ON THE EXISTENCE OF FIVE NONTRIVIAL SOLUTIONS FOR RESONANT PROBLEMS WITH p-LAPLACIAN

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Abstract

In this paper we study a nonlinear Dirichlet elliptic differential equation driven by the p-Laplacian and with a nonsmooth potential. The hypotheses on the nonsmooth potential allow resonance with respect to the principal eigenvalue $\lambda_1 > 0$ of $(-\Delta_p, W^{1,p}_0(Z))$. We prove the existence of five nontrivial smooth solutions, two positive, two negative and the fifth nodal.

Keywords: p-Laplacian, Clarke subdifferential, linking sets, upper-lower solutions, second eigenvalue, nodal and constant sign solutions, second deformation theorem.

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1. Introduction

Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with a $C^2$-boundary $\partial Z$. In this paper we study the following nonlinear elliptic problem with a nonsmooth potential:

\[
\begin{cases}
-\text{div}(|\nabla x(z)|^{p-2}\nabla x(z)) \in \partial j(z, x(z)) & \text{for a.a. } z \in Z, \\
x|_{\partial Z} = 0.
\end{cases}
\]

with $p \in (1, +\infty)$. Here $(z, \zeta) \mapsto j(z, \zeta)$ is a measurable potential which for almost all $z \in Z$, as a function of $\zeta \in \mathbb{R}$, is locally Lipschitz and in general nonsmooth. By $\partial j(z, \zeta)$ we denote the Clarke subdifferential of $\zeta \mapsto j(z, \zeta)$ (see Section 2.). Our goal is to prove a multiplicity result for problem (1.1), when the problem at infinity is resonant with respect to $\lambda_1 > 0$, the principal eigenvalue of $(-\Delta_p, W^{1,p}_0(Z))$. This implies that the corresponding Euler functional of the problem is indefinite. Moreover, we will provide full information about the sign of the solutions we obtain.

Let us mention some recent papers containing multiplicity results for the $p$-Laplacian equation. In Carl-Perera [3], Jiu-Su [12], Zhang-Chen-Li [20], Liu-Liu [15], Liu [16], Papageorgiou-Papageorgiou [18] and Carl-Motreanu [2], the Euler functionals of the problem is coercive and the authors produce at most three nontrivial solutions. Precise information about the sign of their solutions, can be found in Carl-Perera [3], Zhang-Chen-Li [20] and Carl-Motreanu [2]. Papers of Ambrosetti-Garcia Azorero-Peral Alonso [1] and Garcia Azorero-Manfredi-Peral Alonso [9], deal with an indefinite Euler functional.

2. Mathematical Background

Let $X$ be a Banach space and $X^*$ its topological dual. By $\| \cdot \|$ we denote the norm in $X$, by $\| \cdot \|_*$ the norm in $X^*$, and by $\langle \cdot, \cdot \rangle$ the duality brackets for the pair $(X, X^*)$. If $\varphi : X \mapsto \mathbb{R}$ is a locally Lipschitz function, then the generalized directional derivative of $\varphi$ at $x \in X$ in the direction $h \in X$ is defined by

$$
\varphi^0(x; h) = \limsup_{x' \to x} \frac{\varphi(x' + th) - \varphi(x')}{t}.
$$

It is easy to check that the function $X \ni h \mapsto \varphi^0(x; h) \in \mathbb{R}$ is sublinear, continuous, hence it is the support function of a nonempty, convex and
$w^*$-compact set, defined by
$$
\partial \varphi(x) = \{ x^* \in X^* : \langle x^*, h \rangle \leq \varphi^0(x; h) \text{ for all } h \in X \}.
$$

The multifunction $x \mapsto \partial \varphi(x)$ is called Clarke subdifferential of $\varphi$. For a given locally Lipschitz functional $\varphi : X \to \mathbb{R}$, we say that $x$ is a critical point of $\varphi$, if $0 \in \partial \varphi(x)$. It is easy to see that, if $x \in X$ is a local extremum of $\varphi$ (i.e., a local minimum or a local minimum of $\varphi$), then $x \in X$ is a critical point of $\varphi$.

We say that a locally Lipschitz functional $\varphi : X \to \mathbb{R}$ satisfies the Cerami condition at level $c \in \mathbb{R}$ (the $C_c$-condition for short), if every sequence $\{ x_n \}_{n \geq 1} \subseteq X$, such that $\varphi(x_n) \to c$ and $(1 + \| x_n \|) m^\varphi(x_n) \to 0$, with $m^\varphi(x_n) = \inf \{ \| x^* \| : x^* \in \partial \varphi(x_n) \}$, has a strongly convergent subsequence. We say that $\varphi$ satisfies the $C$-condition, if it satisfies the $C_c$-condition for every $c \in \mathbb{R}$.

Let $Y$ be a Hausdorff topological space, $E_0, E, D$ are nonempty closed subsets of $Y$ and $E_0 \subseteq E$. We say that the pair $\{E_0, E\}$ is linking with $D$ in $Y$ if and only if $E_0 \cap D = \emptyset$, and for any $\gamma \in C(E; Y)$ such that $\gamma|_{E_0} = id|_{E_0}$ we have $\gamma(E) \cap D \neq \emptyset$. Using this general topological notion, we can prove the following minimax principle for the critical values of a locally Lipschitz function.

**Theorem 2.1.** If $X$ is a Banach space, $E_0, E$ and $D$ are nonempty closed subset of $X$, such that the pair $\{E_0, E\}$ is linking with $D$ in $X$, $\varphi : X \to \mathbb{R}$ is locally Lipschitz,

$$
\sup_{E_0} \varphi \leq \inf_{D} \varphi,
$$

$$
\Gamma = \{ \gamma \in C(E; X) : \gamma|_{E_0} = id|_{E_0} \},
$$

$$
c = \inf_{\gamma \in \Gamma} \sup_{v \in E} \varphi(\gamma(v))
$$

and $\varphi$ satisfies the $C_c$-condition, then $c \geq \inf_{D} \varphi$ and $c$ is a critical value of $\varphi$. Moreover, if $c = \inf_{D} \varphi$, then there exists a critical point of $\varphi$ in $D$.

**Definition 2.2.** Let $C$ be a subset of the Banach space $X$. A continuous deformation of $C$ is a continuous map $h : [0, 1] \times C \to C$, such that $h(0, \cdot) = id_C$. If $B \subseteq C$, then we say that $B$ is a weak deformation retract (resp. a strong deformation retract) of $C$, if there exists a continuous deformation $h : [0, 1] \times C \to C$, such that $h([0, 1] \times B) \subseteq B$ (resp. $h(t, b) = b$ for all $t \in [0, 1]$ and all $b \in B$) and $h(1, C) \subseteq B$. 
For a given locally Lipschitz functional \( \varphi: X \rightarrow \mathbb{R} \), we introduce the following sets:

\[
\begin{align*}
\varphi^c &= \{ x \in X : \varphi(x) < c \} \quad \text{(the strict sublevel set of } \varphi \text{ at } c); \\
K^\varphi &= \{ x \in X : 0 \in \partial \varphi(x) \} \quad \text{(the critical set of } \varphi); \\
K^\varphi_c &= \{ x \in X : \varphi(x) = c \} \quad \text{(the critical set of } \varphi \text{ at level } c \in \mathbb{R}).
\end{align*}
\]

The next theorem is a nonsmooth version of the so-called “second deformation theorem” (see Chang [4, p. 23] and Gasinski-Papageorgiou [11, p. 628]).

