POWER-ORDERED SETS

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Abstract

We define a natural ordering on the power set \( \mathcal{P}(Q) \) of any finite partial order \( Q \), and we characterize those partial orders \( Q \) for which \( \mathcal{P}(Q) \) is a distributive lattice under that ordering.

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1. Introduction

For an unstructured set \( X \), the power set \( \mathcal{P}(X) \), equipped with the partial order of inclusion, is a Boolean algebra. When we consider a partially ordered (finite) set \( (Q, \leq) \), there is another (perhaps more natural) ordering on \( \mathcal{P}(Q) \):
For $A, B \subseteq Q$, let $A \leq B$ iff there is a 1-1 map $\pi : A \to B$ with $a \leq \pi(a)$ for all $a \in A$.

(For infinite sets this relation $\leq$ is in general not antisymmetric.)

We call the structure $(\mathcal{P}(Q), \leq)$ a “power-ordered set”. We will show that $(\mathcal{P}(Q), \leq)$ is a distributive lattice iff $Q$ is a chain or a horizontal sum (see Definition 3.1) of chains. We also remark that the complement operation on $\mathcal{P}(X)$ is an involutory anti-automorphism of $(\mathcal{P}(Q), \leq)$.

2. Powers of chains

Let $L$ be a linear order. We will show that $\mathcal{P}(L)$ is a distributive lattice. Our proof also gives an explicit description of the lattice operations of the power-ordered set $\mathcal{P}(L)$ by representing $\mathcal{P}(L)$ as a sublattice of a product of chains.

**Setup 2.1.** Let $L$ be a linear order, $n \in \{1, 2, \ldots\}$. Let $-\infty \notin L$, and let $\bar{L} := \{-\infty\} \cup L$, with the obvious order.

Let $L^{(n)}$ be the set of all $n$-tuples $(x_1, \ldots, x_n) \in \bar{L}^n$ which satisfy:

- $x_1 \geq x_2 \geq \cdots \geq x_n$;
- for all $\ell \in \{1, \ldots, n-1\}$: if $x_\ell \neq -\infty$, then $x_\ell > x_{\ell+1}$.

That is, we consider all strictly decreasing $k$-tuples from $L$, for $0 \leq k \leq n$, but we make them into $n$-tuples by appending the necessary number of copies of $-\infty$.

**Fact 2.2.** Let $L, \bar{L}, L^{(n)}$ be as above. Then

- $\bar{L}^n$, as a product of distributive lattices, is again a distributive lattice
- $L^{(n)}$ is a sublattice of $\bar{L}^n$.

**Lemma 2.3.** Let $L$ be a finite linear order.

1. Let $D, E \subseteq L$ be nonempty sets of the same cardinality. Then we can inductively analyse the relation $D \leq E$ in the power-ordered set $\mathcal{P}(L)$ as follows:
\[ D \leq E \iff (D \setminus \{\max D\}) \leq (E \setminus \{\max E\}) \text{ and } \max D \leq \max E; \]

2. If \( D \) and \( E \) are enumerated in decreasing order by \( d_1 > \cdots > d_k \) and \( e_1 > \cdots > e_k \), respectively, then
\[ D \leq E \iff d_1 \leq e_1 \& \cdots \& d_k \leq e_k. \]

**Proof.**

Proof of (1): \( \Leftarrow \) is clear. Conversely, assume that \( \pi \) witnesses \( D \leq E \).

Define a function \( \hat{\pi} : D \to E \) as follows: if \( \pi(\max D) = \max E \), then \( \hat{\pi} = \pi \). Otherwise, let \( \pi(x_0) = \max E \), for some (unique) \( x_0 \in D \setminus \{\max D\} \) and let \( y_0 = \pi(\max D) \). Define \( \hat{\pi}(x_0) = y_0 \), \( \hat{\pi}(\max D) = \max E = \pi(x_0) \), and \( \hat{\pi}(x) = \pi(x) \) otherwise.

