)/\eta(\omega B_n) \cong \eta(\omega G_n)\) has such a property. Therefore, \(\bar{\xi}\eta G_n \cong \bar{\eta}G_n\).

Further, it follows from the relation \(\eta(FQ_n)/\eta(\omega B_n) \cong \eta(FG_n)\) that zero of the algebra \(\eta(FQ_n)/\eta(\omega B_n)\), namely \(\eta(\omega B_n)\), is mapped on zero of the second algebra by the composition of homomorphisms:

\[\eta(FQ_n) \rightarrow \eta(FQ_n)/\eta(\omega B_n) \rightarrow \eta(FG_n)/\eta(FG_n)^{(2)}\].

It means that the homomorphism \(\xi\) does not impose any restrictions on the ideal \(\eta(\omega B_n)\). By Lemma 2, the homomorphism \(\eta(FQ_n) \rightarrow \eta(FQ_n)/\eta(\omega B_n)\) induces the homomorphism of the CML \(\eta Q_n\), whose kernel is the normal subloop \(\{g \in \eta Q_n \mid 1 - g \in \eta(\omega B_n)\}\) = \(\bar{\eta}B_n\). Therefore, we infer that \(\xi\) is an isomorphism of CML \(\eta B_n\). Consequently, we have the isomorphisms \(\bar{\xi}\eta Q_n \cong \bar{\eta}Q_n \cong Q_n\). Further we will identify the CML \(\bar{\xi}\eta Q_n\) with the CML \(Q_n\). We put \(\xi(\omega Q_n) = \omega Q_n\), \(\xi(\omega B_n) = \omega B_n\), and \(\xi(\omega G_n) = \omega G_n\). As for the homomorphism \(\eta\) it is proved that \(\omega Q_n/\omega B_n \cong \omega G_n\). It is obvious that \(\omega B_n \in \mathcal{S}_2\) and \(\omega G_n \in \mathcal{S}_2\). Then the algebra \(\omega Q_n\) belongs to the product of the varieties \(\mathcal{S}_2\mathcal{S}_2\).
Further, $g_i, r_i, s_i$ will denote the elements of CML $\overline{Q}_n$. Let $a_i = 1 - g_i, b_i = 1 - r_i, c_i = 1 - s_i$. We also write

\[
\begin{align*}
\theta &= \theta(x_1, x_2, \ldots, x_{19}) = \{\{x_1, x_2, x_3, x_5\}, \{x_6, x_7, x_8, x_9, x_{10}\}, \\
&\quad \{x_{11}, x_{12}, x_{13}, x_{14}, x_{15}\}, \{x_{16}, x_{17}, x_{18}\}, x_{19}\}; \\
\xi_k &= \xi_k(x_1, x_2, x_3, x_4, x_5; y_1, y_2, \ldots, y_{24k}) = \\
&\quad \{\{x_1, x_2, x_3, x_4, x_5\}, \{y_1, y_2, y_3\}, \ldots, \{y_{12k-2}, y_{12k-1}, y_{12k}\}, \\
&\quad \{\{x_1, x_2, x_3, x_4, x_5\}, \{y_{12k+1}, y_{12k+2}, y_{12k+3}\}, \ldots, \\
&\quad \{y_{24k-2}, y_{24k-1}, y_{24k}\}\}; \\
\eta_k &= \eta_k(x_2, x_3, x_5; y_1, y_2, \ldots, y_{24k}; z_1, z_2, \ldots, z_{12}) = \\
&\quad \{\{z_1, x_2, x_3, z_2, x_5\}, \{y_1, y_2, y_3\}, \ldots, \{y_{12k-2}, y_{12k-1}, y_{12k}\}, \\
&\quad \{\{z_3, x_2, x_3, z_4, x_5\}, \{y_{12k+1}, y_{12k+2}, y_{12k+3}\}, \ldots, \\
&\quad \{y_{24k-2}, y_{24k-1}, y_{24k}\}\}; \\
&\quad \{z_5, x_2, x_3, z_6, x_5\} \} - \\
&\quad -\{\{z_7, x_2, x_3, z_8, x_5\}, \{y_1, y_2, y_3\}, \ldots, \{y_{12k-2}, y_{12k-1}, y_{12k}\}, \\
&\quad \{z_9, x_2, x_3, z_{10}, x_5\}, \{z_{11}, x_2, x_3, z_{12}, x_5\}, \\
&\quad \{y_{12k+1}, y_{12k+2}, y_{12k+3}\}, \ldots, \{y_{24k-2}, y_{24k-1}, y_{24k}\}\};
\end{align*}
\]
or 4) \( z_2 = x_s, z_6 = x_t, z_8 = x_t, z_{10} = x_s, z_1 = z_3 = z_5 = z_7 = z_9 = z_{11} = x_1 \), and \( z_4 = z_{12} = x_4 \)

or 5) \( z_3 = x_s, z_5 = x_t, z_{11} = x_t, z_9 = x_s, z_2 = z_4 = z_6 = z_8 = z_{10} = z_{12} = x_1 \), and \( z_1 = z_7 = x_1 \)

or 6) \( z_4 = x_s, z_6 = x_t, z_{12} = x_7, z_{10} = x_s, z_1 = z_3 = z_5 = z_7 = z_9 = z_{11} = x_1 \), and \( z_2 = z_8 = x_4 \).