**Theorem 2.3.** If \( X \) is a Banach space, \( \varphi: X \rightarrow \mathbb{R} \) is locally Lipschitz and satisfies the \( C \)-condition, \( a \in \mathbb{R}, \ a < b \leq +\infty, \varphi \) has no critical points in \( \varphi^{-1}(a, b) \) and \( K^\varphi_a \) is a finite set consisting of only local minima, then there exists a continuous deformation \( h: [0, 1] \times \varphi^b \rightarrow \varphi^b, \) such that

(a) \( h(t, \cdot)|_{K^\varphi_a} = \text{id}|_{K^\varphi_a} \) for all \( t \in [0, 1]; \)

(b) \( h(1, \varphi^b) \subseteq \varphi^a \cup K^\varphi_a; \)

(c) \( \varphi(h(t, x)) \leq \varphi(x) \) for all \( t \in [0, 1] \) and all \( x \in \varphi^b. \)

In particular, the set \( \varphi^a \cup K^\varphi_a \) is a weak deformation retract of \( \varphi^b. \) Next we recall some basic facts about the spectrum of the negative \( p \)-Laplacian with Dirichlet boundary condition. So, let \( m \in L^\infty(Z)_+, \ m \neq 0 \) and consider the following weighted (with weight \( m \)) nonlinear eigenvalue problem:

\[
\begin{align*}
(2.1) \quad &\begin{cases}
-\text{div}(\|\nabla u(z)\|^{p-2}\nabla u(z)) = \lambda m(z)|u(z)|^{p-2}u(z) \quad \text{for a.a. } z \in Z, \\
u|_{\partial Z} = 0,
\end{cases}
\end{align*}
\]

with \( 1 < p < +\infty, \lambda \in \mathbb{R}. \) In what follows we use the notation \( -\Delta_p u = -\text{div}(\|\nabla u(z)\|^{p-2}\nabla u(z)). \) By an eigenvalue of \( (-\Delta_p, W^{1,p}_0(Z), m), \) we mean a number \( \lambda(m) \in \mathbb{R}, \) such that (2.1) has a nontrivial solution \( u \in W^{1,p}_0(Z). \)

Nonlinear regularity theory implies that \( u \in C^1_0(Z) \) (see for example Gasinski-Papageorgiou [11, p. 737–738]). The least \( \lambda \in \mathbb{R} \) for which problem (2.1) has a nontrivial solution, is the first eigenvalue of \( (-\Delta_p, W^{1,p}_0(Z), m) \) and it is denoted by \( \lambda_1(m). \) We know that: \( \lambda_1(m) > 0; \lambda_1(m) \) is isolated (i.e., there exists \( \varepsilon > 0, \) such that \( (\lambda_1(m), \lambda_1(m) + \varepsilon) \) contains no eigenvalues of \( (-\Delta_p, W^{1,p}_0(Z), m); \) \( \lambda_1(m) \) is simple (i.e., the corresponding eigenspace is one-dimensional). The first eigenvalue \( \lambda_1(m) > 0 \) admits the following
The minimum in (2.2) is attained on the one-dimensional eigenspace of $\hat{\lambda}_1(m)$. By $u_1$ we denote the $L^p$-normalized eigenfunction for $\lambda_1(m)$. We already know that $u_1 \in C_0^1(\bar{Z})$ and from (2.2) it is clear that $u_1(z) \geq 0$ for all $z \in \bar{Z}$. Note that the Banach space $C_0^1(\bar{Z})$ is an ordered Banach space with order cone $C_+ = \{ x \in C_0^1(\bar{Z}) : x(z) \geq 0 \text{ for all } z \in \bar{Z} \}$. This cone has a nonempty interior and in fact

$$\text{int } C_+ = \{ x \in C_+ : x(z) > 0 \text{ for all } z \in Z \text{ and } \frac{\partial x}{\partial n}(z) < 0 \text{ for all } z \in \partial Z \}.$$  

Here by $n$ we denote the unit outward normal on $\partial Z$. By virtue of the nonlinear strong maximum principal of Vazquez [19], we have that $u_1 \in \text{int } C_+$. In addition to $\hat{\lambda}_1(m) > 0$, we obtain a whole strictly increasing sequence $\{\hat{\lambda}_k(m)\}_{k \geq 1} \subseteq \mathbb{R}_+$ of eigenvalues of $(-\Delta_p, W^{1,p}_0(Z), m)$, such that $\hat{\lambda}_k(m) \rightarrow +\infty$ as $k \rightarrow +\infty$. These are the so called Lusternik-Schnirelmann eigenvalues (LS-eigenvalues for short) of $(-\Delta_p, W^{1,p}_0(Z), m)$. If $p = 2$ (linear eigenvalue problem), then these are all the eigenvalues. If $p \neq 2$ (nonlinear eigenvalue problem), we do not know if this is true. Nevertheless, since $\hat{\lambda}_1(m) > 0$ is isolated and the set of eigenvalues of $(-\Delta_p, W^{1,p}_0(Z), m)$ is closed, we can define

$$\hat{\lambda}_2(m) = \inf \{ \hat{\lambda} : \hat{\lambda} \text{ is an eigenvalue of } (-\Delta_p, W^{1,p}_0(Z), m), \hat{\lambda} > \hat{\lambda}_1(m) \} > \hat{\lambda}_1(m),$$

which is an eigenvalue of $(-\Delta_p, W^{1,p}_0(Z), m)$. In fact we have $\hat{\lambda}_2(m) = \hat{\lambda}_2(m)$, i.e., the second eigenvalue and the second LS-eigenvalue coincide.

If $m = 1$, then we write $\hat{\lambda}_k(m) = \lambda_k$ for all $k \geq 1$ and $(-\Delta_p, W^{1,p}_0(Z), m) = (-\Delta_p, W^{1,p}_0(Z))$. Since $\lambda_2 > 0$ is also the second LS-eigenvalue, it admits a variational characterization provided by the Lusternik-Schnirelmann theory. However, for our purposes that characterization is not convenient. Instead, we will use the following characterization of $\lambda_2 > 0$, due to Cuesta-de
Figueiredo-Gossez [6]. Let
\[ \partial B_1^{L^p} = \{ v \in L^p(Z) : \|v\|_p = 1 \}, \]
\[ S = W_0^{1,p}(Z) \cap \partial B_1^{L^p}, \]
\[ \Gamma_0 = \{ \gamma_0 \in C([-1,1]; S) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1 \}. \]

Then
\[ \lambda_2 = \inf_{\gamma_0 \in \Gamma_0} \sup_{x \in \gamma_0([-1,1])} \|\nabla x\|_{L^p}. \]

Our approach also uses truncation techniques coupled with the method of upper-lower solutions. So let us recall the definition of upper and lower solutions for problem (1.1).

