A NOTE ON VARIATIONAL-TYPE INEQUALITIES FOR 
$(\eta, \theta, \delta)$- PSEUDOMONOTONE-TYPE SET-VALUED 
MAPPINGS IN NONREFLEXIVE BANACH SPACES

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Abstract

In this paper the existence of solutions to variational-type inequalities 
problems for $(\eta, \theta, \delta)$- pseudomonotone-type set-valued mappings in nonre- 
flexive Banach spaces introduced in [4] is considered. Presented theorem 
does not require a compact set-valued mapping, but requires a weaker con- 
dition 'locally bounded' for the mapping.

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1. Introduction

Variational inequalities for monotone single-valued mappings in nonreflexive Banach spaces was firstly considered by Chang, Lee and Chen in [1]. After that Watson in [7] and Verma [6] studied variational inequalities for pseudomonotonicity and strong pseudomonotonicity for single-valued mappings in nonreflexive Banach spaces, respectively. Lee and Lee in [3] introduced a notion of $(\eta, \theta)$- pseudomonotonicity for single valued mappings in nonreflexive Banach spaces which extends previously mentioned notions. $(\theta, \eta)$- pseudomonotonicity for set-valued mappings in nonreflexive Banach spaces was considered by Lee and Noh in [5]. Lee, Lee and Lee in [4] introduced $(\eta, \theta, \delta)$- pseudomonotone-type which general- izes $(\eta, \theta)$- pseudomonotonicity to set-valued case in nonreflexive Banach spaces. In this paper we show the existence of solutions to variational-type inequality problems for $(\eta, \theta, \delta)$- pseudomonotone-type set-valued mapping in nonreflexive
Banach spaces without the assumption of a compact set-valued mapping. We change this assumption into locally bounded mapping and the assumption which is quite similar to one of conditions in the definition of pseudomonotonicity in reflexive Banach spaces.

**Definition** [4]. Let $X$ be a real nonreflexive Banach space with the dual $X^*$ and $X^{**}$ the dual of $X^*$. Let $T : K \rightarrow 2^{X^*} \setminus \{\emptyset\}$ be a set-valued mapping, $\eta, \theta : K \times K \rightarrow X^{**}$ be operators and $\delta : K \times K \rightarrow \mathbb{R}$ a function, where $K \subset X^{**}$. $T$ is said to be $(\eta, \theta, \delta)$-pseudomonotone-type, if there exists a constant $r$ (called $(\eta, \theta, \delta)$-pseudomonotone-type constant of $T$) such that for all $x, y \in K$ and $v \in T(y)$ there exists $u \in T(x)$ such that

$$\langle v, \eta(x, y) \rangle + \delta(x, y) \geq 0 \text{ implies } \langle u, \eta(x, y) \rangle + \delta(x, y) \geq r\|\theta(x, y)\|^2.$$ 

**Definition** [4]. A set-valued mapping $T : X^{**} \supset K \rightarrow 2^{X^*}$ is said to be finite-dimensional u.s.c. if for any finite-dimensional subspace $F$ of $X^{**}$ with $K_F = K \cap F \neq \emptyset$, $T : K_F \rightarrow 2^{X^*}$ is u.s.c. in the norm topology.

**Definition**. A set-valued mapping $T : X^{**} \supset K \rightarrow 2^{X^*}$ is said to be locally bounded set-valued mapping if for any $x \in K$ there exist $M > 0$, $r > 0$ that for any $y \in B(x, r)$ and $v \in T(y)$ we have $\|v\| \leq M$.

**Definition** [2]. Let $K \subset X$, where $X$ is a topological vector space. A set-valued mapping $T : K \rightarrow 2^X$ is called a Knaster-Kuratowski-Mazurkiewicz mapping (in short KKM mapping) if for each nonempty finite subset $N$ of $K$, $\text{conv}(N) \subset T(N)$, where $\text{conv}$ denotes convex hull and $T(N) = \bigcup\{T(x) : x \in N\}$.

**Theorem 1** (KKM Theorem, [2]). Let $K$ be an arbitrary nonempty subset of a Hausdorff topological vector space $X$. Let a set-valued mapping $T : K \rightarrow 2^X$ be KKM mapping such that $T(x)$ is closed for all $x \in K$ and compact for at least one $x \in K$. Then

$$\bigcap_{x \in K} T(x) \neq \emptyset.$$ 

2. **Main results**

Let us remind the main result from [4].

**Theorem 2** (Theorem 2.3, [4]). Let $X$ be a real nonreflexive Banach space and $K$ a nonempty bounded closed convex subset of $X^{**}$. Let $T : K \rightarrow 2^{X^*}$ be an $(\eta, \theta, \delta)$-pseudomonotone-type, finite dimensional u.s.c., compact set-valued mapping, and $\eta, \theta : K \times K \rightarrow X^{**}$ be operators and $\delta : K \times K \rightarrow \mathbb{R}$ be a function such that
(i) for all $x \in K$, $\eta(x, x) = 0$, $\delta(x, x) = 0$;
(ii) for all $y \in K$ $x \to \eta(x, y)$, $x \to \theta(x, y)$ are affine and $x \to \delta(x, y)$ is convex;
(iii) for all $x \in K$ $y \to \eta(x, y)$, $y \to \theta(x, y)$ and $y \to \delta(x, y)$ are continuous.

