ON DUALITY OF SUBMODULE LATTICES 1

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Dedicated to the memory of George Hutchinson

Abstract

An elementary proof is given for Hutchinson’s duality theorem, which states that if a lattice identity \( \lambda \) holds in all submodule lattices of modules over a ring \( R \) with unit element then so does the dual of \( \lambda \).

Keywords: submodule lattice, lattice identity, duality.

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Given a ring \( R \), always with unit element \( 1 = 1_R \), the class of left modules over \( R \) is denoted by \( R\text{-Mod} \). Let \( T(R) \) denote the set of all lattice identities that hold in the submodule lattices of all \( R \)-modules, i.e., in the class of \( \{ \text{Sub}(M) : M \in R\text{-Mod} \} \). Using the heavy machinery of abelian category theory and Theorem 4 from [3], G. Hutchinson in [2] and [3] has proved the following duality result.

Main Theorem (G. Hutchinson). For every ring \( R \), \( T(R) \) is a selfdual set of lattice identities. In other words, a lattice identity \( \lambda \) holds in \( \{ \text{Sub}(M) : M \in R\text{-Mod} \} \) iff so does the dual of \( \lambda \).

The goal of the present paper is to give an easy new proof of this theorem. Our elementary approach does not resort to category theory and uses much less from [3] than the original one.

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Proof of the Main Theorem. Let \( \lambda \) be a lattice identity. Since \( \text{Sub}(M) \cong \text{Con}(M) \) for every \( M \in \text{-Mod} \) and \( \text{-Mod} \) is a congruence permutable variety, by results of R. Wille ([5]) or A. Pixley [4] (cf. [3] for more details) there is a strong Mal’cev condition \( U(\lambda) \) such that \( \lambda \in T(R) \) is equivalent to the satisfaction of \( U(\lambda) \) in \( \text{-Mod} \). Using the fact that each \( n \)-ary term \( f(y_1, \ldots, y_n) \) in \( \text{-Mod} \) can uniquely be written in the form \( r_1 y_1 + \ldots + r_n y_n \) with \( r_1, \ldots, r_n \in R \), \( U(\lambda) \) easily turns to a system of linear equations

\[
Ay = b \cdot 1_R
\]

where \( A \) is an integer matrix, \( b \) is a column vector with integer entries, and \( y \) is the column vector of ring variables (cf. [3] for concrete examples). So we obtain that

\[
\lambda \in T(R) \iff Ay = b \cdot 1_R \text{ is solvable in } R.
\]

We can easily infer from this observation that for any rings \( R_i \) (\( i \in I \)) and their direct product we have

\[
T\left( \prod_{i \in I} R_i \right) = \bigcap_{i \in I} T(R_i).
\]

A classical matrix diagonalization method, due to Frobenius ([1], cf. also [3]), asserts that for any integer matrix \( A \) there exist invertible integer matrices \( B \) and \( C \) with integer inverses such that \( BAC \) is a diagonal matrix. Choosing \( B \) and \( C \) according to this result, multiplying (1) by \( B \) from the left and introducing the notations \( M := BAC, z := C^{-1}y, c := Bh \) we easily conclude that the solvability of (1) in \( R \) is equivalent to the solvability of

\[
Mz = c \cdot 1_R
\]

in \( R \). Now, for integers \( m \geq 0 \) and \( n \geq 1 \) let \( D(m, n) \) denote the ”divisibility condition” \( (\exists x)(mx = n \cdot 1) \) where \( mx = x + \ldots + x \) (\( m \) times) and 1 stands for the ring unit. The set \( \{(m, n) : m \geq 0, n \geq 1, \text{ and } D(m, n) \text{ holds in } R\} \) will be denoted by \( D(R) \). Since \( M \) in (4) is a diagonal matrix, the solvability of (4) in \( R \) depends only on \( D(R) \). Hence, combining the previous assertions and (2), we conclude that

\[
D(R) \text{ determines } T(R),
\]
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i.e., \( D(R_1) = D(R_2) \) implies \( T(R_1) = T(R_2) \). Clearly, for arbitrary rings \( R_i, i \in I \),

\[
D \left( \prod_{i \in I} R_i \right) = \bigcap_{i \in I} D(R_i).
\]

Now we claim that for arbitrary rings \( R \) and \( R_i (i \in I) \)

\[
\text{if } D(R) = \bigcap_{i \in I} D(R_i) \text{ then } T(R) = \bigcap_{i \in I} T(R_i).
\]

Indeed, \( \bigcap_{i \in I} T(R_i) = T(\prod_{i \in I} R_i) \) by (3). Since \( D(\prod_{i \in I} R_i) = D(R) \) by (6)

and the premise of (7), (5) yields \( T(\prod_{i \in I} R_i) = T(R) \), proving (7).