We say that \( \overline{x} \in W^{1,p}(Z) \) is an “upper solution” for problem (1.1), if
\[ \overline{x}|_{\partial Z} \geq 0 \quad \text{and} \quad \int_Z \|\nabla \overline{x}\|^{p-2}(\nabla \overline{x}, \nabla v)_{\mathbb{R}^N} \, dz \geq \int_Z \overline{u} v \, dz \]
for all \( v \in W_0^{1,p}(Z), v \geq 0 \) and some \( \overline{x} \in L^p(Z) \), with \( \overline{x}(z) \in \partial j(z, \overline{x}(z)) \) for a.a. \( z \in Z \). We say that \( \overline{x} \) is a “strict upper solution”, if in addition it is not a solution of (1.1). We say that \( \underline{x} \in W^{1,p}(Z) \) is a “lower solution” for problem (1.1), if
\[ \underline{x}|_{\partial Z} \leq 0 \quad \text{and} \quad \int_Z \|\nabla \underline{x}\|^{p-2}(\nabla \underline{x}, \nabla v)_{\mathbb{R}^N} \, dz \leq \int_Z \underline{u} v \, dz \]
for all \( v \in W_0^{1,p}(Z), v \geq 0 \) and some \( \underline{u} \in L^p(Z) \), with \( \underline{u}(z) \in \partial j(z, \underline{x}(z)) \) for a.a. \( z \in Z \). We say that \( \underline{x} \) is a “strict lower solution”, if in addition it is not a solution of (1.1).

If \( X \) is a reflexive Banach space and \( A : X \rightarrow X^* \) is a map, we say that the map \( A \) is of type \((S)_+\), if for every sequence \( \{x_n\}_{n \geq 1} \subseteq X \), such that \( x_n \rightharpoonup x \) weakly in \( X \) and
\[ \limsup_{n \to +\infty} \langle A(x_n), x_n - x \rangle \leq 0, \]
one has that \( x_n \rightharpoonup x \) in \( X \) as \( n \to +\infty \). If \( A : X \rightarrow X^* \) is of type \((S)_+\) and \( B : X \rightarrow X^* \) is compact, then \( A + B \) is of type \((S)_+\).

If \( X \) is an ordered Banach space with order cone \( K \), \( \text{int} \, K \neq \emptyset \) and \( e \in \text{int} \, K \), then for every \( x \in X \), we can find \( \vartheta(x) \geq 0 \), such that \( x \leq \vartheta(x)e. \)
In the sequel we use the notation $r^\pm = \max\{\pm r, 0\}$ and $\|x\| = \|\nabla x\|_p$ for all $x \in W_0^{1,p}(Z)$.

To formulate the hypotheses on the potential $j$, we need to introduce the following notion. Suppose $f: Z \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function, such that for every $r > 0$ there exists $a_r \in L^\infty(Z)_+$, such that

$$|f(z, \zeta)| \leq a_r(z) \text{ for a.a. } z \in Z \text{ and all } |\zeta| \leq r.$$ 

We permit $f(z, \cdot)$ to have jump discontinuities and in order to be able to guarantee a solution, we fill in the discontinuities gaps. For this purpose we define

$$(2.7) \quad f_1(z, \zeta) = \liminf_{\zeta' \to \zeta} f(z, \zeta') \quad \text{and} \quad f_2(z, \zeta) = \limsup_{\zeta' \to \zeta} f(z, \zeta').$$

Note that for almost all $z \in Z$, both limits are finite. We assume that $f_1$ and $f_2$ are superpositionally measurable, meaning that, if $u: Z \rightarrow \mathbb{R}$ is a measurable function, then so are the functions $z \mapsto f_1(z, u(z))$ and $z \mapsto f_2(z, u(z))$. We set

$$j(z, \zeta) = \int_0^\zeta f(z, r) \, dr.$$ 

Evidently for almost all $z \in Z$, the function $j(z, \cdot)$ is locally Lipschitz and

$$\partial j(z, \zeta) = [f_1(z, \zeta), f_2(z, \zeta)]$$

(see Clarke [5]). Clearly $j(z, 0) = 0$ for almost all $z \in Z$ and if $f(z, \cdot)$ is continuous at $\zeta = 0$, then $\partial j(z, 0) = \{0\}$. The set of hypotheses on the nonsmooth potential $j$ is the following.

$H(j): Z \times \mathbb{R} \rightarrow \mathbb{R}$ is a function, such that $j(z, \zeta) = \int_0^\zeta f(z, r) \, dr$ where $f: Z \times \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying

(i) the function $(z, \zeta) \mapsto f(z, \zeta)$ is measurable; the function $f_1, f_2$, defined by (2.7) are superpositionally measurable;

(ii) for almost all $z \in Z$, the function $\zeta \mapsto f(z, \zeta)$ is continuous at $\zeta = 0$;

(iii) for almost all $z \in Z$, all $\zeta \in \mathbb{R}$, we have $|f(z, \zeta)| \leq a(z) + c|\zeta|^{p-1}$ with $a \in L^\infty(Z)_+, c > 0$;
(iv) \( \lim_{|\zeta| \to +\infty} \frac{p j(z, \zeta)}{|\zeta|^p} = \lambda_1 \) uniformly for almost all \( z \in Z \) and
\[
\lim_{\zeta \to +\infty} \left( f_2(z, \zeta) \zeta - p j(z, \zeta) \right) = -\infty,
\]
uniformly for almost all \( z \in Z \);
(v) there exists \( \hat{\eta} \in L^\infty(Z)_+ \), such that
\[
\lambda_2 < \inf_{z \in Z} \liminf_{\zeta \to 0} \frac{f_1(z, \zeta)}{|\zeta|^{p-2} \zeta} \leq \limsup_{\zeta \to 0} \frac{f_2(z, \zeta)}{|\zeta|^{p-2} \zeta} \leq \hat{\eta}(z)
\]
uniformly for almost all \( z \in Z \);
(vi) there exist \( a_- < 0 < a_+ \), such that \( 0 \in \partial j(z, a_-) \) and \( 0 \in \partial j(z, a_+) \) for almost all \( z \in Z \) and there is \( k > 0 \), such that
\[
0 \leq u^* \leq k(a_+ - \zeta)^{p-1}
\]
for almost all \( z \in Z \), all \( \zeta \in [0, a_+] \) and all \( u^* \in \partial j(z, \zeta) \) and
\[
-k(\zeta - a_-)^{p-1} \leq u^* \leq 0
\]
for almost all \( z \in Z \), all \( \zeta \in (a_-, 0] \) and all \( u^* \in \partial j(z, \zeta) \).

**Remark 2.4.** The first limit in hypothesis \( H(j)(iv) \) implies that the problem is resonant at infinity with respect to \( \lambda_1 \). For this reason, we need to introduce an additional asymptotic condition at infinity.

**Example 2.5.** The following function satisfies hypotheses \( H(j) \) (for simplicity we drop the \( z \)-dependence):
\[
j(\zeta) = \int_0^\zeta f(r) \, dr \quad \forall \zeta \in \mathbb{R},
\]
with \( f : \mathbb{R} \to \mathbb{R} \), defined by
\[
f(\zeta) = \begin{cases} 
-c(1 + \zeta)^{p-1} \ln(|\zeta|^{p-1} + 1) & \text{if } \zeta \in [-1, 0], \\
c(1 - \zeta)^{p-1} \ln(|\zeta|^{p-1} + 1) & \text{if } \zeta \in [0, 1], \\
\lambda_1 |\zeta|^{p-2} \zeta - \ln |\zeta| & \text{if } |\zeta| > 1,
\end{cases}
\]
with $c > \lambda_2$. Note that at $\zeta = -1$, the function $f$ exhibits a downward jump discontinuity of $-\lambda_1$ and at $\zeta = 1$, an upward jump discontinuity of $\lambda_1$.