Let \( k \neq n, 2n \) and let \( I \) denote the ideal of the algebra \( \omega Q_n \), generated by the expressions of the forms:

\[
\begin{align*}
\theta(a_1a_1, \ldots, a_19a_19), \\
\xi_k(a_1a_1, a_2a_2, a_3a_3, a_4a_4, a_5a_5; \beta_1b_1, \beta_2b_2, \ldots, \beta_{24k}b_{24k}), \\
\eta_k = \eta_k(a_1a_2a_2, a_3a_3, a_5a_5; \beta_1b_1, \beta_2b_2, \ldots, \beta_{24k}b_{24k}; \gamma_1c_1, \gamma_2c_2, \ldots, \gamma_{12}c_{12}),
\end{align*}
\]

where

either 1) \( c_1 = a_s, c_3 = a_t, c_7 = a_t, c_{11} = a_s, c_2 = c_4 = c_6 = c_8 = c_{10} = c_{12} = a_4, \) and \( c_5 = c_9 = a_1 \)

or 2) \( c_2 = a_s, c_4 = a_t, c_8 = a_t, c_{12} = a_s, c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = a_1, \) and \( c_6 = c_{10} = a_4 \)

or 3) \( c_1 = a_s, c_5 = a_t, c_7 = a_t, c_9 = a_s, c_2 = c_4 = c_6 = c_8 = c_{10} = c_{12} = a_1, \) and \( c_3 = c_{11} = a_4 \)

or 4) \( c_2 = a_s, c_6 = a_t, c_8 = a_t, c_{10} = a_s, c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = a_1, \) and \( c_4 = c_{12} = a_4 \)

or 5) \( c_3 = a_s, c_5 = a_t, c_{11} = a_t, c_9 = a_s, c_2 = c_4 = c_6 = c_8 = c_{10} = c_{12} = a_4, \) and \( c_1 = c_7 = a_1 \)

or 6) \( c_4 = a_s, c_6 = a_t, c_{12} = a_t, c_{10} = a_s, c_1 = c_3 = c_5 = c_7 = c_9 = c_{11} = a_1, \) and \( c_2 = c_8 = a_4. \)

Let \( u, v, w \) denote the images of the elements \( a_s, b_t, c_i \), respectively, under the homomorphism \( \omega Q_n \to \omega Q_n/I \). Then the following equalities hold in the algebra \( \omega Q_n/I \):

\[
\begin{align*}
\theta(a_1u_1, a_2u_2, \ldots, a_19u_{19}) &= 0, \\
\xi_k(a_1u_1, a_2u_2, a_3u_3, a_4u_4, a_5u_5; \beta_1v_1, \beta_2v_2, \ldots, \beta_{24k}v_{24k}) &= 0, \\
\eta_k = \eta_k(a_1u_2a_2, a_3u_3, a_5u_5; \beta_1v_1, \beta_2v_2, \ldots, b_{24k}v_{24k}; \\
\gamma_1w_1, \gamma_2w_2, \ldots, \gamma_{12}w_{12} &= 0,
\end{align*}
\]
where

either 1) \( w_1 = u_s, w_3 = u_t, w_7 = u_t, w_{11} = u_s, w_2 = w_4 = w_6 = w_8 = w_{10} = w_{12} = u_4, \) and \( w_5 = w_9 = u_1 \)

or 2) \( w_2 = u_s, w_4 = u_t, w_8 = u_t, w_{12} = u_s, w_1 = w_3 = w_5 = w_7 = w_9 = w_{11} = u_1, \) and \( w_6 = w_{10} = u_4 \)

or 3) \( w_1 = u_s, w_5 = u_t, w_7 = u_t, w_9 = u_s, w_2 = w_4 = w_6 = w_8 = w_{10} = w_{12} = u_4, \) and \( w_3 = w_{11} = u_1 \)

or 4) \( w_2 = u_s, w_6 = u_t, w_8 = u_t, w_{10} = u_s, w_1 = w_3 = w_5 = w_7 = w_9 = w_{11} = u_1, \) and \( w_4 = w_{12} = u_4 \)

or 5) \( w_3 = u_s, w_5 = u_t, w_{11} = u_t, w_9 = u_s, w_2 = w_4 = w_6 = w_8 = w_{10} = w_{12} = u_4, \) and \( w_1 = w_7 = u_1 \)

or 6) \( w_4 = u_s, w_6 = u_t, w_{12} = u_t, w_{10} = u_s, w_1 = w_3 = w_5 = w_7 = w_9 = w_{11} = u_1, \) and \( w_2 = w_8 = u_4. \)

By Lemma 2, the image \( \overline{Q}_n \) of the CML \( Q_n \) under the homomorphism \( FQ_n \to F\overline{Q}_n/I \) is CML.

