ON THE UNIFORM CONVERGENCE OF SINE, COSINE AND DOUBLE SINE-COSINE SERIES

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Abstract

In this paper we define new classes of sequences $GM(\beta, r)$ and $DGM(\alpha, \beta, \gamma, r)$. Using these classes we generalize and extend the P. Kórus results concerning the uniform convergence of sine, cosine and double sine-cosine series, respectively.

Keywords: sine series, cosine series, double sine-cosine series, uniform convergence, generalized monotonicity.

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1. Introductions

In this section we will present some definitions and notations for series of one and two variables. Main and auxiliary results will be formulated in two next sections and in the last one we will give the proofs of the main results.

1.1. The sine and cosine series

Let $\{c_k\}_{k=1}^{\infty}$ be a non-negative real sequence and consider the series

\[ \sum_{k=1}^{\infty} c_k \sin kx, \]

\[ \sum_{k=1}^{\infty} c_k \cos kx. \]

In 1916 Chaundy and Jolliffe [1] proved the following classical result for sine series.
Theorem 1. If \( \{c_k\}_{k=1}^{\infty} \subset \mathbb{R}_+ \) is non-increasing and tends to zero, then (1) converges uniformly if and only if
\[
kc_k \to 0 \quad \text{as} \quad k \to \infty.
\]

Several classes of sequences have been introduced to generalize Theorem 1 (see [3, 6, 8, 9]). These classes are larger than the class of monotone sequences and contain sequences of complex numbers as well. The definitions of the latest classes are the following:

\[
MVBVS = \left\{ \{c_k\}_{k=1}^{\infty} \subset \mathbb{C}, \exists C > 0, \lambda \geq 2 : \sum_{k=n}^{2n} |\Delta_1 c_k| \leq \frac{C}{n} \sum_{k=\lfloor n/\lambda \rfloor}^{\lfloor n \rfloor} |c_k|, n \geq \lambda \right\},
\]

\[
SBVS = \left\{ \{c_k\}_{k=1}^{\infty} \subset \mathbb{C}, \exists C > 0, \lambda \geq 2 : \sum_{k=n}^{2n-1} |\Delta_1 c_k| \leq \frac{C}{n} \left( \sup_{m \geq \lfloor n/\lambda \rfloor} \sum_{k=m}^{2m} |c_k| \right), n \geq \lambda \right\},
\]

\[
SBVS_2 = \left\{ \{c_k\}_{k=1}^{\infty} \subset \mathbb{C}, \exists C > 0, b(k) \nearrow \infty : \sum_{k=n}^{2n-1} |\Delta_1 c_k| \leq \frac{C}{n} \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} |c_k| \right), n \geq 1 \right\},
\]

where \( \Delta_r c_k = c_k - c_{k+r} \) for \( r \in \mathbb{N} \), and the constants \( C \) and \( \lambda \) depend only on \( \{c_k\} \), a sequence \( \{b(k)\}_{k=1}^{\infty} \subset \mathbb{R}_+ \) is increasing and tends to infinity. It was proved in [3] that \( MVBVS \subsetneq SBVS \subsetneq SBVS_2 \) and a series (1) with coefficients of complex numbers from the classes \( MVBVS, SBVS, SBVS_2 \), is uniformly convergent if (3) is satisfied.

In [3] Kőrűs proved the following theorem:

Theorem 2 [5]. Let \( \{c_k\}_{k=1}^{\infty} \subset \mathbb{C} \) belong to the class \( SBVS_2 \).

(i) If (3) is satisfied, then (1) is uniformly convergent.

(ii) Conversely, if \( \{c_k\}_{k=1}^{\infty} \) is a non-negative sequence and (1) convergent uniformly, then (3) holds.

In [5] Kőrűs showed also that the above theorem is also true for cosine series but with an additional condition.
Theorem 3 [5]. Let $\{c_k\}_{k=1}^{\infty} \subseteq \mathbb{C}$ belongs to the class $SBVS_2$.

(i) If (3) and $\sum_{k=1}^{\infty} c_k < \infty$ are satisfied, then (2) is uniformly convergent.
(ii) Conversely, if $\{c_k\}$ is non-negative and (2) converges uniformly, then (3) and $\sum_{k=1}^{\infty} c_k < \infty$ hold.

In [6] Szal defined new class of sequence in the following way:

Definition 4 [6]. Let $\beta = \beta_k$ be a non-negative sequence and $r$ a natural number. The sequence of complex number $c := \{c_k\}$ is said to be $(\beta, r)$ – general monotone, or $c \in GM(\beta, r)$, if the relation
\[
\sum_{k=n}^{2n-1} |\Delta_r a_k| \leq \mathcal{C}_\beta n
\]
holds for all $n$.

Using the above definition for $r = 1$, we have:

(1) $MVBVS \equiv GM(1 \beta, 1)$, where $\{1\beta\}$ is the sequence defined by the formula on the right hand side of the inequality (4);
(2) $SBVS \equiv GM(2 \beta, 1)$, where $\{2\beta\}$ is the sequence defined by the formula on the right hand side of the inequality (5);
(3) $SBVS_2 \equiv GM(3 \beta, 1)$, where $\{3\beta\}$ is the sequence defined by the formula on the right hand side of the inequality (6).

Analogously as in [6], we can show that
\[
GM(3 \beta, 1) \subsetneq GM(3 \beta, 2).
\]
In this paper we generalize and extend the P. Kórus results [5] to the class $GM(3 \beta, 2)$ concerning the uniform convergence of sine and cosine series.

1.2. The sine-cosine series

We start this section by giving some definitions and notations.

A double series
\[
\sum_{j=1}^{\infty} \sum_{k=1}^{n} z_{jk}
\]
of complex numbers converges regularly if the sums
\[
\sum_{j=1}^{m} \sum_{k=1}^{n} z_{jk}
\]
converge to a finite number as \( m \) and \( n \) tend to infinity independently of each other, moreover, both the column series and row series

\[
\sum_{j=1}^{\infty} z_{jn}, \quad n = 1, 2, \ldots, \quad \text{and} \quad \sum_{k=1}^{\infty} z_{mk}, \quad m = 1, 2, \ldots.
\]

are convergent.

Or equivalently, if for any \( \epsilon > 0 \) there exists a positive number \( m_0 = m_0(\epsilon) \) such that

\[
\left| \sum_{j=m}^{M} \sum_{k=n}^{N} z_{jk} \right| < \epsilon
\]

holds for any \( m, n, M, N \) for which \( m + n > m_0 \), \( 1 \leq m \leq M \) and \( 1 \leq n \leq N \).

A monotonically decreasing double sequence \( \{c_{jk}\}_{j,k=1}^{\infty} \) is a sequence of real numbers such that

\[
\Delta_{10}c_{jk} \geq 0, \quad \Delta_{01}c_{jk} \geq 0, \quad \Delta_{11}c_{jk} \geq 0, \quad j, k = 1, 2, \ldots,
\]

where

\[
\Delta_{10}c_{jk} := c_{jk} - c_{j+1,k}, \quad \Delta_{01}c_{jk} := c_{jk} - c_{j,k+1},
\]

\[
\Delta_{11}c_{jk} := \Delta_{10}(\Delta_{01}c_{jk}) = \Delta_{01}(\Delta_{10}c_{jk}) = c_{jk} - c_{j+1,k} - c_{j,k+1} + c_{j+1,k+1}.
\]

Let \( \{c_{jk}\}_{j,k=1}^{\infty} \) be a double sequence of complex numbers. Consider the sine-cosine series

\[
(8) \quad \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} c_{jk} \sin jx \cos ky.
\]

**Definition 5** [4]. A double sequence \( \{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C} \) belongs to the class \( SBVDS_1 \), if there exist a positive constant \( C \) and an integer \( \lambda \geq 2 \), and sequences \( \{b_1(l)\}_{l=1}^{\infty}, \{b_2(l)\}_{l=1}^{\infty}, \{b_3(l)\}_{l=1}^{\infty} \), each one tends (not necessarily monotonically) to infinity, all of them depend only on \( \{c_{jk}\} \), such that:

\[
(9) \quad \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{10}c_{jn}| \leq \frac{C}{m} \left( \max_{b_1(m) \leq M \leq b_1(M)} \sum_{j=M}^{2M} |c_{jn}| \right), \quad m \geq \lambda, \quad n \geq 1,
\]

\[
(10) \quad \sum_{k=n}^{2n-1} \sum_{j=m}^{2m-1} |\Delta_{01}c_{mk}| \leq \frac{C}{n} \left( \max_{b_2(n) \leq N \leq b_2(N)} \sum_{k=N}^{2N} |c_{mk}| \right), \quad n \geq \lambda, \quad m \geq 1,
\]

\[
(11) \quad \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{11}c_{jk}| \leq \frac{C}{mn} \left( \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right), \quad m, n \geq \lambda.
\]
In [5] Körus proved the following theorem for the sine-cosine series (8).

**Theorem 6** [5]. Suppose that \( \{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C} \) belongs to the class \( SBVDS_1 \).

(i) If condition

\[
jkc_{jk} \to 0 \quad \text{as} \quad j + k \to \infty
\]

holds and there exists an \( m_1 \geq 1 \) such that

\[
\max_{m \leq m_1} \sum_{k=n}^{\infty} c_{mk} \to 0 \quad \text{and} \quad \sup_{m \geq m_1} \sum_{k=n}^{\infty} |c_{mk}| \to 0 \quad \text{as} \quad n \to \infty,
\]

then the regular convergence of the sine-cosine series (8) is uniform in \((x,y)\).