3. Constant sign solutions

In this section we produce four nontrivial smooth solutions of constant sign (two positive and two negative) for problem (1.1).

We consider the following truncations of the nonsmooth potential $j(z, \zeta)$:

$$
\begin{align*}
\hat{j}_+(z, \zeta) &= \begin{cases} 
0 & \text{if } \zeta \leq 0, \\
\{\tau \partial j(z, \zeta) : \tau \in [0, 1]\} = \{0\} & \text{if } \zeta = 0, \\
\partial j(z, \zeta) & \text{if } \zeta \in (0, a_+), \\
\{0\} & \text{if } a_+ < \zeta, \\
\partial j(z, a_+) & \text{if } \zeta = a_+, \\
\hat{j}_-(z, \zeta) &= \begin{cases} 
0 & \text{if } \zeta < a_-, \\
\{\tau \partial j(z, \zeta) : \tau \in [0, 1]\} = \{0\} & \text{if } \zeta = 0, \\
\partial j(z, \zeta) & \text{if } \zeta \in (a_-, 0), \\
\{0\} & \text{if } 0 < \zeta.
\end{cases}
\end{align*}
$$

Note that for all $\zeta \in \mathbb{R}$, the functions $z \mapsto \hat{j}_\pm(z, \zeta)$ are measurable and for almost all $z \in Z$, functions $\zeta \mapsto \hat{j}_\pm(z, \zeta)$ are locally Lipschitz.

From the nonsmooth chain rule (see Clarke [5, p. 42]), we know that

$$
\begin{align*}
\partial \hat{j}_+(z, \zeta) &= \begin{cases} 
0 & \text{if } \zeta < 0, \\
\{\tau \partial j(z, \zeta) : \tau \in [0, 1]\} = \{0\} & \text{if } \zeta = 0, \\
\partial j(z, \zeta) & \text{if } \zeta \in (0, a_+), \\
\{0\} & \text{if } a_+ < \zeta, \\
\partial j(z, a_+) & \text{if } \zeta = a_+, \\
\partial \hat{j}_-(z, \zeta) &= \begin{cases} 
0 & \text{if } \zeta < a_-, \\
\{\tau \partial j(z, \zeta) : \tau \in [0, 1]\} = \{0\} & \text{if } \zeta = 0, \\
\partial j(z, \zeta) & \text{if } \zeta \in (a_-, 0), \\
\{0\} & \text{if } 0 < \zeta.
\end{cases}
\end{align*}
$$

We consider the functionals $\varphi, \hat{\varphi}_\pm : W_0^{1, p}(Z) \rightarrow \mathbb{R}$, defined by

$$
\hat{\varphi}_\pm(x) = \frac{1}{p} \|\nabla x\|^p_p - \int_Z \hat{j}_\pm(z, x(z)) \, dz \quad \forall x \in W_0^{1, p}(Z),
$$
$$
\varphi(x) = \frac{1}{p} \|\nabla x\|^p_p - \int_Z j(z, x(z)) \, dz \quad \forall x \in W_0^{1, p}(Z).
$$
We know that $\tilde{\varphi}_\pm$ and $\varphi$ are Lipschitz continuous on bounded sets, hence locally Lipschitz (see Clarke [5, p. 83]).

In what follows by $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair of spaces $(W_0^{1,p}(Z), W^{-1,p'}(Z))$ (where $\frac{1}{p} + \frac{1}{p'} = 1$). Let $A: W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ be the nonlinear operator defined by

$$
\langle A(x), y \rangle = \int_Z \|\nabla x\|^{p-2}(\nabla x, \nabla y)_{\mathbb{R}^N} \, dz \quad \forall x, y \in W_0^{1,p}(Z).
$$

**Lemma 3.1.** If $A: W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ is the nonlinear operator defined by (3.5), then $A$ is of type $(S)_+$. 

We know that

$$
\begin{align*}
\partial \tilde{\varphi}_\pm(x) &\subseteq A(x) - \hat{N}_\pm(x) \quad \text{and} \\
\partial \varphi(x) &\subseteq A(x) - \hat{N}(x) \quad \forall x \in W_0^{1,p}(Z),
\end{align*}
$$

where

$$
\begin{align*}
\hat{N}_\pm(x) &= \{ u \in L^p(Z) : u(z) \in \partial j_\pm(z, x(z)) \text{ a.e. on } Z \}, \\
\hat{N}(x) &= \{ u \in L^p(Z) : u(z) \in \partial j(z, x(z)) \text{ a.e. on } Z \}
\end{align*}
$$

(see Clarke [5, p. 83]).

**Proposition 3.2.** If hypotheses $H(j)$ hold, then problem (1.1) has at least two solutions $x_0 \in \text{int} \, C_+$ and $v_0 \in -\text{int} \, C_+$, $x_0$ is a minimizer of $\tilde{\varphi}_+$, $v_0$ is a minimizer of $\tilde{\varphi}_-$ and both are local minimizers of $\varphi$.

**Proof.** From (3.3), we see that $\tilde{\varphi}_+$ is coercive. Exploiting the compactness of the embedding $W_0^{1,p}(Z) \subseteq L^p(Z)$, we can check that $\tilde{\varphi}_+$ is weakly lower semicontinuous. So by the Weierstrass theorem, we can find $x_0 \in W_0^{1,p}(Z)$, such that

$$
\tilde{\varphi}_+(x_0) = \hat{m}_+ = \inf \{ \tilde{\varphi}_+(x) : x \in W_0^{1,p}(Z) \}.
$$

We show that without any loss of generality, we can assume that $\hat{m}_+ < 0$. Let $u_1 \in \text{int} \, C_+$ be the $L^p$-normalized principal eigenfunction. Using hypothesis $H(j)(v)$ we can find $\delta > 0$ such that

$$
\lambda_1 \leq \frac{f(z, \zeta)}{|\zeta|^{p-2}\zeta} \quad \text{for all } |\zeta| \leq \delta \quad \text{and a.a. } z \in Z,
$$
so also

\[(3.8) \quad \frac{\lambda_1}{p}|\zeta|^p \leq j(z, \zeta) \quad \text{for all } |\zeta| \leq \delta \quad \text{and a.a. } z \in Z.\]

Next we can find \(\varepsilon_0 > 0\) small enough and such that

\[0 \leq \varepsilon_0 u_1(z) \leq \beta < \min\{\delta, a_+\} \quad \forall z \in \overline{Z},\]

where \(a_+\) is as in hypothesis \(H(j)(vi)\). Then from (3.8), for all \(\varepsilon \in (0, \varepsilon_0]\), we have

\[(3.9) \quad \frac{\lambda_1}{p}\varepsilon^pu_1(z)^p \leq j(z, \varepsilon u_1(z)) = \tilde{j}_+(z, \varepsilon u_1(z)) \quad \text{for a.a. } z \in Z.\]

Hence from (3.3), (2.2), (3.9) and recalling that \(\|u_1\| = 1\), we have

\[\varphi_+'(\varepsilon u_1) = \frac{\varepsilon^p}{p}\|\nabla u_1\|^p_p - \int_Z \tilde{j}_+(z, \varepsilon u_1) \, dz \leq \frac{\varepsilon^p}{p}\lambda_1 - \frac{\lambda_1}{p}\varepsilon^p = 0,\]

so \(\tilde{m}_+ \leq 0\).