**Lemma 8.** The identity \( \tau_n = 1 \) does not hold in the CML \( \overline{Q}_n \).

**Proof.** It follows from Lemma 4 that the identity (1) holds in the algebra \( \omega\overline{Q}_n \). Then, as shown before Lemma 5, the set \( T = 1 - \omega\overline{Q}_n \) forms a CML. It is obvious that \( \overline{Q}_n \subseteq T \). Using (iii) of Lemma 3 it is easy to show that \( \omega T = \omega\overline{Q}_n \). We denote by \( H \) the subloop of the CML \( T \), generated by all the expressions of the form

\[
\overline{\lambda} = \lambda(1 - u_1, 1 - u_2, \ldots, 1 - u_{19}),
\]

\[
\overline{\tau}_k = \tau_k(1 - u_1, 1 - u_2, \ldots, 1 - u_5; 1 - v_1, 1 - v_2, \ldots, 1 - v_{24k}),
\]

\[
\overline{\nu}_k = \nu_k(1 - u_2, 1 - u_3, 1 - u_5; 1 - v_1, 1 - v_2, \ldots, 1 - v_{24k}; 1 - w_1, 1 - w_2, \ldots, 1 - w_{12}),
\]

where \( u_i, v_i, w_i \in \omega\overline{Q}_n \), with \( k \neq n, k \neq 2n \). It follows from (12)-(15) that the identity \( \tau_n = 1 \) is not a corollary to the system of the identities \( \lambda = 1, \tau_k = 1, \nu_k = 1 \) (for \( k \neq n \) and \( k \neq 2n \)). Then it follows from (15) and the isomorphism of the CMLs \( Q_n \) and \( \overline{Q}_n \) that for certain \( g_1, g_2, \ldots, g_5, r_1, r_2, \ldots, r_{24k} \in \overline{Q}_n \)
As $\overline{\lambda}, \tau_k, \nu_k$ are defined by the associators of the CML $T$, it is easy to show that the subloop $H$ is invariant with respect to the inner mapping group of the CML $H$. Therefore, it is normal in $T$. We denote by $\ker \varphi$ the kernel of the homomorphism $F\overline{Q}_n \to F(T/H)$, where $\varphi(\sum_{t \in T} \alpha_t t) = \sum_{t \in T} \alpha_t (tH)$. Let the image of the element $\tau_k \in T$ under the homomorphism $T \to T/H$ has the form $\overline{\alpha_k} \beta_k$ whenever $u_i = \alpha_i (1 - g_i), v_i = \beta_i (1 - r_i), w_i = \gamma_i (1 - s_i)$ $(\alpha_i, \beta_i, \gamma_i \in F)$ in $\overline{\tau}_k$ and $\overline{\nu}_k$. Then $\alpha_k \beta_k = 1, \alpha_k = \beta_k^{-1}$. Here $\alpha_k, \beta_k$ are associators of the CML $T/H$. With the help of (3) we present $\beta_k^{-1}$ in the form $\nu_k$ in which the parenthesis distribution $[,]$ in $\nu_k$ coincides with the parenthesis distribution $[,]$ in the second member of the expression $\eta_k$ (see the notations in (16)). The parenthesis distribution in $\alpha_k$ and in the first member of $\eta_k$ coincide. Now we use the equality (8) for $\alpha_k$ and $\gamma_k$. We assume that $\alpha_k = 1 - \overline{\alpha}_k, \gamma_k = 1 - \overline{\gamma}_k$. As the identity $\alpha_k = \gamma_k$ holds in the CML $T/H$, it follows from the relation $F\overline{Q}_n/\ker \varphi \cong F(T/H)$ that the identity $\overline{\alpha}_k = \overline{\gamma}_k$ holds in the algebra $F\overline{Q}_n/\ker \varphi$. Consequently, $\overline{\alpha}_k - \overline{\gamma}_k \in \ker \varphi$. But $\overline{\alpha}_k - \overline{\gamma}_k = \overline{\eta}_k$. Therefore $\overline{\eta}_k \in \ker \varphi$. By analogy, we obtain $\overline{\beta}_k, \overline{\xi}_k \in \ker \varphi$ from the relations $\overline{\lambda}, \overline{\nu}_k \in H$. Then it follows from the definition of the ideal $I$ that $I \subseteq \ker \varphi$. Finally, it follows from (18) that the identity $\tau_k = 1$ does not hold in the CML, being the image of the CML $\overline{Q}_n$ under the homomorphism $F\overline{Q}_n \to F\overline{Q}_n/\ker \varphi$. Then it follows from the homomorphism $F\overline{Q}_n/I \to F\overline{Q}_n/\ker \varphi$ that the identity $\tau_k = 1$ does not hold in the CML $\overline{Q}_n$ as well, being the image of the CML $\overline{Q}_n$ under the homomorphism $F\overline{Q}_n \to F\overline{Q}_n/I$. This completes the proof of Lemma 8. ■