(ii) Conversely, if \( \{c_{jk}\}_{j,k=1}^{\infty} \) is non-negative and the regular convergence of (8) is uniform in \((x,y)\), then (12) holds and (13) is satisfied for any \( m_1 \).

Now, we shall define the new class of double sequences in the following way:

Let \( r \in \mathbb{N} \), \( \Delta_{0r} c_{jk} := c_{jk} - c_{j+r,k} \), \( \Delta_{0r} c_{jk} := c_{jk} - c_{j,k+r} \) and \( \Delta_{rr} c_{jk} := \Delta_{0r}(\Delta_{0r} c_{jk}) \).

**Definition 7.** A double sequence \( \{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C} \) belongs to the class \( DGM(\alpha, \beta, \gamma, r) \) (called Double General Monotone), if there exist a positive constant \( C \) and an integer \( \lambda \geq 2 \) depend only on \( \{c_{jk}\} \), for which:

\[
\sum_{j=m}^{2m-1} |\Delta_{0r} c_{jn}| \leq C \alpha_{mn}, \quad m \geq \lambda, \quad n \geq 1,
\]

\[
\sum_{k=n}^{2n-1} |\Delta_{0r} c_{mk}| \leq C \beta_{mn}, \quad n \geq \lambda, \quad m \geq 1,
\]

\[
\sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{rr} c_{jk}| \leq C \gamma_{mn}, \quad m, n \geq \lambda
\]

hold, where \( \alpha := \{\alpha_{mn}\}_{m,n=1}^{\infty}, \beta := \{\beta_{mn}\}_{m,n=1}^{\infty}, \gamma := \{\gamma_{mn}\}_{m,n=1}^{\infty} \) are non-negative double sequences and \( r \in \mathbb{N} \).

Using our definition for \( r = 1 \), we have: \( SBVDS_1 \equiv DGM(2\alpha, 2\beta, 2\gamma, 1) \), where \( \{2\alpha\}, \{2\beta\} \) and \( \{2\gamma\} \) are the sequences defined by the formulas on the right hand sides of the inequalities (9), (10) and (11), respectively.

In this paper we shall present some properties of the classe \( DGM(2\alpha, 2\beta, 2\gamma, 2) \). Moreover, we generalize and extend the results of Körus [5] to the class \( DGM(2\alpha, 2\beta, 2\gamma, 2) \).
2. Mean results

2.1. Trigonometric series of one variable

The following results holds true:

**Theorem 8.** Let \( \{c_k\}_{k=1}^{\infty} \subset \mathbb{C} \) belong to the class \( GM(3\beta, 2) \).

(i) If (3) is satisfied, then (1) is uniformly convergent.

(ii) Conversely, if \( \{c_k\}_{k=1}^{\infty} \) is a non-negative sequence and (1) convergent uniformly, then (3) holds.

**Remark 9.** If we confine our attention to the class \( GN(3\beta, 1) \), then by (7) the Kórus result [3] follows from Theorem 8.

**Theorem 10.** Let \( \{c_k\}_{k=1}^{\infty} \subset \mathbb{C} \) belong to the class \( GM(3\beta, 2) \).

(i) If (3),

\[
\sum_{k=1}^{\infty} c_k < \infty
\]

and

\[
\sum_{k=1}^{\infty} (-1)^k c_k < \infty
\]

are satisfied, then (2) is uniformly convergent.

(ii) Conversely, if \( \{c_k\}_{k=1}^{\infty} \) is a non-negative sequence and (2) convergent uniformly, then (3), (14) and (15) holds.

**Remark 11.** If the series \( \sum_{k=1}^{\infty} c_k \) is absolutely convergent, then (14) and (15) hold. Moreover, the converse implication is not true.

**Remark 12.** There exist \( x_0 \in \mathbb{R} \) and a sequence \( \{c_k\}_{k=1}^{\infty} \) belonging to the class \( GM(3\beta, 2) \), with the properties (3) and (14), such that the series (2) is divergent in \( x_0 \).

2.2. Trigonometric series of two variables

We have the following results:

**Theorem 13.** Suppose that \( \{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C} \) belongs to \( DGM(2\alpha, 2\beta, 2\gamma, 2) \).

(i) If (12) holds and there exists an \( m_1 \geq 1 \) such that (13) holds and

\[
\max_{m < m_1} \sum_{k=n}^{\infty} (-1)^k c_{mk} \to 0
\]
are satisfied, then the regular convergence of sine-cosine series (8) is uniform in \((x, y)\).

(ii) Conversely, if \(\{c_{jk}\}_{j,k=1}^{\infty}\) is non-negative and the regular convergence of (8) is uniform in \((x, y)\), then (12) holds and (13), (16) are satisfied for any \(m_1\).

**Theorem 14.**

(i) \(DGM(2\alpha, 2\beta, 2\gamma, 1) \subset DGM(2\alpha, 2\beta, 2\gamma, 2)\) (see [2, Corollary 1]).

(ii) There exists a double sequence \(\{c_{jk}\}_{j,k=1}^{\infty}\), with the property (12), (13) and (16) which belongs to the class \(DGM(2\alpha, 2\beta, 2\gamma, 2)\) but it does not belong to the class \(DGM(2\alpha, 2\beta, 2\gamma, 1)\).

3. Auxiliary results

3.1. Lemmas for one variable

Denote, for \(r \in \mathbb{N}\) and \(k = 0, 1, 2, \ldots\), by

\[
D_{k,r}(x) = \frac{\sin(k + \frac{r}{2})x}{2\sin(\frac{r}{2}x)} \quad \text{and} \quad \tilde{D}_{k,r}(x) = \frac{\cos(k + \frac{r}{2})x}{2\sin(\frac{r}{2}x)}
\]

the Dirichlet type kernels.

**Lemma 15** [7, Lemma 17]. Let \(r \in \mathbb{N}\), \(l \in \mathbb{Z}\) and \(\{a_k\}_{k=1}^{\infty} \subset \mathbb{C}\). If \(x \neq \frac{2\pi l}{r}\), then for all \(m \geq n\)

\[
\sum_{k=n}^{m} a_k \cos kx = \sum_{k=n}^{m} \Delta_r a_k D_{k,r}(x) - \sum_{k=m+1}^{m+r} a_k D_{k,-r}(x) + \sum_{k=n}^{n+r-1} a_k D_{k,-r}(x)
\]

and

\[
\sum_{k=n}^{m} a_k \sin kx = -\sum_{k=n}^{m} \Delta_r a_k \tilde{D}_{k,r}(x) + \sum_{k=m+1}^{m+r} a_k \tilde{D}_{k,-r}(x) - \sum_{k=n}^{n+r-1} a_k \tilde{D}_{k,-r}(x).
\]

**Lemma 16.** If \(\{c_k\}_{k=1}^{\infty}\) is a non-negative sequence belonging to the class \(GM(3\beta, 2)\) with \(C\) and \(\{b(l)\}_{l=1}^{\infty}\), then for any \(n \geq 1\)

\[
nc_n \leq C \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k \right) + 2 \sum_{k=n}^{2n+1} c_k.
\]
Proof. Let \( n \geq 1 \). If \( \{c_k\}_{k=1}^{\infty} \in GM(3\beta,2) \), then for any \( n \leq \nu \leq 2n \)
\[
c_n = \sum_{k=n}^{\nu-1} \Delta_2 c_k + c_\nu + c_{\nu+1} - c_n \leq \sum_{k=n}^{2n-1} |\Delta_2 c_k| + c_\nu + c_{\nu+1}
\]
(19)
\[
\leq \frac{C}{n} \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k \right) + c_\nu + c_{\nu+1}.
\]
Adding up all inequalities in (19) for \( \nu = n+1, n+2, \ldots, 2n \) we obtain
\[
\sum_{\nu=n+1}^{2n} c_n \leq \sum_{\nu=n+1}^{2n} \frac{C}{\nu} \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k \right) + \sum_{\nu=n+1}^{2n} (c_\nu + c_{\nu+1}).
\]
Hence we get
\[
nc_n \leq C \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k \right) + 2 \sum_{k=n}^{2n} c_k.
\]
This ends the proof. ■

3.2. Lemmas for two variables

In this section we give some lemmas for two variables.

Lemma 17. Let \( \{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C} \) and \( m, M, n, N \in \mathbb{N} \) such that \( m \leq M \) and \( n \leq N \).

(i) If \( x \in \left(0, \frac{\pi}{2}\right)\), then
\[
\left| \sum_{j=m}^{M} c_{jk} \sin jx \right| \leq \frac{\pi}{4x} \left( \sum_{j=m}^{M} |\Delta_2 c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right)
\]
(20)
and if \( x \in \left(\frac{\pi}{2}, \pi\right)\), then
\[
\left| \sum_{j=m}^{M} c_{jk} \sin jx \right| \leq \frac{\pi}{4(\pi - x)} \left( \sum_{j=m}^{M} |\Delta_2 c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right)
\]
(21)
for any \( k \in \mathbb{N} \).

(ii) If \( y \in \left(0, \frac{\pi}{2}\right)\), then
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\[ \left| \sum_{k=n}^{N} c_{jk} \cos ky \right| \leq \frac{\pi}{4y} \left( \sum_{k=n}^{N} |\Delta_{02}c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right) \]

and if \( y \in \left( \frac{\pi}{2}, \pi \right) \), then

\[ \left| \sum_{k=n}^{N} c_{jk} \cos ky \right| \leq \frac{\pi}{4(\pi - y)} \left( \sum_{k=n}^{N} |\Delta_{02}c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right) \]

for any \( j \in \mathbb{N} \).