If \(\tilde{m}_+ = 0\), then \(\varphi_+'(\varepsilon u_1) = 0\) for all \(\varepsilon \in (0, \varepsilon_0]\). So \(0 \in \partial \varphi_+'(\varepsilon u_1)\) and \(A(\varepsilon u_1) = u_\varepsilon^*,\) with \(u_\varepsilon^* \in \tilde{N}_+(\varepsilon u_1)\), so

\[(3.10) \quad -\Delta\varepsilon u_1(z) = u_\varepsilon^*(z) \quad \text{for a.a. } z \in Z.\]

Since \(0 < \varepsilon u_1(z) \leq \beta < a_+\) for all \(z \in Z\), using also (3.1), we have

\[\partial \tilde{j}_+(z, \varepsilon u_1(z)) = \partial j(z, \varepsilon u_1(z)) \quad \text{for a.a. } z \in Z\]

and so from (3.10) we deduce that \(\varepsilon u_1\) is a solution of (1.1) and a minimizer of \(\varphi_+\) for any \(\varepsilon \in (0, \varepsilon_0]\). Moreover, since \(\varepsilon u_1 \in \text{int } C_+\) and \(\varepsilon u_1(z) \leq \beta < a_+\) for all \(z \in \overline{Z}\), we can find \(r > 0\) small, such that

\[\varphi_+|_{\pi^0_{\varepsilon u_1}(\varepsilon u_1)} = \varphi_+|_{\pi^0_{\varepsilon u_1}(\varepsilon u_1)},\]

so \(\varepsilon u_1\) is a local \(C^1_0(\overline{Z})\)-minimizer of \(\varphi\), thus also a local \(W^{1,p}_0(Z)\)-minimizer of \(\varphi\) (see Gasinski-Papageorgiou [10, p. 655]). Thus, if \(\tilde{m}_+ = 0\), then we have a whole continuum of functions (the functions \(\{\varepsilon u_1\}_{\varepsilon \in (0, \varepsilon_0]}\)) satisfying the properties claimed by the proposition.
So without any loss of generality we may assume that \( \hat{m}_+ < 0 \), so from (3.7), we have \( \hat{\varphi}_+(x_0) < 0 = \hat{\varphi}_+(0) \), thus \( x_0 \neq 0 \). We have \( 0 \in \partial \hat{\varphi}_+(x_0) \), so

\[
A(x_0) = u_0^* ,
\]

with \( u_0^* \in \hat{N}_+(x_0) \). Acting on (3.11) with \( -x_0^- \in W_0^{1,p}(Z) \) and using (3.1), we obtain \( \| \nabla x_0 \|_p = 0 \), so \( x_0 = 0 \) and thus \( x_0 \geq 0 \), \( x_0 \neq 0 \). From (3.11), we have

\[
\begin{cases}
-\Delta_p x_0(z) = u_0^*(z) \in \partial j_+(z, x_0(z)) & \text{for a.a. } z \in Z \\
x_0 \mid_{\partial Z} = 0 .
\end{cases}
\]

(3.12)

From Theorem 7.1, p. 286 of Ladyzhenskaya-Uraltseva [13], we have that \( x_0 \in L^\infty(Z) \). Then invoking Theorem 1 of Lieberman [14], we have that \( x_0 \in C^1_0(Z) \).

\[
u_0^*(z) = 0 \text{ for a.a. } z \in \{ x > a_+ \}
\]

(3.13)

(see (3.1)). So if we act on (3.11) with \( (x_0 - a_+)^+ \in W_0^{1,p}(Z) \cap C(Z) \), we obtain

\[
\int_{\{x_0 > a_+\}} \| \nabla x_0 \|_p^p \, dz = 0 ,
\]

so \( \| \{x_0 > a_+\} \| = 0 \), and thus \( 0 \leq x_0(z) \leq a_+ \) for almost all \( z \in Z \). Note that (3.12) and hypothesis \( H(j)(vi) \) implies that \( \Delta_p x_0(z) \leq 0 \) for almost all \( z \in Z \). Invoking the strong maximum principle of Vazquez [19], we conclude that \( x_0 \in \text{int } C_+ \). Moreover, using hypothesis \( H(j)(vi) \), we have

\[
-\Delta_p (a_+ - x_0)(z) = \Delta_p x_0(z) = -u_0^*(z) \geq -k(a_+ - x_0)(z)^{p-1} .
\]

Thus

\[
\Delta_p (a_+ - x_0)(z) \leq k(a_+ - x_0)(z)^{p-1} \text{ for a.a. } z \in Z .
\]

(3.14)

Invoking once more the nonlinear strong maximum principle of Vazquez [19], we obtain from (3.14), \( x_0(z) < a_+ \) for all \( z \in Z \). Recalling the definition of \( \hat{j}_+(z, \zeta) \), we see that, if we choose \( r > 0 \) small, we get

\[
\frac{\hat{\varphi}_+}{p^{c_1}(x_0)} \mid_{\pi_{r_0}^c(x_0)} = \varphi \mid_{\pi_{r_0}^c(x_0)} .
\]

Therefore, \( x_0 \) is a local \( C^1_0(Z) \)-minimizer of \( \varphi \) thus also a local \( W_0^{1,p}(Z) \)-minimizer of \( \varphi \) (see Gasiński-Papageorgiou [10, p. 655]).
Similarly, working with $\widehat{\varphi}_-$ and using this time (3.2), we obtain $v_0 \in -\text{int } C_+$, a minimizer of $\widehat{\varphi}_-$, which is also a local minimizer of $\varphi$ and $v_0$ solves problem (1.1).

Next we can produce two more nontrivial smooth solutions of constant sign.

**Theorem 3.3.** If hypotheses $H(j)$ hold, then problem (1.1) has at least four nontrivial solution $x_0, \widehat{x} \in \text{int } C_+, \widehat{v}, v_0 \in -\text{int } C_+$, such that

$$x_0 \leq \widehat{x}, \quad x_0 \neq \widehat{x} \quad \text{and} \quad \widehat{v} \leq v_0, \quad \widehat{v} \neq v_0.$$ 

**Proof.** From Proposition 3.2, we already have two solutions $x_0 \in \text{int } C_+$ and $v_0 \in -\text{int } C_+$. First we will show that we can find $\widehat{x} \in \text{int } C_+$, the solution of (1.1), such that $x_0 \leq \widehat{x}, x_0 \neq \widehat{x}$. For this purpose we introduce the functional $\psi_+: W_{0}^{1,p}(Z) \longrightarrow \mathbb{R}$ defined by

$$(3.15) \quad \psi_+(y) = \frac{1}{p} \|\nabla(x_0 + y)\|_p^p - \int_Z j(z, x_0 + y^+) \, dz + \int_Z u_0^* y^- \, dz + \xi,$$

where $u_0^* \in L^p(Z)$, $u_0^*(z) \in \partial j(z, x_0(z))$ for almost all $z \in Z$ and satisfies

$$-\Delta_p x_0(z) = u_0^*(z) \quad \text{for a.a. } z \in Z$$

(see Proposition 3.2) and $\xi = \int_Z j(z, x_0(z)) \, dz$. Evidently, $\psi_+$ is locally Lipschitz on $W_{0}^{1,p}(Z)$. Note that

$$(3.16) \quad \|\nabla(x_0 + y)\|_p^p = \|\nabla(x_0 + y^+)\|_p^p - \|\nabla x_0\|_p^p + \|\nabla(x_0 - y^-)\|_p^p.$$ 