For \( k > 0 \) let \( \mathbb{Z}_k \) denote the factor ring of the ring \( \mathbb{Z} \) of integers modulo \( k \), and let \( \mathbb{Z}_0 = \mathbb{Q} \), the field of rational numbers. We claim that, for any ring \( R \),

\[
D(R) = \bigcap \{ D(\mathbb{Z}_k) : D(R) \subseteq D(\mathbb{Z}_k) \}
\]

The proof of (8) will implicitly use the fact that for any integers \( m \geq 0, n > 0 \) and \( k > 0 \) the following equivalence holds:

\[
(m, n) \in D(\mathbb{Z}_k) \iff \gcd(m, k) \mid n.
\]

First we deal with the case when \( k := \text{char } (R) > 0 \). Here \( \text{char } (R) \) denotes \( \min \{ i : 0 < i \in \mathbb{Z} \text{ and } i \cdot 1_R = 0 \} \), the characteristic of \( R \), where \( \min \emptyset \) is understood as 0. We assert that

\[
D(R) = D(\mathbb{Z}_k),
\]

which clearly yields (8) for \( \text{char } R > 0 \). The embedding \( \mathbb{Z}_k \rightarrow R, x \cdot 1_{\mathbb{Z}_k} \mapsto x \cdot 1_R (x \in \mathbb{Z}) \) ensures that \( D(\mathbb{Z}_k) \subseteq D(R) \). Now suppose that \( (a, b) \notin D(\mathbb{Z}_k) \), i.e., \( d := \gcd(a, k) \) does not divide \( b \). Let \( k = k_1 d, a = a_1 d \)

and \( b = qd + r, 0 < r < d \). If we had \( ax = b \cdot 1_R \) for some \( x \in R \), then \( 0 = k(a_1 x) = k_1 a x = k_1 b \cdot 1_R = k_1 qd \cdot 1_R + k_1 r \cdot 1_R = k(q \cdot 1_R) + (k_1 r) \cdot 1_R \) would be a contradiction, for \( k_1 r < k_1 d = k = \text{char } (R) \). Hence \( (a, b) \notin D(R) \). This proves \( D(R) = D(\mathbb{Z}_k) \), and (8) follows.

Now let us assume that \( \text{char } (R) = 0 \). Only the \( \supseteq \) part of (8) has to be verified, so suppose

\[
(m, n) \notin D(R),
\]

\( m \geq 0 \) and \( n > 0 \); we have to show that \( (m, n) \) does not belong to the right-hand side of (8). Two cases will be distinguished.
Case 1. \( m = 0 \). Then \((m, n) \notin D(Z_0)\), and \(D(R) \subseteq D(Z_0)\) clearly follows from the implication: \((a, b) \in D(R) \implies a \neq 0\). Hence \((m, n) = (0, n)\) does not belong to the right-hand side of (8).

Case 2. \( m > 0 \). First we claim that for arbitrary \( 0 \leq a_1, \ldots, a_t \in Z \) and \( 1 \leq b_1, \ldots, b_t \in Z \) we have

\[
(a_1, b_1), \ldots, (a_t, b_t) \in D(R) \implies (a_1 \ldots a_t, b_1 \ldots b_t) \in D(R).
\]

Indeed, if \( a_1r_1 = b_1 \cdot 1_R \) and \( a_2r_2 = b_2 \cdot 1_R \) for \( r_1, r_2 \in R \), then \((a_1a_2)(r_1r_2) = a_2(a_1r_1)r_2 = a_2(b_1 \cdot 1_R)r_2 = b_1(a_2r_2) = b_1b_2 \cdot 1_R \). This proves (11) for \( t = 2 \), whence it holds for \( t > 2 \) as well.

Now let \( m = p_1^{f_1} \ldots p_t^{f_t} \) and \( n = p_1^{g_1} \ldots p_t^{g_t} \) with pairwise distinct primes \( p_1, \ldots, p_t \) and nonnegative integers \( f_1, \ldots, f_t, g_1, \ldots, g_t \). We infer from (11) that \((p_1^{f_i}, p_2^{g_i}) \notin D(R)\) for some \( i \in \{1, \ldots, t\} \). With the notations \( p := p_i, f := f_i, g := g_i \) and \( k := p^{g+1} \), \((p^f, p^g) \notin D(R)\) implies \( f > g \). Hence \((m, n) \notin D(Z_k)\), for \( mx = 0 \neq n \cdot 1 \in Z_k \) holds for all \( x \in Z_k \). Now, before showing that \( Z_k \) occurs on the right hand side of (8), let us observe that if \((p^{g+1}, p^g)\) belonged to \( D(R) \), then, choosing an \( r \in R \) with \( p^{g+1}r = p \cdot 1_R \), we could obtain \( p^g \cdot 1_R = p^{g+1}r = p(p^g \cdot 1_R)r = pp^{g+1}r^2 = p^{g+2}r^2 = \ldots = p^{f+g}\), which would contradict \((p^f, p^g) \notin D(R)\). Therefore \((p^{g+1}, p^g) \notin D(R)\).