Then also \( \hat{\pi} \) witnesses \( D \leq E \). [Why? We have to check \( x_0 \leq \hat{\pi}(x_0) \). This follows from \( x_0 \leq \max D \leq \pi(\max D) = \hat{\pi}(x_0) \).] Moreover, we have \( \hat{\pi}(\max D) = \max E \). Now let \( \pi_0 : D \setminus \{\max D\} \to E \setminus \{\max E\} \) be the restriction of \( \pi \). Then \( \pi_0 \) witnesses \( (D \setminus \{\max D\}) \leq (E \setminus \{\max E\}) \).

Proof of (2) : This follows from (1) by induction.

**Fact 2.4.** If \( E \subseteq L \), and \( E \) is enumerated in decreasing order by \( e_1 > \cdots > e_k \), then:

1. for any \( \ell \leq k \), every \( \ell \)-element subset of \( E \) is \( \leq \{e_1, \ldots, e_\ell\} \);
2. for any \( \ell \leq k \), and any \( \ell \)-element set \( D \subseteq L \), we have \( D \leq E \iff D \leq \{e_1, \ldots, e_\ell\} \).

This fact allows us to reduce the question “\( A \leq B \)” to a question “\( A \leq B' \)” where \( B' \) has the same number of elements as \( A \). Lemma 3.3 can then be used to compare \( A \) and \( B' \):

**Conclusion 2.5.** Let \( L \) be a finite linear order with \( n \) elements, and let \( L^{(n)} \) be defined as above. Then \( \mathfrak{P}(L) \) is (as a partial order, hence also as a lattice) isomorphic to \( L^{(n)} \).

So \( \mathfrak{P}(L) \) is a distributive lattice.

We can compute meet and join in \( \mathfrak{P}(L) \) as follows: If \( D = \{d_1, \ldots, d_\ell\} \subseteq L \) and \( E = \{e_1, \ldots, e_\ell\} \subseteq L \), both in decreasing order, and \( \ell \leq k \), then
• $D \land E = \{d_1 \land e_1, \ldots, d_\ell \land e_\ell\}$;

• $D \lor E = \{d_1 \lor e_1, \ldots, d_\ell \lor e_\ell, e_{\ell+1}, \ldots, e_k\}$.

**Proof.** The map $h : (x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\} \setminus \{-\infty\}$ is a bijection from $L^{(n)}$ onto $\mathcal{P}(L)$. We have to check that $h$ and $h^{-1}$ preserve order:

Let $(d_1, \ldots, d_n), (e_1, \ldots, e_n) \in L^{(n)}$, and let $D := h(d_1, \ldots, d_n)$, $E := h(e_1, \ldots, e_n)$. If $(d_1, \ldots, d_n) \leq (e_1, \ldots, e_n)$ in the product partial order, then the map $\pi : D \to E$ defined by $\pi(d_i) = e_i$ for $d_i \neq -\infty$ witnesses $D \leq E$.

(Note that $d_i \neq -\infty$ implies $e_i \neq -\infty$.)

Conversely, if $D \leq E$, then Lemma 2.3 and Fact 2.4 show that $(d_1, \ldots, d_n) \leq (e_1, \ldots, e_n)$.

3. Sums of chains

**Definition 3.1.** Let $(Q_1, \leq_1)$ and $(Q_2, \leq_2)$ be disjoint partially ordered sets. The “horizontal sum” of $Q_1$ and $Q_2$ is the following partial order $(Q, \leq)$:

$$Q = Q_1 \cup Q_2, \text{ and } \leq = \leq_1 \cup \leq_2, \text{ i.e., } x \leq y \text{ in } Q \text{ iff for some }$$

$$\ell \in \{1, 2\} \text{ we have: } x, y \in Q_\ell \text{ and } x \leq_\ell y.$$ We write $(Q_1, \leq_1) + (Q_2, \leq_2)$ [or just $Q_1 + Q_2$] for the horizontal sum of $Q_1$ and $Q_2$.