**Proof.** We will prove only the part (ii) because the part (i) was proved in [2, Lemma 5]. By (17), we have

\[ \left| \sum_{k=n}^{N} c_{jk} \cos ky \right| = \left| \sum_{k=n}^{N} \Delta_{02}c_{jk}D_{k,2}(y) - \sum_{k=N+1}^{N+2} c_{jk}D_{k,-2}(y) + \sum_{k=n}^{n+1} c_{jk}D_{k,-2}(y) \right| \]

\[ \leq \sum_{k=n}^{N} |\Delta_{02}c_{jk}| \cdot |D_{k,2}(y)| + \sum_{k=N+1}^{N+2} |c_{jk}| \cdot |D_{k,-2}(y)| + \sum_{k=n}^{n+1} |c_{jk}| \cdot |D_{k,-2}(y)|. \]

If \( y \in \left( 0, \frac{\pi}{2} \right) \), then using the inequality \( \sin y \geq \frac{2}{\pi}y \) we obtain the following estimation:

\[ |D_{k,\pm 2}(y)| \leq \frac{\sin(k \pm 1)y}{2 \sin(\pm y)} \leq \frac{1}{2 \sin y} \leq \frac{1}{\frac{\pi}{4}} = \frac{\pi}{4y}. \]

From this we get

\[ \left| \sum_{k=n}^{N} c_{jk} \cos ky \right| \leq \frac{\pi}{4y} \left( \sum_{k=n}^{N} |\Delta_{02}c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right). \]

If \( y \in \left( \frac{\pi}{2}, \pi \right) \), then using the inequality \( \sin y \geq 2 - \frac{2}{\pi}y \) we have the estimation:

\[ |D_{k,\pm 2}(y)| \leq \frac{1}{2(2 - \frac{2}{\pi}y)} \leq \frac{\pi}{4(\pi - y)} \]

and consequently

\[ \left| \sum_{k=n}^{N} c_{jk} \cos ky \right| \leq \frac{\pi}{4(\pi - y)} \left( \sum_{k=n}^{N} |\Delta_{02}c_{jk}| + \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{k=n}^{n+1} |c_{jk}| \right). \]

This ends the proof. \( \blacksquare \)
Lemma 18. Suppose \( \{c_{jk}\}_{j,k=1}^{\infty} \subset \mathbb{C} \), a constant \( C > 0 \), an integer \( \lambda \geq 2 \), sequences \( \{b_1(l)\}_{l=1}^{\infty} \), \( \{b_2(l)\}_{l=1}^{\infty} \), each one tends to infinity, all of them depend only on \( \{c_{jk}\} \) and the condition (12) is satisfied.

(i) If the inequality

\[
2^{m-1} \sum_{j=m}^{\infty} |\Delta_2 c_{jn}| \leq \frac{C}{m} \left( \max_{b_1(m) \leq M \leq b_1(m)} \sum_{j=M}^{2M} |c_{jn}| \right), \quad m \geq \lambda, \quad n \geq 1
\]

holds, then

\[
m \sup_{k \geq n} \sum_{j=m}^{\infty} |\Delta_2 c_{jk}| \to 0
\]

(ii) if the inequality

\[
2^{n-1} \sum_{k=n}^{\infty} |\Delta_2 c_{mk}| \leq \frac{C}{n} \left( \max_{b_2(n) \leq N \leq b_2(n)} \sum_{k=N}^{2N} |c_{mk}| \right), \quad n \geq \lambda, \quad m \geq 1
\]

holds, then

\[
n \sup_{j \geq m} \sum_{k=n}^{\infty} |\Delta_2 c_{jk}| \to 0
\]

as \( m + n \to \infty \), where \( m \geq \lambda \) and \( n \geq 1 \) or \( n \geq \lambda \) and \( m \geq 1 \), respectively.

Proof. Part (i). Set \( \epsilon > 0 \) arbitrarily. By condition (12) and from the fact that \( \{b_1(l)\}_{l=1}^{\infty} \) tends to infinity, there exists an \( m_1 = m_1(\epsilon) \) such that

\[
jk|c_{jk}| < \epsilon \quad \text{for all } j, k \text{ where } j + k > b_1(m_1).
\]

Using (24), for \( m \geq \max\{m_1, \lambda\} \) and \( n > m_1 \), we have

\[
\sup_{k \geq n} \sum_{j=m}^{\infty} |\Delta_2 c_{jk}| = \sup_{k \geq n} \sum_{r=0}^{2^r+1} \sum_{j=2^rm}^{2^{r+1}m-1} |\Delta_2 c_{jk}|
\]

\[
\leq \sup_{k \geq n} \sum_{r=0}^{\infty} \frac{C}{2^r m} \left( \max_{b_1(2^r m) \leq M \leq b_1(2^{r+1} m)} \sum_{j=M}^{2M} |c_{jk}| \right)
\]

\[
\leq \sup_{k \geq n} \frac{C}{m} \sum_{r=0}^{\infty} \frac{1}{2^r} \left( \max_{b_1(2^r m) \leq M \leq b_1(2^{r+1} m)} \sum_{j=M}^{2M} jk|c_{jk}| \frac{1}{jk} \right)
\]

\[
< \frac{Ce}{m} \sum_{k \geq n} \sum_{r=0}^{\infty} \frac{1}{2^r} \left( \max_{b_1(2^r m) \leq M \leq b_1(2^{r+1} m)} \frac{M+1}{M} \right) \leq \frac{2Ce}{m} \sum_{r=0}^{\infty} \frac{1}{2^r} < \frac{4Ce}{m}.
\]
This implies that (25) holds.

**Part** (ii). Set $\epsilon > 0$ arbitrarily. By condition (12) and from the fact that \( \{b_2(l)\}_{l=1}^{\infty} \) tends to infinity, there exists an \( n_1 = n_1(\epsilon) \) such that
\[
jk |c_{jk}| < \epsilon \quad \text{for all } j, k \quad \text{where } j + k > b_2(n_1).
\]
Using (26), for \( n \geq \max\{n_1, \lambda\} \) and \( m > n_1 \), we have
\[
\sup_{j \geq m} j \sum_{k=n}^{\infty} |\Delta_{02}c_{jk}| = \sup_{j \geq m} j \sum_{r=0}^{\infty} \sum_{k=2^r n}^{2^{r+1} n-1} |\Delta_{02}c_{jk}|
\]
\[
\leq \sup_{j \geq m} j \sum_{r=0}^{\infty} \frac{C}{2^r n} \left( \max_{b_2(2^r n) \leq N \leq \lambda b_2(2^r n)} \sum_{k=N}^{2N} |c_{jk}| \right)
\]
\[
\leq \sup_{j \geq m} j \sum_{r=0}^{\infty} \frac{C}{2^r} \left( \max_{b_2(2^r n) \leq N \leq \lambda b_2(2^r n)} \sum_{k=N}^{2N} jk |c_{jk}| \frac{1}{jk} \right)
\]
\[
< \frac{C\epsilon}{n} \sup_{j \geq m} j \sum_{r=0}^{\infty} \frac{1}{2^r} \left( \max_{b_2(2^r n) \leq N \leq \lambda b_2(2^r n)} N + 1 \frac{N}{n} \right) \leq \frac{2C\epsilon}{n} \sum_{r=0}^{\infty} \frac{1}{2^r} \leq 4C \epsilon.
\]
Hence (27) is satisfied.

Now, our proof is completed.

\[\text{Lemma 19 [2, Lemma 1]. If } \{c_{jk}\} \subset \mathbb{C} \text{ is such that the condition (12) is satisfied, a constant } C > 0, \text{ an integer } \lambda \geq 2, \text{ a sequences } \{b_3(l)\}_{l=1}^{\infty} \text{ tends to infinity, all of them depend only on } \{c_{jk}\} \text{ and the inequality}
\]
\[
2^{m-1} \sum_{k=n}^{2n-1} |\Delta_{22}c_{jk}| \leq \frac{C}{mn} \left( \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} |c_{jk}| \right), \quad m, n \geq \lambda
\]
holds, then
\[
\sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{22}c_{jk}| \to 0 \quad \text{as } m + n \to \infty, \quad m, n \geq \lambda.
\]

\[\text{Lemma 20 [2, Lemma 3]. If } \{c_{jk}\}_{j,k=1}^{\infty} \text{ is a non-negative sequence belonging to the class } DGM(\{\alpha, 2, \beta, 2, \gamma, 2\} \text{ with } C, \lambda \text{ and } \{b_1(l)\}_{l=1}^{\infty}, \{b_2(l)\}_{l=1}^{\infty}, \{b_3(l)\}_{l=1}^{\infty} \text{ then for any } m, n \geq \lambda
\]
\[
\text{for } \{b_1(l)\}_{l=1}^{\infty}, \{b_2(l)\}_{l=1}^{\infty}, \{b_3(l)\}_{l=1}^{\infty} \text{ then for any } m, n \geq \lambda
\]
\[
\text{for } \{b_1(l)\}_{l=1}^{\infty}, \{b_2(l)\}_{l=1}^{\infty}, \{b_3(l)\}_{l=1}^{\infty} \text{ then for any } m, n \geq \lambda
\]
\[
\text{for } \{b_1(l)\}_{l=1}^{\infty}, \{b_2(l)\}_{l=1}^{\infty}, \{b_3(l)\}_{l=1}^{\infty} \text{ then for any } m, n \geq \lambda
\]
\[
\text{for } \{b_1(l)\}_{l=1}^{\infty}, \{b_2(l)\}_{l=1}^{\infty}, \{b_3(l)\}_{l=1}^{\infty} \text{ then for any } m, n \geq \lambda
\]
4. Proofs of the main results

In this section we shall prove our main results.