Let $f = f(x_1, x_2, \ldots, x_t)$ be one of the polynomials $\theta, \xi_k, \eta_k$ appeared in (16). By the definition of $[,]$, we pass to the operations $(+), (\cdot)$ in $f$ and we introduce in the natural way the notions of degree on every variable $x_i$, degree and homogeneity of polynomials for the obtained polynomials. Let us write $f$ in the form $f = f_0 + f_1 + \ldots + f_{r_1}$, where $f_i$ is the sum of all the monomials of the polynomial $f$ that have the degree $i$ on $x_1$. Let $u_1, u_2, \ldots, u_t$ be the elements of the algebra $\omega\overline{C}_n/I$, determined above. For simplicity we write $f(u)$ instead of $f(u_1, u_2, \ldots, u_t)$. If $\alpha \in F$, then $f(\alpha u_1, u_2, \ldots, u_t) = f_0(u) + \alpha f_1(u) + \alpha^2 f_2(u) + \ldots + \alpha^{r_1} f_{r_1}(u)$. Let $\alpha_1, \alpha_2, \ldots, \alpha_{r_1}$ be arbitrary elements from $F$. Then, by (17), we get a system consisting of the $r_1$ equations

\begin{equation}
\tau_k(g_1, g_2, \ldots, g_5; r_1, r_2, \ldots, r_{24k}) \notin H.
\end{equation}
Lemma 9. Let \( f = f_1(x_1, x_2, \ldots, x_t) + \ldots + f_i(x_1, x_2, \ldots, x_t) + \ldots + f_r(x_1, x_2, \ldots, x_t) \) be the decomposition of the polynomial \( f \) into homogeneous components \( f_i(x_1, x_2, \ldots, x_t) \) and let \( u_1, u_2, \ldots, u_t \) be the elements of the algebra \( \omega \mathcal{Q}_n/I \) determined above. Then \( f_i(u_1, u_2, \ldots, u_t) = 0. \) ■

In particular, examining the homogeneous components of the least degree in each of the cases \( \overline{\theta}, \overline{\xi}_k, \overline{\eta}_k \) and taking into account the identity \((x, y, z) = -(x, z, y)\) of the alternative algebra, we infer that the equalities:

\[
\begin{align*}
&((u_1, u_2, u_3, u_4, u_5), (u_6, u_7, u_8, u_9, u_{10})), \\
&(u_{11}, u_{12}, u_{13}, u_{14}, u_{15}), (u_{16}, u_{17}, u_{18}), u_{19}) = 0; \\
&((u_1, u_2, u_3, u_4, u_5), (v_1, v_2, v_3), \ldots, (v_{12k-2}, v_{12k-1}, v_{12k}), \\
&(u_1, u_2, u_3, u_4, u_5), (v_{12k+1}, v_{12k+2}, v_{12k+3}), \ldots, \\
&\ldots, (v_{24k-2}, v_{24k-1}, v_{24k})), (u_1, u_2, u_3, u_4, u_5)) = 0; \\
&((u_1, u_2, u_3, u_2), (v_1, v_2, v_3), \ldots, (v_{12k-2}, v_{12k-1}, v_{12k}), \\
&(u_3, u_2, u_3, u_4, u_5), (v_{12k+1}, v_{12k+2}, v_{12k+3}), \ldots, \\
&\ldots, (v_{24k-2}, v_{24k-1}, v_{24k})), (w_5, u_2, u_3, u_6, u_5) + \\
+ ((u_7, u_2, u_3, w_8, u_5), (v_1, v_2, v_3), \ldots, (v_{12k-2}, v_{12k-1}, v_{12k}), \\
&(w_9, u_2, u_3, w_{10}, u_5), (v_{12k+1}, v_{12k+2}, v_{12k+3}), \ldots, \\
&\ldots, (v_{24k-2}, v_{24k-1}, v_{24k})), (w_{11}, u_2, u_3, w_{12}, u_5)) = 0,
\end{align*}
\]
hold in the algebra $\omega Q_n/I$ (for $k \neq n, k \neq 2n$), where $w_1, w_2, \ldots, w_{12}$ take values $u_1, u_4$ exactly as in the previous case.

The algebra $\omega Q_n$ is the homomorphic image of the “augmentation ideal” $\omega Q_n$. Then it follows from (iii) of Lemma 3 that $\omega Q_n$ is generated as $F$-module by the elements of the form $a_i = 1 - g_i$, where $g_i \in Q_n$. We denote by $u_i$ the image of the element $a_i$ under the homomorphism $\omega Q_n \to \omega Q_n/I$. Then any element $v$ from $\omega Q_n/I$ has the decomposition $v = \alpha_1 u_1 + \ldots + \alpha_t u_t$.

Now, by induction on length $t$ from the last equalities it is easy to prove the statement.

**Lemma 10.** The identities (2) and $\mu_k = 0$ hold in the algebra $\omega Q_n/I$ for $k \neq n, k \neq 2n$.

Let $G$ be CML and $a_1, a_2, \ldots, a_{2i+1}, b_1, b_2, \ldots, b_{2j+1}, c_1, c_2, \ldots, c_{2m+1}$ be elements in $G$. We will inductively define the associator of multiplicity $k$ with the $\beta^k$ parenthesis distribution. The associators of multiplicity 0 are the elements of the CML $G$, and the associators of multiplicity 1 with the $\beta^1$ parenthesis distribution are the associators from $G$ of the form $[a_1, a_2, a_3]$. If $\beta^i(a_1, a_2, \ldots, a_{2i+1})$, $\beta^j(b_1, b_2, \ldots, b_{2j+1})$, $\beta^m(c_1, c_2, \ldots, c_{2m+1})$ are, respectively, associators of multiplicity $i, j, m$ with the $\beta^i$, $\beta^j$, $\beta^m$ parenthesis distribution, then $[\beta^i(a_1, a_2, \ldots, a_{2i+1}), \beta^j(b_1, b_2, \ldots, b_{2j+1}), \beta^m(c_1, c_2, \ldots, c_{2m+1})]$ is an associator of multiplicity $i + j + m + 1$ with the $\beta^{i+j+m+1}$ parenthesis distribution.