Using (3.16) in (3.15), we obtain

$$\psi_+(y) = \varphi(x_0 + y^+) - \varphi(x_0) + \frac{1}{p} \|\nabla(x_0 - y^-)\|_p^p + \int_Z u_0^* y^- \, dz.$$

Since $u_0^*(z) \in \partial j(z, x_0(z))$ for almost all $z \in Z$ and $x_0(z) \in [0,a_+]$ for all $z \in Z$, from hypothesis $H(j)(vi)$, we have $u_0^*(z) > 0$ for almost all $z \in Z$. Also recall that $x_0 \in \text{int } C_+$ is a local minimizer of $\varphi$. So from (3.17), it can be shown that the origin is a local minimizer of $\psi_+$ (cf. Gasiński-Papageorgiou [10, p. 675]). Next following the ideas of Gasiński-Papageorgiou [10, Chapter 4], we can show the following:
(1) there exists $\rho > 0$ small, such that
\[ \inf \{ \psi_+(y) : \|y\| = \rho \} \geq \frac{1}{p} \| \nabla x_0 \|_p = \psi_+(0). \]

(2) $\psi_+$ satisfies the C-condition.

(3) there exists $y \in W^{1,p}_0(Z)$ with $\|y\| > \rho$, such that $\psi_+(y) < \psi_+(0) \leq \beta_+$. Then the three above facts permit the use of the nonsmooth mountain pass theorem (with relaxed boundary condition; see Gasiński-Papageorgiou [10, p. 140] and Theorem 2.1). Then, there exists $y_0 \in W^{1,p}_0(Z)$, $y_0 \neq 0$, such that
\[ \beta_+ \leq \psi_+(y_0) \quad \text{and} \quad 0 \in \partial \psi_+(y_0). \]

From the inclusion in (3.18), we obtain
\[ A(x_0 + y_0) = u_0^* - \tilde{u}_0^*, \]
with $u_0^* \in L^p(Z)$, $u_0^*(z) \in \partial \tilde{j}_+(z, y_0(z))$ for almost all $z \in Z$ and $\tilde{u}_0^* \in L^p(Z)$, $\tilde{u}_0^*(z) \in \partial \tilde{i}-(y_0(z))$ for almost all $z \in Z$, where
\[ \tilde{j}_+(z, \zeta) = j(z, x_0(z) + \zeta^+) \quad \text{and} \quad \tilde{i}-(\zeta) = \zeta^-.
\]

Acting with $-y_0^- \in W^{1,p}_0(Z)$ we obtain
\[ \langle A(x_0 + y_0), -y_0^- \rangle = \int_Z u_0^*(-y_0^-) \, dz - \int_Z \tilde{u}_0^*(-y_0^-) \, dz \]
so $\langle A(x_0 - y_0^-), -y_0^- \rangle = \langle A(x_0), -y_0^- \rangle$ and thus
\[ \langle A(x_0 - y_0^-) - A(x_0), -y_0^- \rangle = 0. \]

But clearly $A$ is strictly monotone (strongly monotone if $p \geq 2$). So from (3.18) we infer that $y_0^- = 0$, i.e.: $y_0 \geq 0$, $y_0 \neq 0$. Moreover, from (3.19), we have
\[ \begin{cases} 
-\Delta_p(x_0 + y_0)(z) = u_0^*(z) - \tilde{u}_0^*(z) & \text{for a.a. } z \in Z, \\
(x_0 + y_0)|_{\partial Z} = 0. 
\end{cases} \]

So from nonlinear regularity theorem, we obtain $x_0 + y_0 \in \text{int } C_+$. Let us set $\hat{x} = x_0 + y_0$. Then $\hat{x} \in \text{int } C_+$, $x_0 \leq \hat{x}$ and $x_0 \neq \hat{x}$. Also from Stampacchia
theorem, for almost all $z \in \{y_0 = 0\}$, we have

$$-\Delta_p \hat{x}(z) = -\Delta_p x_0(z) = u_0(z) \in \partial j(z, x_0(z)) = \partial j(z, \hat{x}(z)).$$

On the other hand, from (3.21), we have

$$-\Delta_p \hat{x}(z) = u_0^*(z) \in \partial j(z, \hat{x}(z)) \quad \text{for a.a.} \ z \in \{y_0 > 0\}.$$

Therefore $\hat{x} \in \text{int } C_+$ is a solution of problem (1.1).

In a similar way, we obtain $\hat{v} \in W_0^{1,p}(Z)$ with $\hat{v} \in -\text{int } C_+$, $\hat{v} \leq v_0$, $\hat{v} \neq v_0$ and $\hat{v}$ is a solution of problem (1.1).

4. Nodal solution

The strategy to produce a fifth nontrivial smooth nodal solution, was inspired by work of Dancer-Du [7] the semilinear case ($p = 2$) and smooth $j(z, \cdot) \in C^1(\mathbb{R})$. The first step in the execution of our solution plan, is to establish certain lattice-type properties of the sets of upper-lower solutions for problem (1.1). Let $S \subseteq W^{1,p}(Z)$ be a nonempty set. We say that $S$ is downward (respectively upward) directed, if for any $u_1, u_2 \in S$, we can find $u_3 \in S$, such that $u_3 \leq \min\{u_1, u_2\}$ (respectively $u_3 \geq \max\{u_1, u_2\}$). We can prove the following lemma.

**Lemma 4.1.** If hypotheses $H(j)$ hold, then the set of upper solutions for problem (1.1) is downward directed. In fact, if $y_1, y_2$ are two upper solutions for problem (1.1), then $\min\{y_1, y_2\}$ is an upper solution too. Similarly the set of lower solutions for problem (1.1) is upward directed. In fact, if $v_1, v_2$ are two lower solutions for problem (1.1), then $\max\{v_1, v_2\}$ is a lower solution too.

Next we can show that problem (1.1) admits positive lower solutions and negative upper solutions.

**Proposition 4.2.** If hypotheses $H(j)$ hold, then we can find $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0]$, the function $\underline{x}_\varepsilon = \varepsilon u_1 \in \text{int } C_+$ is a strict lower solution for problem (1.1) and $\overline{x}_\varepsilon = -\varepsilon u_1 \in \text{int } C_+$ is a strict upper solution for problem (1.1).
Proof. By virtue of hypothesis $H(j)(v)$, we can find $\beta > \lambda_2$ and $\delta \in (0, a_+)$, such that $\beta \zeta^{p-1} \leq f_1(z, \zeta)$ for almost all $z \in \mathbb{Z}$ and all $\zeta \in [0, \delta]$. Since $u_1 \in \text{int} C_+$, we can find $\varepsilon_0 > 0$ small, such that $\varepsilon u_1(z) \in [0, \delta]$ for all $z \in \mathbb{Z}$ and all $\varepsilon \in (0, \varepsilon_0)$. Thus, using the fact that $\beta > \lambda_2$, we have

$$-\Delta_p(\varepsilon u_1)(z) = \lambda_1 \varepsilon^{p-1} u_1(z)^{p-1} < \beta \varepsilon^{p-1} u_1(z)^{p-1}$$

(4.1)

for almost all $z \in \mathbb{Z}$ and all $u^* \in L^p(\mathbb{Z})$, $u^*(z) \in \partial j(z, \varepsilon u_1(z))$ for almost all $z \in \mathbb{Z}$. So $\varepsilon u_1 \in \text{int} C_+$ is a strict lower solution for problem (1.1). Similarly we show that $-\varepsilon u_1 \in -\text{int} C_+$ is a strict upper solution for problem (1.1).