4.1. Proof of Theorem 8

Part (i). We will show that the series (1) is uniform convergent. Let \( \epsilon > 0 \) be arbitrary fixed. We will prove that for any \( N \geq n > \eta(\epsilon) \) and all \( x \in \mathbb{R} \) we have

\[
|s(n,N;x)| < (\pi C + \pi + 1)\epsilon
\]

where

\[
s(n,N;x) := \sum_{k=n}^{N} c_k \sin kx.
\]

The inequality (30) is trivial, when \( x = 0 \). We have the same situation, if \( x = \pi \).

Suppose \( x \in (0,\frac{\pi}{2}) \), set \( \nu := \left\lceil \frac{1}{x} \right\rceil \), where \( \lceil \cdot \rceil \) means the integer part of a real number. We have two cases:

Case (a). \( \eta < n \leq N < \nu \). Using the inequality \( \sin x \leq x \) and (3), we have

\[
|s(n,N;x)| = \left| \sum_{k=n}^{N} c_k \sin kx \right| \leq \frac{\pi}{4x} \left( \sum_{k=n}^{N} |\Delta_2 c_k| + \sum_{k=N+1}^{n+2} |c_k| + \sum_{k=n}^{n+1} |c_k| \right) \leq \frac{\pi}{4} \sum_{k=n}^{\infty} |\Delta_2 c_k| + \pi \sup_{k \geq n} |c_k|.
\]

Since \( (c_k)_{k=1}^{\infty} \in GM(3,\beta,2) \) and using (3), we have

\[
\sum_{k=n}^{\infty} |\Delta_2 c_k| = \sum_{r=0}^{\infty} \sum_{k=2^r n}^{2^{r+1} n-1} |\Delta_2 c_k| \leq \sum_{r=0}^{\infty} \frac{C}{2^{r+1} n} \left( \sup_{m \geq b(2^r n)} \sum_{k=m}^{2m} |c_k| \right) \leq \frac{C}{n} \sum_{r=0}^{\infty} \frac{1}{2^r} \left( \sup_{m \geq b(2^r n)} \sum_{k=m}^{2m} \frac{1}{k} \right) \leq \frac{4C}{n} \epsilon.
\]

and consequently

\[
|s(n,N;x)| < (\pi C + \pi)\epsilon.
\]
Let \( x \in \left( \frac{\pi}{2}, \pi \right) \), set \( \nu = \left\lceil \frac{1}{\pi - x} \right\rceil \). We consider two cases

\textbf{Case (a\*).} \( \eta < n \leq N < \nu \). Using the inequality \( \sin x \leq \pi - x \) and (3), we obtain

\[
|s(n, N; x)| = \left| \sum_{k=n}^{N} c_k \sin kx \right| \leq (\pi - x) \sum_{k=n}^{N} k |c_k| < \frac{1}{\nu} \sum_{k=n}^{\nu} \epsilon \leq \epsilon.
\]

\textbf{Case (b\*).} \( \max \{\eta, \nu\} < n \leq N \). Applying Lemma 15 and the inequality \( \sin x \geq \frac{2}{\pi}(\pi - x) \), analogously as in the case (b), we have

\[
|s(n, N; x)| = \left| \sum_{k=n}^{N} c_k \sin kx \right| \leq \frac{\pi}{4(\pi - x)} \left( \sum_{k=n}^{N} |\Delta_2 c_k| + \sum_{k=N+1}^{N+2} |c_k| + \sum_{k=n}^{n+1} |c_k| \right)
\]

\[
< \left( \pi C + \pi \right) \epsilon.
\]

Summing up all partial estimations, we get (30) and this ends the proof of the part (i).

\textbf{Part (ii).} Suppose that \( \{c_k\}_{k=1}^{\infty} \) is non-negative and let \( \epsilon > 0 \) be arbitrarily fixed. Using the uniform convergence of (1), we find that there exists an integer \( \eta = \eta(\epsilon) \) for which

\[
(31) \quad \left| \sum_{k=n}^{N} c_k \sin ky \right| < \epsilon
\]

holds for any \( n > \eta \) and all \( x \in \mathbb{R} \). Set \( x(n) = \frac{\pi}{4n} \), we have

\[
\sin(kx(n)) \geq \sin \frac{\pi}{4} \quad \text{if} \quad n \leq k \leq 2n + 1.
\]

Since \( \{b(l)\}_{l=1}^{\infty} \) tends to infinity, there exists \( \nu \in \mathbb{N} \) such that \( b(n) > \eta \) for \( n > \nu \). Then, for \( n > \max \{\eta, \nu\} \) by (31) and Lemma 16, we have

\[
(C + 2)\epsilon > C \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k \sin(kx(n)) \right) + 2 \sum_{k=n}^{2n+1} c_k \sin(kx(n)).
\]

Next

\[
(C + 2)\epsilon > C \sin \frac{\pi}{4} \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k \right) + 2 \sin \frac{\pi}{4} \sum_{k=n}^{2n+1} c_k
\]

and finally, we have

\[
(C + 2)\epsilon > \left( \sin \frac{\pi}{4} \right) nc \quad \text{whenever} \quad n > \max \{\eta, \nu\}.
\]

Whence it completes the proof of the part (ii). \( \blacksquare \)
4.2. Proof of Theorem 10

Part (i). Let $\epsilon > 0$ be arbitrary fixed. We will prove that for any $N \geq n > \eta(\epsilon)$ and all $x \in \mathbb{R}$ we have

$$|c(n, N; x)| < (\pi C + \pi + 2)\epsilon$$

where

$$c(n, N; x) := \sum_{k=n}^{N} c_k \cos kx.$$

Using (14) and (15) we get that the inequality (32) is true for $x = 0$ and $x = \pi$, respectively. Let $x \in (0, \frac{\pi}{2})$, set $\left\lceil \frac{1}{x} \right\rceil$. Two cases are considered

Case (a). $\eta < n \leq N < \nu$. Using (14), the inequality $\sin x \leq x$ and (3), we obtain

$$|c(n, N; x)| = \left| \sum_{k=n}^{N} c_k \cos kx \right| = \left| \sum_{k=n}^{N} c_k - \sum_{k=n}^{N} c_k(1 - \cos kx) \right|$$

$$\leq \sum_{k=n}^{N} |c_k| + \sum_{k=n}^{N} 2c_k \sin^2 \frac{kx}{2} < \epsilon + \sum_{k=n}^{N} 2|c_k| \left| \sin \frac{kx}{2} \right|$$

$$\leq \epsilon + x \sum_{k=n}^{N} k|c_k| \leq \epsilon + \frac{1}{\nu} \sum_{k=n}^{\nu} k|c_k| < 2\epsilon.$$

Case (b). $\max \{\eta, \nu\} < n \leq N$. By Lemma 15 and analogously as in the proof of Theorem 8, we have

$$|c(n, N; x)| = \left| \sum_{k=n}^{N} c_k \cos kx \right| \leq \frac{\pi}{4x} \left( \sum_{k=n}^{N} |\Delta_2 c_k| + \sum_{k=n+1}^{N} |c_k| + \sum_{k=n}^{n+1} |c_k| \right)$$

$$< (\pi C + \pi)\epsilon.$$

Let $x \in \left( \frac{\pi}{2}, \pi \right)$ set $\nu = \left\lceil \frac{1}{\pi - x} \right\rceil$. We have two cases:

Case (a*). $\eta < n \leq N < \nu$. Using (15), the inequality $\sin x \leq \pi - x$ and (3), we get

$$|c(n, N; x)| = \left| \sum_{k=n}^{N} c_k \cos kx \right| = \left| \sum_{k=n}^{N} (-1)^k c_k - \sum_{k=n}^{N} c_k((-1)^k - \cos kx) \right|$$

$$\leq \sum_{k=n}^{N} (-1)^k |c_k| + \sum_{k=n}^{N} c_k(\cos k\pi - \cos kx)$$
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\[< \epsilon + \sum_{k=n}^N c_k \left( -2 \sin \frac{k(\pi - x)}{2} \sin \frac{k(\pi + x)}{2} \right)\]

\[\leq \epsilon + \sum_{k=n}^N \left| c_k \left( -2 \sin \frac{k(\pi - x)}{2} \right) \right| \leq \epsilon + (\pi - x) \sum_{k=n}^N k|c_k|\]

\[\leq \epsilon + \frac{1}{\nu} \sum_{k=n}^\nu k|c_k| < 2\epsilon.\]

**Case (b**). \[\max\{\eta, \nu\} < n \leq N.\] By Lemma 15 and analogously as in the proof of Theorem 8, we have

\[|c(n, N; x)| = \left| \sum_{k=n}^N c_k \cos kx \right| \leq \frac{\pi}{4(\pi - x)} \left( \sum_{k=n}^N |\Delta_2 c_k| + \sum_{k=N+1}^n |c_k| + \sum_{k=n}^{n+1} |c_k| \right)\]

\[< (\pi C + \pi)\epsilon.\]

It completes the proof of the part (i).

**Part (ii).** Suppose that \(\{c_k\}_{k=1}^\infty\) is non-negative and let \(\epsilon > 0\) be arbitrarily fixed. Using the uniform convergence of (2), we find that there exists an integer \(\eta = \eta(\epsilon)\) for which

\[|\sum_{k=n}^N c_k \cos kx| < \epsilon \quad (33)\]

holds for any \(n > \eta\) and all \(x \in \mathbb{R}\). If \(x = 0\) or \(x = \pi\), then from (33) we obtain that (14) and (15) hold immediately, respectively.