Then there exists $x_0 \in K$ such that for all $x \in K$ there exists $v_0 \in T(x_0)$ such that
$$\langle v_0, \eta(x, x_0) \rangle + \delta(x, x_0) \geq 0.$$ 

The assumption of a compact set-valued mapping $T$ is quite restrictive. In the following theorem it will be proved that this assumption can be weaken.

**Theorem 3.** Let $X$ be a real nonreflexive Banach space and $K$ a nonempty bounded closed, convex subset of $X^{**}$. Let $T : K \to 2^{X^*}$ be an $(\eta, \theta, \delta)$- pseudo-monotone-type, finite dimensional u.s.c., locally bounded and such that the following condition is fulfilled

(a) for any $\{x_n\} \subset K$, the conditions $x_n \rightharpoonup x$ in $X^{**}$ and
$$\liminf_{n \to \infty} \langle v_n, \eta(x, x_n) \rangle + \delta(x, x_n) \geq 0,$$
with $v_n \in T(x_n)$ imply that for any $y \in K$ there exists $v(y) \in T(x)$ such that
$$\limsup_{n \to \infty} \langle v_n, \eta(y, x_n) \rangle + \delta(x, y) \leq \langle v(y), \eta(y, x) \rangle + \delta(y, x).$$

Moreover, let $\eta, \theta : K \times K \to X^{**}$ be operators and $\delta : K \times K \to \mathbb{R}$ be a functional such that

(i) for all $x \in K$, $\eta(x, x) = 0$, $\delta(x, x) = 0$;
(ii) for all $y \in K$ $x \to \eta(x, y)$, $x \to \theta(x, y)$ are affine and $x \to \delta(x, y)$ is convex;
(iii) for all $x \in K$ $y \to \eta(x, y)$, $y \to \theta(x, y)$ and $y \to \delta(x, y)$ are continuous.

Then there exists $x_0 \in K$ such that for all $x \in K$ there exists $v_0 \in T(x_0)$ such that
$$\langle v_0, \eta(x, x_0) \rangle + \delta(x, x_0) \geq 0.$$ 

**Proof.** The proof is very similar to the proof of the Theorem 2.3 in [4]. We prove that for each finite dimensional subspace $F$ of $X^{**}$ with $K_F = F \cap K \neq \emptyset$ there exists $x_0 \in K_F$ such that for all $x \in K_F$ there exists $v_0 \in T(x_0)$ such that
$$\langle v_0, \eta(x, x_0) \rangle + \delta(x, x_0) \geq 0.$$ 

The rest of the proof is the same as in [4].
Let $F$ be a subspace of $X^{**}$ with $K_F = F \cap K \neq \emptyset$. Define a set valued mapping $G : K_F \to 2^F$ for any $y \in K_F$ by

$$G(y) := \{ x \in K_F : \text{there exists } v \in T(x) \text{ such that } \langle v, \eta(y, x) \rangle + \delta(y, x) \geq 0 \}.$$

In [4] it was proved that the mapping $G$ is a KKM mapping.

Now, we show that $G$ is a closed valued mapping. Let $y \in K_F$ and $\{ x_n \} \in G(y)$ be such that $\lim_{n \to \infty} x_n = x_0$ in $F$. Then for any $n \in \mathbb{N}$ there exists $v_n \in T(x_n)$ such that

$$(1) \quad \langle v_n, \eta(y, x_n) \rangle + \delta(y, x_n) \geq 0.$$

From the fact that $T$ is a locally bounded mapping, the continuity of $\eta, \delta$ and the assumption (i) we get that

$$\liminf_{n \to \infty} \langle v_n, \eta(x_0, x_n) \rangle + \delta(x_0, x_n) = 0.$$

Then from the assumption (a) we get there exists $v(y) \in T(x_0)$ such that

$$\limsup_{n \to \infty} \langle v_n, \eta(y, x_n) \rangle + \delta(x_n, y) \leq \langle v(y), \eta(y, x_0) \rangle + \delta(y, x_0).$$

From (1) we obtain

$$0 \leq \langle v(y), \eta(y, x_0) \rangle + \delta(y, x_0),$$

which means that $x_0 \in G(y)$. Hence $G(y)$ is closed in $F$.

$G$ is compact valued mapping, because of the compactness of $K_F$. Consequently, from the KKM theorem $\bigcap_{x \in K_F} G(x) \neq \emptyset$.

**Remark 4.** It is quite obvious that if $T : K \to X^*$, $K \subset X^{**}$ is a compact set-valued mapping then $T$ is locally bounded and fulfilled the condition

(a) for any $\{ x_n \} \subset K$, the conditions $x_n \rightharpoonup x$ in $X^{**}$ and

$$\liminf_{n \to \infty} \langle v_n, \eta(x, x_n) \rangle + \delta(x, x_n) \geq 0,$$

with $v_n \in T(x_n)$ imply that for any $y \in K$ there exists $v(y) \in T(x)$ such that

$$\limsup_{n \to \infty} \langle v_n, \eta(y, x_n) \rangle + \delta(x_n, y) \leq \langle v(y), \eta(y, x) \rangle + \delta(y, x).$$

These conditions seem to be easier to check than the compactness of graph of mapping $T$. 


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References


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