**Lemma 11.** Let a CML $G$ with the lower central series $G = G_0 \supseteq G_1 \supseteq \ldots$ be generated by the elements $a_1, a_2, \ldots$ and let $\beta^k(b_1, b_2, \ldots, b_{2k+1})$ be the associator of the CML $G$ of multiplicity $k$ with a certain $\beta^k$ parenthesis distribution. Then:

1) $\beta^k(b_1, b_2, \ldots, b_{2k+1}) \in G_k$;

2) the quotient loop $G_k/G_{k+1}$ is generated by those cosets that contain associators of the form

\begin{equation}
[a_{i_1}, a_{i_2}, \ldots, a_{i_{2k+1}}],
\end{equation}

where $a_{i_j} \in \{a_1, a_2, \ldots\}$.

**Proof.** The first assertion follows easily from (6) by induction on $k$. The second assertion will be also proved by induction on $k$. Under $k = 0$ the elements of form (19) are generators of CML and, as a consequence, the
cosets that contain these elements generate the quotient loop $G_0/G_1$. Let us assume that the quotient loop $G_k/G_{k+1}$ is generated by the cosets that contain elements of form (19). As $G_{k+1} = [G_k, G, G]$ is generated by the elements $[h, g, f]$, where $h \in G_k$ and $g, f \in G$, it is obvious that the quotient loop $G_{k+1}/G_{k+2}$ is generated by the cosets that contain these elements. Moreover, by induction hypotheses, $h = \prod_{i=1}^{n} h_i^\epsilon_i \cdot h'$, where $\epsilon_i = \pm 1$, $h' \in G_{k+1}$, $h \in G_k$ and every $h_i$ is an associator of form (19). It follows from (5), (6) and (3) that

$$[h, g, f] = [\prod_{i=1}^{n} h_i^\epsilon_i \cdot h', g, f] = [\prod_{i=1}^{n} h_i^\epsilon_i, g, f][h', g, f] \mod G_{k+2} = \prod_{i=1}^{n} [h_i^\epsilon_i, g, f] \mod G_{k+2} = \prod_{i=1}^{n} [h_i^\epsilon_i, g, f] \mod G_{k+2}.$$

Further, suppose that $g = \prod_{j=1}^{m} a_j^{\tilde{m}_j}$, $f = \prod_{m=1}^{\tilde{m}_m} a_m^{\tilde{m}_m}$. Therefore it follows again from (5), (6), and (3) that

$$[h, g, f] = [\prod_{i=1}^{n} h_i, g, f]^{\epsilon_i} \mod G_{k+2} = \prod_{i=1}^{n} [h_i, a_j, f]^{\epsilon_i} \mod G_{k+2} = \prod_{i=1}^{n} [h_i, a_j, \prod_{m=1}^{\tilde{m}_m} a_m^{\tilde{m}_m}]^{\epsilon_i} \mod G_{k+2} = \prod_{i=1}^{n} [h_i, a_j, \prod_{m=1}^{\tilde{m}_m} a_m^{\tilde{m}_m}]^{\epsilon_i} \mod G_{k+2}.$$

Thus $[h_i, a_j, a_m]$ have the form indicated in (19). This completes the proof of Lemma 11.

We remind that a 3-Lie algebra $(L; (\cdot, \cdot))$ is a linear space $L$ over the associative and commutative ring with identity with a certain 3-linear operation $(\cdot, \cdot)$ on $Q$ which satisfies the identities (see [3]):

$$(x, y, y) = 0, \ (x, x, x) = 0, \ (y, x, x) = 0,$$

(20) $$((x, y, z), u, v) = ((x, u, v), y, z) + (x, (y, u, v), z) + (x, y, (z(u, v))).$$

In an arbitrary alternative commutative algebra $A$ the identity $((x, y, z), u, v) = ((x, u, v), y, z) + (y, u, v), z) + (z, u, v), x, y)$ holds, where $(x, y, z) = xy \cdot z - x \cdot yz$ (see [9]). Then, by the bi-associativity of alternative algebra (cf. [15]), the set $A$ with respect to the ternary operation $(x, y, z)$ becomes a 3-Lie algebra. Let us denote it by $\Lambda(A)$.