Since \(\{b(l)\}_{l=1}^\infty\) tends to infinity, there exists \(\nu \in \mathbb{N}\) such that \(b(n) > \eta\) for \(n > \nu\). Then, for \(n > \max\{\eta, \nu\}\) by (33) with \(x = 0\) and Lemma 16, we have

\[(C + 2)\epsilon > C \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} c_k \right) + 2 \sum_{k=n}^{2n+1} c_k\]

whence

\[(C + 2)\epsilon > nc_n \quad \text{whenever} \quad n > \max\{\eta, \nu\}.\]

It completes the proof of the part (ii).

### 4.3. Proof of Remark 11

First implication is immediately. Let \(c_k = a_k \cdot b_k\) where

\[a_k = \frac{1}{k \ln(k + 1)} \quad \text{and} \quad b_k = 1, 1, -1, -1, 1, 1, \ldots\]
Using Dirichlet’s test, we have
\[ \sum_{k=1}^{\infty} a_k b_k < \infty \quad \text{because} \quad a_k \to \infty \quad \text{as} \quad k \to \infty \quad \text{and} \quad \left| \sum_{k=1}^{n} b_k \right| \leq 2 \quad \text{for all} \quad n \in \mathbb{N}, \]
and \[ \sum_{k=1}^{\infty} a_k b_k (-1)^k < \infty \quad \text{because} \quad a_k \to \infty \quad \text{as} \quad k \to \infty \quad \text{and} \quad \left| \sum_{k=1}^{n} b_k (-1)^k \right| \leq 1 \]
for all \( n \in \mathbb{N} \).

Moreover,
\[ \sum_{k=1}^{\infty} |c_k| = \sum_{k=1}^{\infty} \frac{1}{k \ln(k+1)} = \infty. \]
This ends the proof.

4.4. Proof of Remark 12

Let \( c_k = \frac{(-1)^k}{k \ln k} \) and \( x_0 = \pi \). It is easy to see that (3) and (14) hold. Now, we shall prove that \( (c_k)_{k=1}^{\infty} \in GM(3\beta, 2) \). We have
\[
\sum_{k=n}^{2n-1} |\Delta_2 c_k| = \sum_{k=n}^{2n-1} \left| \frac{(-1)^k}{k \ln(k)} - \frac{(-1)^{k+2}}{(k+2) \ln(k+2)} \right|
\]
\[
= \sum_{k=n}^{2n-1} \frac{2 k \ln(k+2) - \ln(k) + 2 \ln(k+2)}{k(k+2) \ln(k) \ln(k+2)}.
\]
Applying the Lagrange mean value theorem, for the function \( y = \ln(x) \), there exists \( c \in (k, k+2) \) such that
\[ \ln(k+2) - \ln(k) = \frac{2}{c} \leq \frac{2}{k} \]
and
\[
\sum_{k=n}^{2n-1} |\Delta_2 c_k| \leq \sum_{k=n}^{2n-1} \frac{k \ln(k) \ln(k+2)}{k(k+2) \ln(k) \ln(k+2)} \leq \sum_{k=n}^{2n-1} \frac{4 \ln(k+2)}{k^2 \ln(k) \ln(k+2)}
\]
\[ \leq \frac{4}{n} \sum_{k=n}^{2n-1} \frac{1}{k \ln(k)} = \frac{4}{n} \sum_{k=n}^{2n-1} |c_k| \leq \frac{4}{n} \left( \sup_{m \geq b(n)} \sum_{k=m}^{2m} |c_k| \right). \]
Hence \( \{c_k\}_{k=1}^{\infty} \in GM(3\beta, 2) \).

Moreover, we have
\[
\sum_{k=1}^{n} c_k \cos kx_0 = \sum_{k=1}^{n} \frac{(-1)^k}{k \ln k} \cos k\pi = \sum_{k=1}^{n} \frac{(-1)^{2k}}{k \ln k} = \sum_{k=1}^{n} \frac{1}{k \ln k} \to \infty \quad \text{as} \quad n \to \infty.
\]
This ends the proof.
4.5. **Proof of Theorem 13**

**Part (i).** Using Theorem 8 and Theorem 10 we deduce that the single series:

\[
\sum_{j=1}^{\infty} c_{jn} \sin jx, \quad n = 1, 2, \ldots, \quad \sum_{k=1}^{\infty} c_{mk} \cos ky, \quad m = 1, 2, \ldots
\]

are uniformly convergent, because \(\{c_{jn}\}_{j=1}^{\infty} \in GM(3\alpha, 2)\) for any \(n \in \mathbb{N}\) and \(\{c_{mk}\}_{k=1}^{\infty} \in GM(3\beta, 2)\) for any \(m \in \mathbb{N}\), respectively.

Let \(\epsilon > 0\) be arbitrarily fixed. We shall prove that for any \(M \geq m > \eta, N \geq n > \eta\) and all \((x, y) \in \mathbb{R}^2\) we have

\[
|s(m, M; n, N; x, y)| < (2 + \pi \lambda C + 2\pi C + 3\pi + 3\pi^2 C + \pi^2)\epsilon
\]

where

\[
s(m, M; n, N; x, y) := \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky
\]

and \(\eta = \eta(\epsilon)\) is the natural number such that for \(m, n > \eta\),

\[
mn|c_{mn}| < \epsilon, \quad mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{22} c_{jk}| < 16\epsilon,
\]

\[
m \sum_{j=m}^{\infty} \sup_{k \geq n} |\Delta_{20} c_{jk}| < 4\epsilon, \quad n \sum_{k=n}^{\infty} \sup_{j \geq m} |\Delta_{02} c_{jk}| < 4\epsilon
\]

and for \(m \geq m_1\)

\[
\sum_{k=n}^{\infty} m|c_{mk}| < \epsilon.
\]

The inequality (35) is trivial, when \(x = 0\) and \(y\) is arbitrary. We have the same situation, if \(x = \pi\) and \(y\) is arbitrary. Suppose \(x \in (0, \frac{\pi}{2})\), set \(\mu := \left\lfloor \frac{1}{x} \right\rfloor\) and \(y = 0\) or \(y = \pi\). We discuss two cases:

**Case (a).** \(\eta < m \leq M < \mu\) and \(\eta < n \leq N\). Using the inequality \(\sin x \leq x\) and (36), we have

\[
|s(m, M; n, N; x, 0)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \right| \leq x \sum_{j=m}^{M} \sum_{k=n}^{N} j|c_{jk}|
\]

\[
\leq \frac{1}{\mu} \sum_{j=m}^{\mu} \sup_{j \geq m} \sum_{k=n}^{N} j|c_{jk}| < \epsilon
\]
and

\[ |s(m, M; n, N; x, \pi)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} (-1)^{j} c_{jk} \sin jx \right| < \epsilon. \]

**Case (b0).** \( \max \{\eta, \mu\} < m \leq M \) and \( \eta < n \leq N \). Using (20) and (36), we get

\[ |s(m, M; n, N; x, 0)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \right| \leq \left| \sum_{k=n}^{N} \sum_{j=m}^{M} c_{jk} \sin jx \right| \]

\[ \leq \frac{\pi}{4x} \sum_{k=n}^{N} \left( \sum_{j=m}^{M} |\Delta_{20}c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) \]

\[ \leq \frac{\pi \mu}{4} \sum_{k=n}^{\infty} \left( \sum_{j=m}^{\infty} |\Delta_{20}c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) \]

\[ \leq \frac{\pi}{4} m \sum_{k=n}^{\infty} \sum_{j=m}^{\infty} |\Delta_{20}c_{jk}| + \pi \sup_{j \geq m} \sum_{k=n}^{\infty} j|c_{jk}| < \frac{\pi}{4} m \sum_{k=n}^{\infty} \left( \sum_{r=0}^{\infty} \sum_{j=2r+1}^{2r+2} |\Delta_{20}c_{jk}| \right) + \pi \epsilon \]

\[ \leq \frac{\pi}{4} \sum_{k=n}^{\infty} \left( \sum_{r=0}^{\infty} \frac{C}{2r} \left( \max_{b_{1}(2r \leq M \leq \lambda b_{1}(2r \leq M))} \sum_{j=M}^{2M} j|c_{jk}| \right) \right) \]

\[ \leq \frac{\pi}{4} \sum_{k=n}^{\infty} \sum_{r=0}^{\infty} \frac{C}{2} \left( \max_{b_{1}(m) \leq M \leq \lambda b_{1}(m)} \sum_{j=M}^{2M} \frac{j|c_{jk}|}{j} \right) \]

\[ \leq \frac{\pi}{2} \sum_{k=n}^{\infty} \sum_{j=b_{1}(m)}^{\lambda b_{1}(m)} j|c_{jk}| + \pi \epsilon \leq \frac{\pi}{2} \sum_{j=b_{1}(m)}^{\lambda b_{1}(m)} \frac{1}{j} \sum_{j=k-n}^{\infty} j|c_{jk}| + \pi \epsilon < \pi \lambda \epsilon + \pi \epsilon \]

and

\[ |s(m, M; n, N; x, \pi)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} (-1)^{j} c_{jk} \sin jx \right| \leq \sum_{k=n}^{N} \sum_{j=m}^{M} c_{jk} \sin jx \leq \pi \lambda \epsilon + \pi \epsilon. \]