Now, to show \( D(R) \subseteq D(Z_k)\), let \((c, d) \notin D(Z_k)\), \( 0 \leq c \), and \( 1 \leq d \); we have to show that \((c, d) \notin D(R)\). If \( c = 0 \) then \((c, d) \notin D(R)\) follows from \( \text{char}(R) = 0 \), so \( c > 0 \) can be supposed. Let \( c = p^v c_1 \) and \( d = p^r d_1 \) such that \( p \) does not divide \( c_1d_1 \). We infer from (9) that \( u > v \) and \( v \leq g \). Hence there are integers \( x \) and \( y \) with \( p^v = \text{g.c.d.}(p^u, d) = xp^u + yd \). If \((c, d)\) belonged to \( D(R)\), i.e., if there was an element \( r \in R \) with \( cr = d \cdot 1_R \), then we would have

\[
\begin{align*}
p^g \cdot 1_R &= p^{g-v}(p^v \cdot 1_R) = p^{g-v}(xp^u + yd) \cdot 1_R = \\
&= p^{g+v-u}x \cdot 1_R + p^{g-v}yd \cdot 1_R = p^{g+u-v}x \cdot 1_R + p^{g-v}yc \cdot r = \\
&= p^{g+1}((xp^{u-v-1} \cdot 1_R + p^{u-v-1}yc_1 \cdot r)),
\end{align*}
\]

which would contradict \((p^{g+1}, p^g) \notin D(R)\). Thus \((c, d) \notin D(R)\), proving (8).

By (7) and (8), \( T(R) \) is the intersection of some \( T(Z_k) \). Therefore it suffices to show that

\[
T(Z_k) \quad \text{is selfdual for every} \quad k \geq 0.
\]
The mentioned strong Mal’cev conditions of Wille and Pixley easily imply that, for any lattice identity \( \lambda \), we have \( \lambda \in T(\mathbb{Z}_k) \) iff \( \lambda \) holds in \( \text{Sub}(\mathbb{Z}_k^t) \) for all positive integers \( t \) where \( \mathbb{Z}_k^t \) is considered a \( \mathbb{Z}_k \)-module in the natural way. (In fact, \( \mathbb{Z}_k^t \) is the free \( \mathbb{Z}_k \)-module on \( t \) generators.) Hence (12) and the Main Theorem will prompt follow from

\[ (13) \quad \text{for all } k \geq 0, \quad \text{Sub}(\mathbb{Z}_k^t) \text{ is a selfdual lattice.} \]

Although there are deep module theoretic results implying (13), the tools we have already listed make a short elementary proof possible. The elements of \( \mathbb{Z}_t^k \) will be row vectors, and for \( \vec{x} = (x_1, \ldots, x_t) \in \mathbb{Z}_t^k \) the transpose of \( \vec{x} \) will be denoted by \( \vec{x}^* \). Standard matrix notations like \( \vec{x} \vec{y}^* = x_1 y_1 + \cdots + x_t y_t \) will be in effect. We claim that

\[ \varphi : \text{Sub}(\mathbb{Z}_k^t) \to \text{Sub}(\mathbb{Z}_k^t), \quad S \mapsto S^\perp := \{ \vec{x} \in \mathbb{Z}_k^t : (\forall \vec{y} \in S)(\vec{x} \vec{y}^* = 0) \} \]

is a dual lattice automorphism and, in addition, an involution. All the necessary properties of \( \varphi \) can be checked very easily except that

\[ (14) \quad (S^\perp)^\perp \subseteq S. \]

Assume that \( k > 0 \), and let \( 1_k \) denote the ring unit of \( \mathbb{Z}_k \). First we prove (14) for the case when \( t = 1 \). Since \( \mathbb{Z} \) is a principal ideal domain, we easily conclude that \( S \) is necessarily of the form \( \{ xu \cdot 1_k : x \in \mathbb{Z} \} \) for some positive divisor \( u \) of \( k \) in \( \mathbb{Z} \). The same holds for the submodule \( S^\perp \), so it is of the form \( \{ vx \cdot 1_k : x \in \mathbb{Z} \} \) for an appropriate positive divisor \( v \) of \( k \) in \( \mathbb{Z} \). Since \( (u \cdot 1_k)(v \cdot 1_k) = 0 \), we obtain

\[ (15) \quad k \mid uv. \]