Let now $G$ be an arbitrary centrally nilpotent CML that satisfies the identity (7) and let $G = G_0 \supset G_1 \supset \ldots G_s = \{1\}$ be its lower central series. As in the case of groups and Lie algebras [5], we tie the 3-Lie algebra $L(G)$ with CML $G$. By (6) we have $G_{i+1} \supset G_{3i+1} = [G_i, G_i, G_i]$; then
\( C_i = G_i/G_{i+1} \) is an abelian group. Let \( L(C) \) be a direct sum of groups \( C_1, C_2, \ldots, C_{n-1} \). We define the addition \( \oplus \) on \( L(G) \) by the formula

\[
(21) \quad g \oplus h = g_1h_1 + g_2h_2 + \ldots + g_{s-1}h_{s-1},
\]

where \( g = g_1 + g_2 + \ldots + g_{s-1}, h = h_1 + h_2 + \ldots + h_{s-1} \). It is obvious that “zero” of the group \( L(G) \) is the element \( 1 + 1 + \ldots \) and the element \( g_1^{-1} + g_2^{-1} + \ldots \) is “opposite” to \( g \).

We introduce on group \( L(G) \) the ternary operation (, ,). Let \( a \in G_i, b \in G_j, c \in G_k, u \in G_{i+1}, v \in G_{j+1}, w \in G_{k+1} \). Then it follows from (5) and (6) that

\[
[a, b, c]_{G_{i+j+k+2}} = [a, b, c]_{G_{i+j+k+1}}.
\]

Now it is clear that if \( g_i = aG_{i+1}, g_j = bG_{j+1}, g_k = cG_{k+1} \), then \( (g_i, g_j, g_k) = [a, b, c]_{G_{i+j+k+2}} \) is a certain element of group \( C_{i+j+k+1} = G_{i+j+k+2}/G_{i+j+k+2} \).

We extend operation (, ,) on the whole group \( L(G) \) by the formula

\[
(22) \quad (g, h, r) = \sum_{i,j,k=1}^{s-1} (g_i, h_j, r_k),
\]

where \( g, h, r \) are elements in \( L(G) \), and \( \sum \) means addition \( \oplus \) in the group \( L(G) \). Let us show that the operation (, ,) is distributive with respect to \( \oplus \). Let \( f, g, h, r \in L(G) \). Let us show that the expressions

\[
(23) \quad (f, g, h \oplus r), \quad (f, g, h) \oplus (f, g, r)
\]

are equal in \( L(G) \). The first of the expressions (23) is, by definition, equal to

\[
\sum_{i,j,k=1}^{s-1} (f_i, g_j, h_k r_k).
\]

Let \( f_i = aG_{i+1}, g_j = bG_{j+1}, h_k = cG_{k+1}, r_k = dG_{k+1} \). Then by (5) and (6)

\[
(f_i, g_j, h_k r_k) = (aG_{i+1}, bG_{j+1}, (cd)G_{k+1}) = (a, b, cd)_{G_{i+j+k+2}} =\]

\[
(a, b, c)_{G_{i+j+k+2}} = (a, b, d)_{G_{i+j+k+2}} = (a, b, d)_{G_{i+j+k+2}} =\]

\[
(aG_{i+1}, bG_{j+1}, cG_{k+1}) \cdot (aG_{i+1}, bG_{j+1}, dG_{k+1}) = (f_i, g_j, h_k) \cdot (f_i, g_j, r_k).
\]
Consequently, we obtain

$$\sum_{i,j,k=1}^{s-1} (f_i, g_j, h_k r_k) = \sum_{i,j,k=1}^{s-1} (f_i, g_j, h_k) \cdot (f_i, g_j, r_k) = (f, g, h) \cdot (f, h, r).$$

In such a way we have seen that both expressions (23) coincide. Other relations of distributivity can be proved by analogy. Finally, it follows from the di-associativity of CML (cf. [1]) and the identity (4) that $L(G)$ is a 3-Lie algebra. Consequently, we have proved

**Proposition 1.** Let $G$ be an arbitrary centrally nilpotent CML with the lower central series $G = G_0 \supset G_1 \supset \ldots \supset G_s = \{1\}$. Then the direct sum $L(G)$ of the modules $G_i/G_{i+1}$, $i = 0, 1, \ldots, s-1$, on the operations (21) and (22) will be a 3-Lie algebra.

Let us now suppose that a CML $G$, that satisfies the identity (7) is generated by the set $X = \{x_1, x_2, \ldots, x_t\}$. We put $y_i = 1 - x_i$. It follows from the definition of the “augmentation ideal” $\omega G$ of the “loop algebra” $FQ$ that $y_i \in \omega G$. We denote by $A$ the subalgebra of algebra $\omega G$, generated by the elements $y_1, y_2, \ldots, y_t$. By Lemma 4, the algebra $A$ satisfies the identity (1), so is nilpotent [14]. Then for every monomial $v \in A$ there exists a number $m$ such that $v \in A^m \setminus A^{m+1}$. The number $m$ will be called the weight of the monomial $v$. The polynomial that only consists of monomials of the weight $m$ will be called **homogeneous of the weight** $m$. Let $U$ be a word of CML $G$ from the generating set $X$. We consider in $U$ the generators $y_i$ of the algebra $A$, using the relation $x_i = 1 - y_i$. Let us assume that $U$ has the decomposition

$$U = 1 - (u_m + u_{m+1} + \ldots, u_r)$$

in $A$, where $u_i$ is a homogeneous polynomial from $A$ of the weight $i$ and $u_m$ is a polynomial of the smallest weight. We determine the mapping $\delta : G \to A$ by the formula: $\delta(U) = 0$ if $U = 1$, and $\delta(U) = u_m$ for all the other cases.