Let \( x \in \left( \frac{\pi}{2}, \pi \right) \) and \( y = 0 \) or \( y = \pi \), set \( \mu := \left[ \frac{1}{\pi - x} \right] \). Two cases are possible to hold:

**Case (a1).** \( \eta < m \leq M < \mu \) and \( \eta < n \leq N \). Using (36) and the inequality \( \sin x \leq \pi - x \), we have
\[ |s(m, M; n, N; x, 0)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \right| \leq (\pi - x) \sum_{j=m}^{M} \sum_{k=n}^{N} |c_{jk}| \]
\[ \leq \frac{1}{\mu} \sum_{j=m}^{M} \sup_{j \geq m} \sum_{k=n}^{N} j|c_{jk}| < \epsilon \]

and
\[ |s(m, M; n, N; x, \pi)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} (-1)^{k} c_{jk} \sin jx \right| < \epsilon. \]

Case \((b_0)\). \(\max \{\eta, \mu\} < m \leq M\) and \(\eta < n \leq N\). By (21) and analogously as in the case \((b_0)\), we get
\[ |s(m, M; n, N; x, 0)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \right| \leq \sum_{k=n}^{N} \left( \sum_{j=m}^{M} |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| \right) < \pi \lambda C \epsilon + \pi \epsilon \]

and
\[ |s(m, M; n, N; x, \pi)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} (-1)^{k} c_{jk} \sin jx \right| < \pi \lambda C \epsilon + \pi \epsilon. \]

Now, let \(x, y \in (0, \frac{\pi}{2})\), set \(\mu := \left\lceil \frac{1}{x} \right\rceil\) and \(v := \left\lceil \frac{1}{y} \right\rceil\). Four cases can occur:

Case \((a)\). \(\eta < m \leq M < \mu\) and \(\eta < n \leq N < v\). Using the inequality \(\sin x \leq x\) and (12), we get
\[ |s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| \leq \sum_{j=m}^{M} \sin jx \sum_{k=n}^{N} c_{jk} \cos ky \]
\[ = \sum_{j=m}^{M} \sin jx \left( \sum_{k=n}^{N} c_{jk} - \sum_{k=n}^{N} c_{jk} (1 - \cos ky) \right) \]
\[ = \sum_{j=m}^{M} \sin jx \left( \sum_{k=n}^{N} c_{jk} - \sum_{k=n}^{N} 2c_{jk} \sin^{2} \frac{ky}{2} \right) \]
\[
\begin{align*}
&= \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx - \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \right| \\
&< |s(m, M; n, N; x, 0)| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \right| \\
&< \varepsilon + xy \sum_{j=m}^{M} \sum_{k=n}^{N} |c_{jk}| < \varepsilon + \frac{1}{\mu v} \sum_{j=m}^{M} \sum_{k=n}^{N} \epsilon < 2\epsilon.
\end{align*}
\]

**Case (b).** $\max \{\eta, \mu\} < m \leq M$ and $\eta < n \leq N < v$. We have

\[
|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| \\
\leq \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx - \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \right| \\
< |s(m, M; n, N; x, 0)| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \right| \\
< \pi \lambda C\epsilon + \pi\epsilon + y \sum_{k=n}^{N} k \left| \sum_{j=m}^{M} c_{jk} \sin jx \right|.
\]

Applying (12), (20) and (28), we obtain

\[
|s(m, M; n, N; x, y)| \\
\leq \pi \lambda C\epsilon + \pi\epsilon + y \sum_{k=n}^{N} k \frac{\pi}{4x} \left( \sum_{j=m}^{M} |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) \\
< \pi \lambda C\epsilon + \pi\epsilon + \frac{1}{\nu} \sum_{k=n}^{N} \frac{\pi}{4} \left( m \sup_{k \geq n} \sum_{j=m}^{\infty} |\Delta_{20} c_{jk}| + 4 \sup_{j \geq m, k \geq n} |c_{jk}| \right) \\
< \pi \lambda C\epsilon + \pi\epsilon + \frac{\pi}{4} \left( m \sup_{k \geq n} \sum_{j=m}^{\infty} |\Delta_{20} c_{jk}| \right) < \pi \lambda C\epsilon + \pi\epsilon + 2\pi\epsilon.
\]

**Case (c).** $\eta < m \leq M < \mu$ and $\max \{\eta, v\} < n \leq N$. We have
Using (12), (22) and (29), we get

\[ |s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| = \left| \sum_{k=n}^{N} c_{jk} \cos ky \sum_{j=m}^{M} \sin jx \right| \left| s \right| \leq x \sum_{j=m}^{M} \left| \sum_{k=n}^{N} c_{jk} \cos ky \right| \leq \frac{1}{\mu} \sum_{j=m}^{M} \left| \sum_{k=n}^{N} c_{jk} \cos ky \right|.

Using (12), (22) and (29), we get

\[
|s(m, M; n, N; x, y)| \leq \frac{1}{\mu} \sum_{j=m}^{M} \frac{\pi}{4} \left( n \sup_{j} \sum_{k=n}^{\infty} |\Delta_{02} c_{jk}| + 4 \sup_{j} \sup_{k} j |c_{jk}| \right) < \pi \varepsilon + \frac{\pi}{4} \left( n \sup_{j} \sum_{k=n}^{\infty} |\Delta_{02} c_{jk}| \right) < \pi \varepsilon + \pi \varepsilon.
\]

Case (d). \( \max \{\eta, \mu\} < m \leq M \) and \( \max \{\eta, v\} < n \leq N \). Using Lemma 15, we get

\[
|s(m, M; n, N; x, y)| = \sum_{k=n}^{N} \left( - \sum_{j=m}^{M} \Delta_{02} (\Delta_{20} c_{jk}) \tilde{D}_{j,2}(x) + \sum_{j=M+1}^{M+2} \Delta_{02} c_{jk} \tilde{D}_{j,-2} \right.
\]
\[
- \sum_{j=m}^{M+1} \Delta_{02} c_{jk} \tilde{D}_{j,-2}(x) \right) D_{k,2}(y) - \sum_{k=N+1}^{N+2} \left( - \sum_{j=m}^{M} \Delta_{20} c_{jk} \tilde{D}_{j,2}(x) \right.
\]
\[
+ \sum_{j=M+1}^{M+2} c_{jk} \tilde{D}_{j,-2} - \sum_{j=m}^{M+1} c_{jk} \tilde{D}_{j,-2}(x) \right) D_{k,-2}(y)
\]
\[
+ \sum_{k=n+1}^{n+1} \left( - \sum_{j=m}^{M} \Delta_{20} c_{jk} \tilde{D}_{j,2}(x) + \sum_{j=M+1}^{M+2} c_{jk} \tilde{D}_{j,-2} - \sum_{j=m}^{M+1} c_{jk} \tilde{D}_{j,-2}(x) \right)
\]
\[
\cdot D_{k,-2}(y)\]
\[
\leq \sum_{j=m}^{M} \sum_{k=n}^{N} |\Delta_{22} c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \cdot |D_{k,2}(y)| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{N} |\Delta_{02} c_{jk}| \cdot |\tilde{D}_{j,-2}(x)|
\]
\[
\cdot |D_{k,2}(y)| + \sum_{j=m}^{M+1} \sum_{k=n}^{N} |\Delta_{02} c_{jk}| \cdot |\tilde{D}_{j,-2}(x)| \cdot |D_{k,2}(y)| + \sum_{j=m}^{M} \sum_{k=N+1}^{N+2} |\Delta_{20} c_{jk}| \cdot |\tilde{D}_{j,2}(x)|
\]
\[
\cdot |D_{k,-2}(y)|
\]
\[ + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \cdot |D_{k,2}(y)| + \sum_{j=m}^{m+1} \sum_{k=n}^{N+2} |c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \]
\[ \cdot |D_{k,2}(y)| \]
\[ + \sum_{j=m+1}^{M} \sum_{k=n}^{n+1} |\Delta_{20}c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \cdot |D_{k,2}(y)| + \sum_{j=M+1}^{M+2} \sum_{k=n+1}^{n+1} |c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \]
\[ \cdot |D_{k,2}(y)| \]
\[ + \sum_{j=m}^{m+1} \sum_{k=n}^{n+1} |c_{jk}| \cdot |\tilde{D}_{j,2}(x)| \cdot |D_{k,2}(y)|. \]

By Lemma 18, Lemma 19 and (12), we obtain
\[ |s(m, M; n, N; x, y)| \]
\[ \leq \frac{\pi^2}{16xy} \left( \sum_{j=m}^{M} \sum_{k=n}^{N} |\Delta_{22}c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{N} |\Delta_{02}c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^{N} |\Delta_{02}c_{jk}| \right) \]
\[ + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+2} |\Delta_{20}c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| \]
\[ + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+1} |c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n+1}^{n+1} |\Delta_{20}c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^{N} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=n}^{N} |c_{jk}| \right) \]
\[ \leq \frac{\pi^2}{16} \left( mn \sum_{j=m}^{\infty} \sum_{k=n}^{\infty} |\Delta_{22}c_{jk}| + 4m \sup_{k \geq n} j \sum_{j=m}^{\infty} |\Delta_{20}c_{jk}| + 4n \sup_{j \geq n} j \sum_{k=n}^{\infty} |\Delta_{02}c_{jk}| \right) \]
\[ + 16 \sup_{j \geq m} j |c_{jk}| \]
\[ < \frac{\pi^2}{16} (16\epsilon + 16\epsilon + 16\epsilon + 16\epsilon) = 3\pi^2 \epsilon + \pi^2 \epsilon. \]