Observe that $\overline{x} \equiv a_+$ is an upper solution for problem (1.1) and $\overline{v} = a_-$ is a lower solution for problem (1.1). Then, using the lower-upper solution pairs $\{x = \varepsilon u_1, \overline{x} = a_+\}$ and $\{v = a_-, \overline{v} = -\varepsilon u_1\}, 0 < \varepsilon \leq \varepsilon_0$, we define the following order intervals:

$$[x, \overline{x}] = \{x \in W^{1,p}_0(\mathbb{Z}) : \underline{x}(z) \leq x(z) \leq \overline{x}(z) \text{ for a.a. } z \in \mathbb{Z}\},$$

$$[v, \overline{v}] = \{v \in W^{1,p}_0(\mathbb{Z}) : \underline{v}(z) \leq v(z) \leq \overline{v}(z) \text{ for a.a. } z \in \mathbb{Z}\}.$$
Proposition 4.4. If hypotheses $H(j)$ hold, then problem (1.1) has the smallest positive solution $x_+ \in \text{int } C_+$ and the greatest negative solution $v_- \in -\text{int } C_+$.

Proof. Let $\varepsilon_n \searrow 0$ and set $\tilde{x}_n = \varepsilon_n u_1$. Then by Proposition 4.3, we can find $\tilde{x}_n \in \text{int } C_+$, the smallest solution of (1.1) in the ordered interval $[\tilde{x}_n, a_+]$. The sequence $\{\tilde{x}_n\}_{n \geq 1} \subseteq W^{1,p}_0(Z)$ is bounded and so we may assume that $\tilde{x}_n \rightharpoonup x_+$, weakly in $W^{1,p}_0(Z)$. Next following some ideas of Gasiński-Papageorgiou [10, Chapter 4], we can show that $x_+$ is the smallest positive solution of (1.1).

To produce $v_- \in -\text{int } C_+$, we work with the pair $[v = a_-, \tilde{x}_n = \varepsilon_n(-u_1)]$ in a similar way.

Now we are ready to produce the nodal solution.

Theorem 4.5. If hypotheses $H(j)$ hold, then problem (1.1) has at least five nontrivial solutions $x_0, \tilde{x}, v_0, \hat{v}, y_0 \in C^1_0(Z)$, such that

$$x_0 \in \text{int } C_+, \quad x_0 \leq \tilde{x}, \quad x_0 \neq \tilde{x}, \quad v_0 \in -\text{int } C_+, \quad \tilde{v} \leq v_0, \quad v_0 \neq \hat{v}$$

and $y_0$ is nodal.

Proof. From Theorem 3.3, we already have the four solutions of constant sign $x_0, \tilde{x}, v_0, \hat{v} \in C^1_0(Z)$. It remains to produce the nodal solution $y_0 \in C^1_0(Z)$. Let $x_+ \in \text{int } C_+$ be the smallest positive solution and $v_- \in -\text{int } C_+$ the greatest negative solution obtained in Proposition 4.4. We have $A(x_+) = u_+^*$ and $A(v_-) = u_-^*$, with $u_+^* \in L^p(Z)$, $u_-^*(z) \in \partial j(z, x_+(z))$ for almost all $z \in Z$ and $u_-^* \in L^p(Z)$, $u_-^*(z) \in \partial j(z, v_-(z))$ for almost all $z \in Z$.

We introduce the following truncations of the nonlinearity $f(z, \zeta)$:

$$f_+(z, \zeta) = \begin{cases} 0 & \text{if } \zeta < 0, \\ f(z, \zeta) & \text{if } 0 \leq \zeta \leq x_+(z), \\ u_+^*(z) & \text{if } x_+(z) < \zeta, \end{cases}$$

$$f_-(z, \zeta) = \begin{cases} u_-^*(z) & \text{if } \zeta < v_-(z), \\ f(z, \zeta) & \text{if } v_-(z) \leq \zeta \leq 0, \\ 0 & \text{if } 0 < \zeta, \end{cases}$$

$$\hat{f}(z, \zeta) = \begin{cases} u_+^*(z) & \text{if } \zeta < v_-(z), \\ f(z, \zeta) & \text{if } v_-(z) \leq \zeta \leq x_+(z), \\ u_-^*(z) & \text{if } x_+(z) < \zeta. \end{cases}$$
Then we define the corresponding potential functions by
\[
\begin{align*}
    j_{\pm}(z, \zeta) &= \int_0^\zeta f_{\pm}(z, r) \, dr \quad \text{and} \quad \widehat{j}(z, \zeta) = \int_0^\zeta \widehat{f}(z, r) \, dr.
\end{align*}
\]
Finally, we introduce the corresponding functionals \( \varphi_{\pm}, \widehat{\varphi} : W_0^{1,p}(Z) \to \mathbb{R} \),
\[
\begin{align*}
    \varphi_{\pm}(x) &= \frac{1}{p} \|\nabla x\|^p_p - \int_Z j_{\pm}(z, x(z)) \, dz \quad \forall x \in W_0^{1,p}(Z),
    \\
    \widehat{\varphi}(x) &= \frac{1}{p} \|\nabla x\|^p_p - \int_Z \widehat{j}(z, x(z)) \, dz \quad \forall x \in W_0^{1,p}(Z).
\end{align*}
\]
We use the following order intervals in \( W_0^{1,p}(Z) \):
\[
I_+ = [0, x_+], \quad I_- = [v_-, 0] \quad \text{and} \quad \widehat{I} = [v_-, x_+].
\]
We can show the following:
\begin{enumerate}[leftmargin=*,noitemsep]
    \item The critical points of \( \varphi_+ \) are in \( I_+ \), of \( \varphi_- \) are in \( I_- \) and of \( \widehat{\varphi} \) are in \( \widehat{I} \).
    \item The set of critical points of \( \varphi_+ \) is \( \{0, x_+\} \) and the set of the critical points of \( \varphi_- \) is \( \{0, v_-\} \).
    \item Both \( x_+ \in \text{int} \, C_+ \) and \( v_- \in -\text{int} \, C_+ \) are local minimizers of \( \widehat{\varphi} \).
\end{enumerate}
Based on the last fact, we may assume that \( x_+ \) and \( v_- \) are isolated critical points (in fact isolated local minimizers) of \( \widehat{\varphi} \), because, if this is not the case, then we can find a sequence \( \{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z) \) of critical points of \( \widehat{\varphi} \), \( x_n \not\in \{0, v_-, v_+\} \), such that, for example \( x_n \to x_+ \) in \( W_0^{1,p}(Z) \). Then \( x_n \in \widehat{I} \) for all \( n \geq 1 \) and so \( x_n \) must be nodal (due to the extremality of \( v_-, x_+ \)). We have produced a whole sequence of distinct nodal solutions for problem (1.1) and so we are done. Assume without any loss of generality that \( \widehat{\varphi}(v_-) \leq \widehat{\varphi}(x_+) \). Similarly as in the proof of Theorem 3.3, we can find \( g > 0 \), such that \( \widehat{\varphi}_+(x_+) < \inf \{\widehat{\varphi}(x) : \|x - x_+\| = g\} \leq 0 \). Let \( D = \partial B_g(x_+) \) and \( E_0 = \{v_-, x_+\} \), \( E = \widehat{I} \). It is easily seen that the pair \( \{E_0, E\} \) links with \( D \) in \( W_0^{1,p}(Z) \). Since \( \widehat{\varphi} \) is coercive, we can easily check that it satisfies the C-condition. Therefore, we can apply Theorem 2.1 and obtain \( y_0 \in W_0^{1,p}(Z) \), such that
\[
\widehat{\varphi}(v_-) \leq \widehat{\varphi}(x_+) < \widehat{\varphi}(y_0) = \inf_{(\tau(t) \in \widehat{\Gamma} \cap [0,1]) \tau \in \widehat{\Gamma}} \max_{t \in [0,1]} \widehat{\varphi}((\tau(t))),
\]
where \( \widehat{\Gamma} = \{\gamma \in C([-1,1]; W_0^{1,p}(Z)) : \gamma(-1) = v_-, \gamma(1) = x_+\} \).
Note that (4.2) implies that \( y_0 \neq v_-, y_0 \neq x_+ \). So, if we show that \( y_0 \neq 0 \), then \( y_0 \) is a nodal solution. According to (4.2), we can show the nontriviality of \( y_0 \) by producing a path \( \varphi_{\ast} \) in \( \Gamma \), such that \( \varphi_{\ast} \neq 0 \). To do this we proceed as follows.