**Lemma 12.** Let $U, V, W$ be words ($\neq 1$) of CML $G$ from the generating set $X$ and let $\delta(U) = u_m$, $\delta(V) = v_k$, $\delta(W) = w_n$. Then for every integer $l$

$$\delta(U^l) = lu_m.$$
If \( m < k \), then

\[
\delta(UV) = u_k. 
\]

If \( m = k \) and \( u_k + v_k \neq 0 \), then

\[
\delta(UV) = u_k + v_k. 
\]

If \( m = k \) and \( u_k + v_k = 0 \), then \( UV = 1 \) or \( \delta(UV) \) belongs to \( A^t \), where \( t > m \). If \( (u_m, v_k, w_n) \neq 0 \), then

\[
\delta([U, V, W]) = (u_m, v_k, w_n). 
\]

If \( (u_m, v_k, w_n) = 0 \), then \([U, V, W] = 1 \) or \( \delta([U, V, W]) \) belongs to \( A^t \), where \( t > m + k + n \).

**Proof.** We put \( u_m + m_{m+1} + \ldots + u_r = u \). Then \( U = 1 - u \). To prove (25) we use the decomposition \((1 - u)^t = \sum_{l=0}^{t} (-1)^l \binom{t}{l} u^l \), where \( \binom{t}{l} = \frac{(t-1)\ldots(t-l+1)}{l!} \). As \( u \in A \), all non-constant members of the smallest weight of the element \((1 - u)^t\) are of the form \(-lu\). Then (25) is proved.

The assertions (26), (27) follow from the multiplication rules, and the remaining assertions follow from Lemma 5.

We put \( D_k = \{ g \in G \mid 1 - g \in (\omega G)^k \} \). It is easy to see that \( D_k \) is the kernel of homomorphism, induced on the CML \( G \) by the natural homomorphism \( FG \to FG/(\omega G)^k \). By Lemma 2, \( G/D_k \) is a CML, so \( D_k \) is a normal subloop of the CML \( G \) (see [1]). It follows from Lemma 12.

**Lemma 13.** If \( G_m \) is the \( m \)-th member of the lower central series of a CML \( G \), then \( G_m \subseteq D_{2m+1} \).

**Proof.** We will use the induction on \( m \). We have \( G_0 = G = D_1 \). Let us suppose that \( G_m \subseteq D_{2m+1} \) and let \( a \in G_m, u, v \in G \). Then \( [a, u, v] = 1 \), or \( \delta([a, u, v]) \) has a weight not less than \( 2m + 3 \), as \( \delta(a) \) has a weight not less than \( 2m + 1 \). In any case \([a, u, v] \in D_{2m+3} \), and therefore \( G_{m+1} \subseteq D_{2m+3} \). This completes the proof of Lemma 13.
The CML $G$ is generated by the finite set $X$. Then by Lemma 1 it is centrally nilpotent. Assume that its lower central series has the form $G = G_0 \supset G_1 \supset \ldots \supset G_s = \{1\}$.

**Proposition 2.** Let $G$ and $A$ be the algebras considered above. Then the mapping $x_i G_1 \rightarrow y_i$ induces the monomorphism of 3-Lie algebra $L(G)$ into 3-Lie algebra $A \subseteq \Lambda(\omega G)$. Obviously, the monomorphism is determined in the following way:

Let $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{2k+1})$, where $x_{i_j} \in X$, be an associator of multiplicity $k$ of the CML $G$ with the $\beta^k$ parenthesis distribution and let $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{2k+1}) \in G_{\mu(k)}/G_{\mu(k)+1}$. Then the mapping

$$\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{2k+1})G_{\mu(k)+1} \rightarrow \beta^k(y_{i_1}, y_{i_2}, \ldots, y_{2k+1})$$

is a monomorphism of quotient loop $G_{\mu(k)}/G_{\mu(k)+1}$ in the additive group $\Lambda_{\mu(k)}(A)$, where $\Lambda_{\mu(k)}(A)$ is a submodule of module $\Lambda(A)$, consisting of homogeneous polynomials of the weight $\mu(k)$, and the parenthesis distribution $\beta^k$ means multiplicity in $\Lambda(A)$.

**Proof.** By the definition (22) of the multiplication operation in the algebra $L(G)$ and (6), and also by the relation between the operation of taking the associator into the group $G_k/G_{k+1}$ and the multiplication in the algebra $\Lambda(\omega G)$, indicated in (28), the expression $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{2k+1})$ obviously turns into an element $\beta^k(y_{i_1}, y_{i_2}, \ldots, y_{2k+1})$ of the algebra $\Lambda(A)$.