Let \( x \in \left( \frac{\pi}{2}, \pi \right) \) and \( y \in (0, \frac{\pi}{2}) \), set \( \mu : = \left\lfloor \frac{1}{\pi - x} \right\rfloor \) and \( v := \left\lfloor \frac{1}{y} \right\rfloor \). We have four cases:

Case (a*). \( \eta < m \leq M < \mu \) and \( \eta < n \leq N < v \). Using the inequality \( \sin x \leq \pi - x \) and (12), we get
\[ |s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| \]
\[ = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx - \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \right| \]
On the uniform convergence of sine, cosine series ...

\[ < |s(m, M; n, N; x, 0)| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \right| \]

\[ \leq \epsilon + (\pi - x)y \sum_{j=m}^{M} \sum_{k=n}^{N} j|c_{jk}| < \epsilon + \frac{1}{\mu v} \sum_{j=m}^{M} \sum_{k=n}^{N} \epsilon < 2\epsilon. \]

**Case (b"):** \( \max \{\eta, \mu\} < m \leq M \) and \( \eta < n \leq N < v \). Applying (21) we get

\[ |s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| \]

\[ < |s(m, M; n, N; x, 0)| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin^2 \frac{ky}{2} \right| \]

\[ \leq \pi \lambda C \epsilon + \pi \epsilon + y \sum_{k=n}^{N} \frac{\pi}{4(\pi - x)} \left( \sum_{j=m}^{M} |\Delta_{2c_{jk}}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) \]

\[ < \pi \lambda C \epsilon + \pi \lambda C + 2\pi \epsilon. \]

**Case (c"):** \( \eta < m \leq M < \mu \) and \( \max \{\eta, v\} < n \leq N \). Analogously as in the case (c), we have

\[ |s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| = \left| \sum_{k=n}^{N} c_{jk} \cos ky \sum_{j=m}^{M} \sin jx \right| \]

\[ \leq (\pi - x) \sum_{j=m}^{M} j \sum_{k=n}^{N} c_{jk} \cos ky \] \( < \pi \lambda C \epsilon + \pi \epsilon. \)

**Case (d"):** \( \max \{\eta, \mu\} < m \leq M \) and \( \max \{\eta, v\} < n \leq N \). Using (21) and (22) analogously as in the case (d), we obtain

\[ |s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| \]

\[ \leq \frac{\pi \epsilon}{16(\pi - x)y} \left( \sum_{j=m}^{M} \sum_{k=n}^{N} |\Delta_{22c_{jk}}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{N} |\Delta_{02c_{jk}}| + \sum_{j=m}^{m+1} \sum_{k=n}^{N} |\Delta_{02c_{jk}}| \right) \]

\[ + \sum_{j=m}^{M} \sum_{k=N+1}^{N+2} |\Delta_{20c_{jk}}| + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^{m+1} \sum_{k=N+1}^{N+2} |c_{jk}| \]
Let \( x \in (0, \frac{\pi}{2}) \) and \( y \in (\frac{\pi}{2}, \pi) \), set \( \mu := \left\lceil \frac{1}{x} \right\rceil \) and \( v := \left\lfloor \frac{1}{\pi - y} \right\rfloor \). Now, we have also four cases:

**Case (a**). \( \eta < m \leq M < \mu \) and \( \eta < n \leq N < v \). Using the inequality \( \sin x \leq x \), we obtain similarly as in the case (a*):

\[
|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| = \left| \sum_{j=m}^{M} \sin jx \sum_{k=n}^{N} c_{jk} \cos ky \right|
\]

\[
= \left| \sum_{j=m}^{M} \sin jx \left( \sum_{k=n}^{N} (-1)^k c_{jk} - \sum_{k=n}^{N} c_{jk} \left( -1 \right)^k - \cos ky \right) \right|
\]

\[
= \left| \sum_{j=m}^{M} \sin jx \left( \sum_{k=n}^{N} (-1)^k c_{jk} + \sum_{k=n}^{N} 2c_{jk} \sin \frac{k(\pi - y)}{2} \sin \frac{k(\pi + y)}{2} \right) \right|
\]

\[
= \left| \sum_{j=m}^{M} \sum_{k=n}^{N} (-1)^k c_{jk} \sin jx + \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin \frac{k(\pi - y)}{2} \sin \frac{k(\pi + y)}{2} \right|
\]

\[
\leq |s(m, M; n, N; x, \pi)| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin \frac{k(\pi - y)}{2} \sin \frac{k(\pi + y)}{2} \right| < \epsilon + x(\pi - y) \sum_{j=m}^{M} \sum_{k=n}^{N} jk|c_{jk}| < \epsilon + \frac{1}{\mu v} \sum_{j=m}^{\mu} \sum_{k=n}^{v} \epsilon < 2\epsilon.
\]

**Case (b**). \( \max \{\eta, \mu\} < m \leq M \) and \( \eta < n \leq N < v \). We obtain, similarly as in the case (b),

\[
|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right|
\]

\[
< |s(m, M; n, N; x, \pi)| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin \frac{k(\pi - y)}{2} \sin \frac{k(\pi + y)}{2} \right|
\]

\[
< \pi \lambda C \epsilon + \pi \epsilon + (\pi - y) \sum_{k=n}^{N} k \frac{\pi}{4x} \left( \sum_{j=m}^{M} |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} |c_{jk}| + \sum_{j=m}^{m+1} |c_{jk}| \right) < \pi \lambda C \epsilon + \pi C \epsilon + 2\pi \epsilon.
\]
Case (c**). $\eta < m \leq M < \mu$ and $\max \{\eta, v\} < n \leq N$. Applying (22) we get

$$|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| = \left| \sum_{k=n}^{M} c_{jk} \cos ky \sum_{j=m}^{M} \sin jx \right|$$

$$\leq x \sum_{j=m}^{M} \sum_{k=n}^{N} |c_{jk}| \cos ky < \pi \epsilon + \pi \epsilon.$$

Case (d**). $\max \{\eta, \mu\} < m \leq M$ and $\max \{\eta, v\} < n \leq N$. Using (20) and (23) and analogously as in the case (d), we get

$$|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right|$$

$$\leq \frac{\pi^2}{16x(\pi - y)} \left( \sum_{j=m}^{M} \sum_{k=n}^{N} |\Delta_{22} c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{N} |\Delta_{02} c_{jk}| + \sum_{j=m}^{M+1} \sum_{k=n}^{N} |\Delta_{02} c_{jk}| \right)$$

$$+ \sum_{j=m}^{M} \sum_{k=N+1}^{N+2} |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=N+1}^{N+2} |c_{jk}| + \sum_{j=m}^{M+1} \sum_{k=N+1}^{N+2} |c_{jk}|$$

$$+ \sum_{j=m}^{M} \sum_{k=n+1}^{n+2} |\Delta_{20} c_{jk}| + \sum_{j=M+1}^{M+2} \sum_{k=n+1}^{n+2} |c_{jk}| + \sum_{j=m}^{M+1} \sum_{k=n+1}^{n+2} |c_{jk}| \right) < 3\pi^2 \epsilon + \pi^2 \epsilon.$$

Finally, let $x \in \left(\frac{\pi}{2}, \pi\right)$ and $y \in \left(\frac{\pi}{2}, \pi\right)$, set $\mu := \left\lceil \frac{1}{\pi - x} \right\rceil$ and $v := \left\lceil \frac{1}{\pi - y} \right\rceil$. Analogously as before we have four cases:

Case (a***) (ii). $\eta < m \leq M < \mu$ and $\eta < n \leq N < v$. Using the inequality $\sin x \leq \pi - x$, we get

$$|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right|$$

$$< \left| s(m, M; n, N; x, \pi) \right| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin \frac{k(\pi - y)}{2} \sin \frac{k(\pi + y)}{2} \right|$$

$$< \epsilon + (\pi - x)(\pi - y) \sum_{j=m}^{M} \sum_{k=n}^{N} jk |c_{jk}| < 2\epsilon.$$
Case (b**). \( \max \{ \eta, \mu \} < m \leq M \) and \( \eta < n \leq N < v \). We have, similarly as in the case (b*),

\[
|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right|
\]

\[
< |s(m, M; n, N; x, \pi)| + \left| \sum_{j=m}^{M} \sum_{k=n}^{N} 2c_{jk} \sin jx \sin \frac{k(\pi - y)}{2} \sin \frac{k(\pi + y)}{2} \right|
\]

\[
< \pi \lambda C \epsilon + \pi \epsilon + (\pi - y) \sum_{k=n}^{N} \frac{\pi}{4(\pi - x)} \left( \sum_{j=m}^{M} \left| \Delta_{20} c_{jk} \right| + \sum_{j=M+1}^{M+2} \left| c_{jk} \right| + \sum_{j=m}^{M+1} \left| c_{jk} \right| \right)
\]

\[
< \pi \lambda C \epsilon + \pi \epsilon + 2\pi \epsilon.
\]

Case (c**). \( \eta < n \leq N \leq \mu \) and \( \max \{ \eta, v \} < n \leq N < v \). We obtain, similarly as in the case (c*),

\[
|s(m, M; n, N; x, y)| = \left| \sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky \right| = \left| \sum_{k=n}^{N} \sum_{j=m}^{M} c_{jk} \cos ky \sum_{j=m}^{M} \sin jx \right|
\]

\[
\leq (\pi - x) \sum_{j=m}^{M} \left| \sum_{k=n}^{N} c_{jk} \cos ky \right| < \pi \epsilon \epsilon + \pi \epsilon.
\]