**Fact 3.2.** Let $Q = Q_1 + Q_2$. Then the partial order $\mathcal{P}(Q)$ is naturally isomorphic to the product $\mathcal{P}(Q_1) \times \mathcal{P}(Q_2)$ (with the pointwise or “product” partial order).

**Proof.** The map $(E_1, E_2) \mapsto E_1 \cup E_2$ is a bijection from $\mathcal{P}(Q_1) \times \mathcal{P}(Q_2)$ onto $\mathcal{P}(Q_1 + Q_2)$, and it is easy to check that it is also an order isomorphism.

**Definition 3.3.** We write $V$ for the 3-element partial order with a unique minimal and two maximal elements, and $\Lambda$ for the dual order.
Lemma 3.4. If $Q$ is a partial order containing an isomorphic copy of $\Lambda$, then the power-ordered set $\mathfrak{P}(Q)$ is not a lattice.

Proof. Let $a < b, c < b$ in $Q$, $a$ and $c$ be incomparable. We will show that in the partial order $\mathfrak{P}(Q)$ the elements $\{a, c\}$ and $\{b\}$ have no least upper bound.

Assume $E = \{a, c\} \lor \{b\}$. So, we have:

1. $\{a, c\} \leq E$.
2. $\{b\} \leq E$.
3. $E \leq \{a, b\}$ as $\{a, b\}$ is also an upper bound.
4. $E \leq \{c, b\}$, similarly.
5. By (1) and (3), $E$ has exactly 2 elements.
6. By (3), both elements of $E$ are $\leq b$, so by (2), $b \in E$.
7. Let $E = \{b, e\}$, $e \neq b$.
8. $e \leq a$, as $\{b, e\} \leq \{a, b\}$ (by (3)).
9. $e \leq c$, similarly. Hence $e < a$, $e < c$.
10. $a \leq e$ or $c \leq e$, as $\{a, c\} \leq \{b, e\}$ (by (1)).

Now (9) and (10) yield the desired contradiction. □

Lemma 3.5. If $Q$ is a finite partial order containing an isomorphic copy of $V$, then $\mathfrak{P}(Q)$ is either not a lattice, or a nondistributive lattice.
**Proof.** Assume that \( \mathcal{P}(Q) \) is a lattice. By Lemma 3.4, every principal ideal \((a] \) in \( Q \) is linearly ordered (and finite, since \( Q \) is finite). Hence, for any \( a, c \in Q \), \((a] \cap (c] \) is either empty or has a greatest element, in other words: if \( a \) and \( c \) have a common lower bound, then they have a greatest lower bound.

Assume that \( V \) embeds into \( Q \), then there are incomparable elements \( a, c \) in \( Q \) with a greatest lower bound \( b = a \land c \). As \( \Lambda \) does not embed into \( Q \), \( a \) and \( c \) have no common upper bound, hence in \( \mathcal{P}(Q) \) we have

\[
\begin{array}{ccc}
a & \lor & c \\
\downarrow & & \downarrow \\
b
\end{array}
\]

Also, \( b = a \land c \) in \( Q \) implies that in the lattice \( \mathcal{P}(Q) \) we have

\[
\{a\} \lor \{c\} = \{a, c\}
\]

Proof: If \( \{x\} \leq \{a\} \) and \( \{x\} \leq \{b, c\} \), then \( x \leq a \) and \( x \leq c \), so \( x \leq b \), \( \{x\} \leq \{b\} \).

Hence the pentagon

\[
\begin{array}{ccc}
\{a, c\} & \lor & \{b, c\} \\
\downarrow & & \downarrow \\
\{a\} & \lor & \{c\} \\
\downarrow & \lor & \downarrow \\
\{b\}
\end{array}
\]

is a sublattice of \( \mathcal{P}(Q) \), so \( \mathcal{P}(Q) \) is not distributive. 

\[\blacksquare\]
**Remark 3.6.** \(\mathcal{P}(V)\) is in fact a lattice. In contrast, \(\mathcal{P}(\Lambda)\) is not a lattice.