By hypothesis \( H(j)(v) \), we can find \( \delta, \delta_0 > 0 \) small enough, such that \( \lambda_2 + \delta < \frac{u^2}{|\zeta|^p} \) for almost all \( z \in Z \), all \( |\zeta| \leq \delta_0 \) and all \( u^\ast \in \partial j(z, \zeta) \). Recall that for almost all \( z \in Z \), the function \( j(z, \cdot) \) is differentiable almost everywhere on \( \mathbb{R} \) (Rademacher theorem) and at a point of differentiability we have \( \frac{d}{dz} j(z, \zeta) \in \partial j(z, \zeta) \), so

\[
\frac{1}{p}(\lambda_2 + \delta)|\zeta|^p < j(z, \zeta) \quad \text{for a.a. } z \in Z, \text{ all } 0 \leq |\zeta| \leq \delta_0.
\] (4.3)

Let \( \partial B_{1}^{j} = \{x \in L^p(Z) : \|x\|_p = 1 \} \) and \( S = W_0^{1,p}(Z) \cap \partial B_{1}^{j} \) furnished with the relative \( W_0^{1,p}(Z) \)-topology. Similarly, let \( S_c = W_0^{1,p}(Z) \cap C_0^1(\mathbb{Z}) \cap \partial B_{1}^{j} \) furnished with the relative \( C_0^1(\mathbb{Z}) \)-topology. Evidently \( S_c \) is dense in \( S \) and this implies that \( C([-1, 1]; S_c) \) is dense in \( C([-1, 1]; S) \). Let

\[
\Gamma_0^c = \{ \gamma_0 \in C([-1, 1]; S_c) : \gamma_0(-1) = -u_1, \gamma_0(1) = u_1 \}.
\]

It follows that \( \Gamma_0^c \) is dense in \( \Gamma_0 \) (see (2.5)) and because of (2.6), we can find \( \tilde{\gamma}_0 \in \Gamma_0^c \), such that

\[
\frac{1}{p}\|\nabla x\|_p^p \leq \lambda_2 + \delta.
\] (4.4)

Since \( \tilde{\gamma}_0 \in \Gamma_0^c \) and \( -v_+, x_+ \in \text{int } C_+ \), we can find \( \varepsilon > 0 \) small enough, and such that \( \varepsilon|x(z)| \leq \delta \) for all \( z \in \mathbb{Z} \), all \( x \in \tilde{\gamma}_0([-1, 1]) \) and \( \varepsilon x \in \hat{I} \) for all \( x \in \tilde{\gamma}_0([-1, 1]) \). Therefore, from (4.3) and (4.4), if \( x \in \tilde{\gamma}_0([-1, 1]) \), then

\[
\hat{\varphi}(\varepsilon x) = \frac{\varepsilon^p}{p}\|\nabla x\|_p^p - \int_Z j(z, \varepsilon x) \text{d}z < \frac{\varepsilon^p}{p}\left[(\lambda_2 + \delta) - (\lambda_2 + \delta)\right] = 0.
\]

So, if \( \gamma_0 = \varepsilon x \), then \( \hat{\varphi}|_{\gamma_0} < 0 \). Next, we produce a continuous path joining \( \varepsilon u_1 \) and \( x_+ \), along which \( \hat{\varphi} \) is strictly negative. For this purpose, let

\[
a = m_+ = \inf \varphi_+ = \varphi_+(x_+) < 0 = \varphi_+(0) = b.
\]

Since \( \varphi_+ \) is coercive it satisfies the C-condition. Also \( K^{p+}_2 = \{x_+\} \). So we can apply Theorem 2.3 and obtain a continuous deformation \( h: [0, 1] \times \varphi^b_+ \rightarrow \varphi^b_+ \), such that:
Then we set \( \gamma(t) = h(t, \varepsilon u_1) \) for all \( t \in [0, 1] \). Since \( \varphi_+(\varepsilon u_1) < 0 \) then \( \gamma_+ \) is well defined and of course continuous. Moreover, since \( h \) is a deformation, we have \( \gamma_+(0) = \varepsilon u_1 \) and since \( \varphi^a \cup K_\delta^{\pm} = K_\delta^{\pm} = \{x_+\} \), we have that \( \gamma_+(1) = x_+ \). Therefore, \( \gamma_+ \) is a continuous path which joins \( \varepsilon u_1 \) with \( x_+ \).

Note that \( \varphi_+(\gamma_+(t)) = \varphi_+(h(t, \varepsilon u_1)) \leq \varphi_+(\varepsilon u_1) < 0 \), so \( \varphi_+|_{\gamma_+} < 0 \). But due to the sign condition (see hypothesis \( H(j)(v_i) \)), we have \( \hat{\varphi}|_{\gamma_+} \leq \varphi_+|_{\gamma_+} \), so \( \hat{\varphi}|_{\gamma_+} < 0 \). In a similar fashion, we produce a continuous path \( \gamma_- \) which joins \(-\varepsilon u_1 \) with \( v_- \) and such that \( \hat{\varphi}|_{\gamma_-} < 0 \). Concatenating \( \gamma_- \), \( \gamma_0 \) and \( \gamma_+ \), we have a continuous path \( \gamma^* \) in \( \Gamma \), such that \( \hat{\varphi}|_{\gamma^*} < 0 \). From (4.2), we conclude that \( \hat{\varphi}(y_0) < 0 = \hat{\varphi}(0) \), so \( y_0 \neq 0 \) and so \( y_0 \) is nodal. Nonlinear regularity theory implies that \( y_0 \in \mathcal{C}_1^0(Z) \).

References

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