Further, by Lemma 13, the arbitrary element $U$ from $G_k/G_{k+1}$, under the mapping $x_i \rightarrow y_i$, turns into an element of the algebra $A$ of the form

$$1 + u_{2k+1} + u_{2k+2} + \ldots + u_t,$$

where $u_j$ has the weight $j$ or equals zero, and $j > 2k + 1$. This lemma also shows that the equality

$$\delta(U G_{k+1}) = \delta(U) = u_{2k+1}$$

determines the mapping $\delta_{2k+1}$ of group $C_k = G_k/G_{k+1}$ in the set of homogeneous elements of the weight $2k + 1$ of the algebra $A$. 
Moreover, by the definition (21) of the addition operation of the algebra $L(G)$ and (24)-(27), $\delta_{2k+1}$ is a linear mapping $C_k$ in $A^{2k+1}$. By Lemma 11 the associators of the form $[x_1, x_2, \ldots, x_{2k+1}]$ generate the subloop $G_k$, therefore the mapping

$$\delta(V) = \delta_1(v_1) + \delta_3(v_3) + \ldots + \delta_{2k+1}(v_{2k+1}) + \ldots$$

is a linear mapping of $Z_3$-module $L(G)$ into $Z_3$-module $A$, where $Z_3$ means the field of three elements. Consequently, the mapping $x_iG_1 \rightarrow y_i$ induces the homomorphism of 3-Lie algebra $L(G)$ in $A$.

By [10] the subloop $G_1$, generated by all the associators of the CML $G$, belongs to the Frattini subloop. Therefore the mapping $x_iG_1 \rightarrow y_i$ is one-to-one. If $a, b, c$ are elements in $G$, then it follows from Lemma 5 that $[a, b, c] = 1 - (a^{-1} \cdot b^{-1} \cdot c^{-1})(a, b, c)$. Therefore, if $[a, b, c] \neq 1$ then $(a, b, c) \neq 0$. Consequently, it is easy to show by induction that if $\beta^k(x_{i_1}, x_{i_2}, \ldots, x_{i_{2k+1}}) \neq 0$, then $\beta^k(y_{i_1}, y_{i_2}, \ldots, x_{i_{2k+1}}) \neq 0$. Then it follows from (28) that the mapping $x_iG_1 \rightarrow y_i$ induces the monomorphism of 3-Lie algebra $L(G)$ into the 3-Lie algebra $A$. This completes the proof of Proposition 2.

It follows from Lemma 8 and Proposition 2 that

**Lemma 14.** In the algebra $\omega Q_n/I$ the identity $\mu_n = 0$ does not hold. □

It is obvious that the identities $xy = yx$ and $x^2 \cdot yx = x^2y \cdot x$ hold in the algebra $\omega Q_n/I$, i.e. the algebra $\omega Q_n/I$ is Jordan. Then from Lemmas 10 and 14 we immediately obtain

**Theorem 1.** The infinite system of identities $\{\mu_k = 0\} (k = 1, 2, \ldots)$ is independent in the variety $A_3 \cap S_2S_2$ of alternative commutative (Jordan) algebras over the infinite field of characteristic 3. □

If a certain identity is deduced from the system of identities $\{\mu_k = 0\}, k = 1, 2, \ldots,$ then in its deduction only a finite number of identities of this system can be used. Therefore, if the system of identities $\{\mu_k = 0\}$ were equivalent to a certain finite system of identities, then it would be equivalent to one of its finite subsystem. Consequently, from Theorem 1 we obtain

**Corollary 1.** Any infinite subset of the system of identities $\{\mu_k = 0\}, k = 1, 2, \ldots,$ is not equivalent to any finite system of identities.
Corollary 2. In the variety $A_3 \cap S_2 S_2$ of alternative commutative (Jordan) algebras over the infinite field of characteristic 3 there exists an algebra, given by the enumerable set of relations, in which the word problem is unsolvable.

Proof. Let $S$ be some recursively enumerable and non-recursive set of numbers. Let us examine the algebra $A$ of the variety $A_3 \cap S_2 S_2$, defined by the identical relations $\{\mu_n = 0\}$ for $n \in S$. It is obvious that each relation of the algebra $A$ is an identical relation. By Theorem 1 the arbitrary identity from $\{\mu_n = 0\}$ under a given $n$ is true in $A$ if and only if $n \in S$. Therefore, in $A$ the problem of word equality is unsolvable.

Corollary 3. The variety $A_3 \cap S_2 S_2$ of alternative commutative (Jordan) algebras over the infinite field of characteristic 3 contains a continuum of different infinite based subvarieties.

This statement follows directly from Theorem 1 and Corollary 1.

Added in proof (by the editors): A.V. Badeev’s thesis: “On Spechtness of varieties of commutative alternative algebras over a field of characteristic 3 and commutative Moufang loops” (Moscow 1999) is closely related to topics of the paper. See also the paper: A.V. Badeev, On the Specht property of varieties of commutative alternative algebras over a field of characteristic 3 and of commutative Moufang loops, Sibirsk. Mat. Zh. 41 (2000), 1252–1268.

References

[7] Yu.A. Medvedev, *Example of a variety of alternative at algebras over a field of characteristic two, that does not have a finite basis of identities* (Russian), Algebra i Logika 19 (1980), 300–313.


Received 27 August 2001
Revised 8 December 2002
Revised 16 June 2004
Revised 14 July 2004