**Conclusion 3.7.** Let \(Q\) be a partial order. The following are equivalent:

1. Comparability is an equivalence relation on \(Q\);
2. \(Q\) is a horizontal sum of chains;
3. Neither \(V\) nor \(\Lambda\) embeds into \(Q\);
4. \(\mathcal{P}(Q)\) is a distributive lattice.

**Proof.** (1) \(\iff\) (2): The chains are just the equivalence classes.

(1) \(\iff\) (3) is clear.

(2) \(\Rightarrow\) (4) was proved in 2.5.

(4) \(\Rightarrow\) (3) follows from 3.4 and 3.5.

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**4. Complements**

**Fact 4.1.** Let \(Q\) be a partial order, \(A, B \subseteq Q\). Then:

\[ A \leq B \iff A \setminus B \leq B \setminus A. \]

**Proof.** Let \(A_0 = A \setminus B = A \setminus (A \cap B), B_0 = B \setminus A\).

If \(\pi_0 : A_0 \to B_0\) witnesses \(A_0 \leq B_0\), then we can extend \(\pi_0\) by the identity function on \(A \cap B\) to a map \(\pi : A \to B\) witnessing \(A \leq B\).

Conversely, let \(\pi : A \to B\) witness \(A \leq B\). Let \(\pi^n\) be the \(n\)-fold iterate of \(\pi\) (a partial function from \(A\) to \(B\); e.g., \(\pi^2(a)\) is only defined if \(\pi(a) \in A \cap B\)).

For each \(a \in A_0 = A \setminus B\) let \(n_a \geq 1\) be the first natural number such that \(\pi^{n_a}(a) \notin A\). [Why does \(n_a\) exist? Note that \(a\) is not a fixpoint of \(\pi\), \(\pi(a) \neq a\), so no \(\pi^n(a)\) can be a fixpoint of \(\pi\), hence all \(\pi^n(a)\) are distinct: \(a < \pi(a) < \cdots\). But \(A\) is finite, so for some \(n\) we must have \(\pi^n(a) \notin A\).]

Now define (for each \(a \in A_0\)): \(\hat{\pi}(a) = \pi^{n_a}(a)\). Clearly \(\hat{\pi} : A_0 \to B_0\), and \(a < \hat{\pi}(a)\). To show that \(\hat{\pi}\) is 1-1, assume \(\hat{\pi}(a) = \hat{\pi}(a')\), and \(n_a' = n_a + \ell\) for some \(\ell \geq 0\). Since \(\pi\) is 1-1, \(\pi^{n_a}(a) = \pi^{n_a + \ell}(a')\) implies \(a = \pi^\ell(a')\), so since \(a \notin B\) we must have \(\ell = 0, a = a'\). 

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**Lemma 4.2.** Let \(Q\) be a finite partial order. We will write \(-X\) for \(Q \setminus X\). Let \(A, B \subseteq Q\). Then: \(A \leq B\) iff \(-B \leq -A\).
**Proof.** By fact 4.1,

\[-B \leq -A \iff -B \setminus (-A) \leq -A \setminus (-B)\]

Now \(-B \setminus (-A) = A \setminus B\), similarly \(-A \setminus (-B) = B \setminus A\), so we can rewrite this as

\[-B \leq -A \iff A \setminus B \leq B \setminus A.\]

Again using Fact 4.1, we see that this is equivalent to \(A \leq B.\)

Hence the complement operation is an involutory anti-automorphism of \(\mathcal{P}(Q)\). If \(Q\) is an antichain, then \(A \leq B\) iff \(A \subseteq B\), so the power-ordered set \(\mathcal{P}(Q)\) is a Boolean algebra.

In general, the equation \(A \land (-A) = \emptyset\) need not hold in the power-ordered set \(\mathcal{P}(Q)\). Indeed, if \(a < b\) in \(Q\), then \(\{a\} \leq \{b\} \leq -\{a\}\).

**References**


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