Case (d**). \( \max \{ \eta, \mu \} < m \leq M \) and \( \max \{ \eta, v \} < n \leq N \). Using (21) and (23), we have

\[
|s(m, M; n, N; x, y)| \leq \frac{\pi^2}{16(\pi - x)(\pi - y)} \left( \sum_{j=m}^{M} \sum_{k=n}^{N} \left| \Delta_{22} c_{jk} \right| + \sum_{j=M+1}^{M+2} \sum_{k=n}^{N} \left| \Delta_{02} c_{jk} \right| \right)
\]

\[
+ \sum_{j=m}^{M+1} \sum_{k=n}^{N+2} \left| \Delta_{02} c_{jk} \right| + \sum_{j=m}^{M+2} \sum_{k=n+1}^{N+2} \left| \Delta_{20} c_{jk} \right| + \sum_{j=M+1}^{M+2} \sum_{k=n+1}^{N+2} \left| c_{jk} \right|
\]

\[
+ \sum_{j=m}^{M+1} \sum_{k=n+1}^{N+2} \left| c_{jk} \right| + \sum_{j=m+1}^{M+2} \sum_{k=n+1}^{N+2} \left| \Delta_{20} c_{jk} \right| + \sum_{j=M+1}^{M+2} \sum_{k=n+1}^{N+2} \left| c_{jk} \right| + \sum_{j=m}^{M+1} \sum_{k=n}^{N+1} \left| c_{jk} \right|
\]

\[
< 3\pi^2 \epsilon \epsilon + \pi^2 \epsilon.
\]

Summing up all partial estimations, we get (35). This ends the proof of the part (i).
Therefore
\[
\sum_{j=m}^{M} \sum_{k=n}^{N} c_{jk} \sin jx \cos ky < \epsilon
\]
holds for any \(m + n > m_0\) and all \((x, y) \in \mathbb{R}^2\). Set \(x_1(m) = \frac{\pi}{4m}, x_2(m) = \frac{\pi}{4\lambda b_1(m)}\), we have
\[
\sin(jx_1(m)) \geq \sin \frac{\pi}{4} \quad \text{if} \quad m \leq j \leq 2m + 1;
\]
\[
\sin(jx_2(m)) \geq \sin \frac{\pi}{4\lambda} \quad \text{if} \quad 0 < b_1(m) \leq j \leq 2\lambda b_1(m) + 1.
\]
Since \(\{b_1(l)\}_{l=1}^{\infty}, \{b_2(l)\}_{l=1}^{\infty}, \{b_3(l)\}_{l=1}^{\infty}\) tends to infinity, there exists an \(m_1\) such that for any \(m, n: m + n > m_1\) implies \(m + n > m_0, b_1(m) + n > m_0, m + b_2(n) > m_0\) and \(b_3(m + n) > m_0\). Then by (37) and Lemma 20, we have for \(m + n > m_1\)
\[
(5C + 8)\epsilon > C \left( \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \sin(jx_1(M)) \right)
+ 2C \sum_{j=b_1(m)}^{2\lambda b_1(m) + 2n + 1} \sum_{k=n}^{2m + 1} c_{jk} \sin(jx_2(m)) + 2C \sum_{j=m}^{2m + 1} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} \sin(jx_1(m))
+ 8 \sum_{j=m}^{2m + 1} \sum_{k=b_2(n)}^{2n + 1} c_{jk} \sin(jx_1(m)).
\]
Therefore
\[
(5C + 8)\epsilon > C \sin \frac{\pi}{4} \left( \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} c_{jk} \right) + 2C \sin \frac{\pi}{4\lambda} \sum_{j=b_1(m)}^{2\lambda b_1(m) + 2n + 1} \sum_{k=n}^{2m + 1} c_{jk}
+ 2C \sin \frac{\pi}{4} \sum_{j=m}^{2m + 1} \sum_{k=b_2(n)}^{2\lambda b_2(n)} c_{jk} + 8 \sin \frac{\pi}{4} \sum_{j=m}^{2m + 1} \sum_{k=b_2(n)}^{2n + 1} c_{jk}
\]
and finally, we have
\[
(5C + 8)\epsilon > \left( \sin \frac{\pi}{4\lambda} \right) mnc_{mn} \quad \text{whenever} \quad m + n > m_1 \quad \text{and} \quad m, n > \lambda.
\]
Hence (12) is satisfied when \(j + k \to \infty\) and \(j, k \geq \lambda\). If \(j \to \infty\) and \(k < \lambda\) or \(j < \lambda\) and \(k \to \infty\), (12) follows from the uniform convergence of the series in (34). It completes the proof of the part (ii).
4.6. Proof of Theorem 14

Let

\[ c_{jk} = 1 + \frac{(-1)^j}{j^2} \cdot \frac{1 + (-1)^k}{k^2} \quad \text{for} \quad j, k \in \mathbb{N}. \]

We show that \( \{c_{jk}\}_{j, k = 1}^{\infty} \in \text{DGM}(2\alpha_2, 2\beta_2, 2\gamma, 2) \). It easy to see that

\[ \Delta_{20}c_{jk} = c_{jk} - c_{j+2,k} = c_{jk} \cdot \frac{4(j+1)}{(j+2)^2} \]

and

\[ \sum_{j=m}^{2m-1} |\Delta_{20}c_{jn}| = \sum_{j=m}^{2m-1} \left| c_{jn} \cdot \frac{4(j+1)}{(j+2)^2} \right| \leq \sum_{j=m}^{2m-1} |c_{jn}| \frac{4}{j+1} \]

\[ \leq 4 \sum_{j=m}^{2m-1} \frac{|c_{jn}|}{j} \leq \frac{4}{m} \sum_{j=m}^{2m-1} |c_{jn}| \leq \frac{4}{m} \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} |c_{jn}|. \]

Similarly as above

\[ \Delta_{02}c_{jk} = c_{jk} - c_{j,k+2} = c_{jk} \cdot \frac{4(k+1)}{(k+2)^2} \]

and

\[ \sum_{k=n}^{2n-1} |\Delta_{02}c_{mk}| \leq \frac{4}{n} \left( \max_{b_2(n) \leq N \leq \lambda b_2(n)} \sum_{k=N}^{2N} |c_{mk}| \right). \]

By elementary calculations

\[ \Delta_{22}c_{jk} = c_{jk} - c_{j+2,k} - c_{j,k+2} + c_{j+2,k+2} = c_{jk} \cdot \frac{4(j+1)}{(j+2)^2} \cdot \frac{4(k+1)}{(k+2)^2} \]

and

\[ \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} |\Delta_{22}c_{jk}| = \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} \left| c_{jk} \cdot \frac{4(j+1)}{(j+2)^2} \cdot \frac{4(k+1)}{(k+2)^2} \right| \]

\[ \leq 16 \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} \left| c_{jk} \right| \cdot \frac{1}{j+1} \cdot \frac{1}{k+1} \leq 16 \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} \left| c_{jk} \right| \frac{1}{jk} \]

\[ \leq \frac{16}{mn} \sum_{j=m}^{2m-1} \sum_{k=n}^{2n-1} \left| c_{jk} \right| \leq \frac{16}{mn} \left( \sup_{M+N \geq b_3(m+n)} \sum_{j=M}^{2M} \sum_{k=N}^{2N} \left| c_{jk} \right| \right). \]
Therefore $\{c_{jk}\}_{j,k=1}^{\infty} \in DGM(2\alpha, 2\beta, 2\gamma, 2)$. 

Now, we show that $\{c_{jk}\}_{j,k=1}^{\infty} \notin DGM(2\alpha, 2\beta, 2\gamma, 1)$. We have

$$
\sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| = \sum_{j=m}^{2m-1} \left| \frac{(-1)^j + 1}{j^2} \cdot \frac{(-1)^n + 1}{n^2} \right| = \frac{(-1)^n + 1}{n^2} \sum_{j=m}^{2m-1} \left| \frac{1 + (-1)^j}{j^2} - \frac{1 - (-1)^j}{(j+1)^2} \right|.
$$

Let $n$ be even and $A_m = \{ j : m \leq j \leq 2m-1 \text{ and } j \text{ is even} \}$, then

$$
\sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| \geq \frac{2}{n^2} \sum_{j \in A_m} \frac{2}{j^2} \geq \frac{4}{n^2} \sum_{j \in A_m} \frac{1}{j^2} \geq \frac{4m-1}{2n^2m^2}
$$

and since

$$
\frac{C}{m} \left( \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} |c_{jn}| \right) 
= \frac{C}{m} \left( \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} \left| \frac{(-1)^j + 1}{j^2} \cdot \frac{(-1)^n + 1}{n^2} \right| \right) 
\leq \frac{C}{m} \cdot \frac{2}{n^2} \left( \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} \frac{2}{j^2} \right) \leq \frac{4C}{mn^2} \left( \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} \frac{1}{j^2} \right) 
\leq \frac{8C}{mn^2} \left( \max_{b_1(m) \leq M \leq \lambda b_1(m)} \frac{1}{M} \right) \leq \frac{8C}{mn^2 b_1(m)}
$$

the inequality

$$
\sum_{j=m}^{2m-1} |\Delta_{10} c_{jn}| \leq \frac{C}{m} \left( \max_{b_1(m) \leq M \leq \lambda b_1(m)} \sum_{j=M}^{2M} |c_{jn}| \right)
$$

does not hold because $\frac{1}{b_1(m)} \to 0$ as $m \to \infty$. This ends the proof. 

References


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