On the other hand, \( (k/u) \cdot 1_k \) is clearly orthogonal to all members of \( S \), so it is in \( S^\perp \), whence \( (k/u) \cdot 1_k = vx \cdot 1_k = v(x \cdot 1_k) \) for some \( x \in \mathbb{Z} \). Therefore \( (v, k/u) \in D(\mathbb{Z}_k) \), and (9) gives \( v \mid k/u \), i.e.,

\[ (16) \quad uv \mid k. \]

From (15) and (16), we have \( v = k/u \). Hence, giving the role of \( u \) to \( v \) we obtain \( (S^\perp)^\perp = \{ x(k/(k/u)) \cdot 1_k : x \in \mathbb{Z} \} = \{ xu \cdot 1_k : x \in \mathbb{Z} \} = S \).

Now let \( t > 1 \), and let \( S \) be a submodule of \( \mathbb{Z}_k^t \). Since \( S \) is finite, we can consider a matrix \( A \) of size \( s \times t \) for some \( s \geq t \) such that each vector of \( S \)
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coincides with at least one row of $A$. Although $A$ is a matrix over $\mathbb{Z}_k$, not over $\mathbb{Z}$, using the natural ring homomorphism $\mathbb{Z} \rightarrow \mathbb{Z}_k$ for matrix entries we can easily conclude from Frobenius’ afore-mentioned result that there are square matrices $B$ and $C$ over $\mathbb{Z}_k$ with respective sizes $s \times s$ and $t \times t$ such that $BAC$ is a diagonal matrix, and $B$ resp. $C$ has an inverse in the ring of $s \times s$ resp. $t \times t$ matrices over $\mathbb{Z}_k$. For any $\vec{y} \in \mathbb{Z}_k^t$ we have

$$\vec{y} \in S^\perp \iff A\vec{y}^* = 0.$$  

Now let $\vec{v}$ be an arbitrary member of $S^{\perp \perp}$. Then

$$(\forall \vec{y} \in \mathbb{Z}_k^t) \ (A\vec{y}^* = 0 \implies \vec{v}\vec{y}^* = 0).$$

Resorting to the above-mentioned $B$ and $C$ and multiplying by $B$ from the left we obtain

$$(\forall \vec{y} \in \mathbb{Z}_k^t) \ ((BAC)(C^{-1}\vec{y}^*) = 0 \implies (\vec{v}C)(C^{-1}\vec{y}^*) = 0).$$

Since $C^{-1}\vec{y}^*$ takes all (transposed) values from $\mathbb{Z}_k^t$, with the notations $M = BAC$ and $\vec{w} = \vec{v}C$ we obtain

$$\forall \vec{z} \in \mathbb{Z}_k^t: (M\vec{z}^* = 0 \implies \vec{w}\vec{z}^* = 0).$$

We know that $M$ is a diagonal matrix, let $m_{11}, \ldots, m_{tt}$ be its diagonal entries. Choosing $\vec{z}$ such that all but one of its components are zero we obtain from (17) that

$$(\forall z_i \in \mathbb{Z}_k) \ (m_{ii}z_i = 0 \implies w_i z_i = 0) \ (i = 1, \ldots, t).$$

Let $S_i = \{ um_{ii} : u \in \mathbb{Z}_k \} \in \text{Sub}(\mathbb{Z}_k)$; condition (18), in other words, says that $w_i \in S_i^{\perp \perp}$. Since (14) has already been proved for $t = 1$, we have $w_i \in S_i$, and we can choose an $r_i \in \mathbb{Z}_k$ such that

$$w_i = r_i m_{ii} \ (i = 1, \ldots, t).$$

Letting $\vec{r} = (r_1, \ldots, r_t, 0, \ldots, 0)$ (with $s$ components) we have $\vec{r}M = \vec{w}$. Hence

$$\vec{v} = \vec{w}C^{-1} = \vec{r}MC^{-1} = \vec{r}BACC^{-1} = (\vec{r}B)A,$$

showing that $\vec{v}$ is a linear combination of the rows of $A$, i.e., $\vec{v} \in S$. This proves (14) for the case $k > 0$.

When $k = 0$, $\mathbb{Z}_0 = \mathbb{Q}$, and the rudiments of linear algebra yield $\dim S^\perp = t - \dim S$. Hence (14) follows from the evident $\supseteq$ inclusion and the fact that both sides have the same dimension. This completes the proof of the the Main Theorem.
References


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