ENUMERATION OF Γ-GROUPS OF FINITE ORDER

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Abstract

The concept of Γ-semigroups is a generalization of semigroups. In this paper, we consider Γ-groups and prove that every Γ-group is derived from a group then, we give the number of Γ-groups of small order.

Keywords: Γ-semigroup, Γ-group.

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1. Introduction

The concept of Γ-semigroups was introduced by Sen in [14] and [15] that is a generalization of a semigroups. Many classical notions of semigroups have been extended to Γ-semigroups (see, for example, [6, 10, 13, 16] and [17]). Dutta and Adhikari have found operator semigroups of a Γ-semigroup to be a very effective tool in studying Γ-semigroups [5]. Recently, Davvaz et al. introduced the notion of Γ-semihypergroups as a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups [2, 8, 9].
The determination of all groups of a given order up to isomorphism is a very old question in group theory. It was introduced by Cayley who constructed the groups of order 4 and 6 in 1854, see [4]. In this paper, we prove that a Γ-group is derived from a group. Also, we give the number of Γ-groups of small order.

2. Preliminaries

We begin this section by the definition of a Γ-semigroup.

**Definition** [14]. Let $S$ and $\Gamma$ be nonempty sets. Then $S$ is called a Γ-semigroup if there exists a mapping $S \times \Gamma \times S \to S$, written $(a, \gamma, b)$ by $a\gamma b$, such that satisfies the identities $(a\alpha b)\beta c = a\alpha(b\beta c)$, for all $a, b, c \in S$ and $\alpha, \beta \in \Gamma$.

Let $S$ be a Γ-semigroup and $\alpha$ be a fixed element in $\Gamma$. We define $a.b = a\alpha b$, for all $a, b \in S$. It is easy to check that $(S, \cdot)$ is a semigroup and we denote this semigroup by $S_\alpha$.

Let $A$ and $B$ be subsets of a Γ-semigroup $S$ and $\Delta \subseteq \Gamma$. Then $A\Delta B$ is defined as follows

$$A\Delta B = \{a\delta b \mid a \in A, b \in B, \delta \in \Delta\}.$$  

For simplicity we write $a\Delta B$ and $A\Delta b$ instead of $\{a\}\Delta B$ and $A\Delta \{b\}$, respectively. Also, we write $A\delta B$ in place of $A\{\delta\}B$.

Let $S$ be an arbitrary semigroup and $\Gamma$ any nonempty set. Define a mapping $S \times \Gamma \times S \to S$ by $a\alpha b = ab$, for all $a, b \in S$ and $\alpha \in \Gamma$. It is easy to see that $S$ is a Γ-semigroup. Thus a semigroup can be considered to be a Γ-semigroup.

In the following some examples of Γ-semigroups are presented.

**Example 1.** Let $S = \{i, 0, -i\}$ and $\Gamma = S$. Then $S$ is a Γ-semigroup under the multiplication over complex number while $S$ is not a semigroup under complex number multiplication.

**Example 2.** Let $S$ be the set of all $m \times n$ matrices with entries from a field $F$ and $\Gamma$ be a set of $n \times m$ matrices with entries from $F$. Then $S$ is a Γ-semigroup with the usual product of matrices.

**Example 3.** Let $(S, \leq)$ be a totally ordered set and $\Gamma$ be a nonempty subset of $S$. We define

$$x\gamma y = \max\{x, \gamma, y\},$$

for every $x, y \in S$ and $\gamma \in \Gamma$. Then $S$ is a Γ-semigroup.
Example 4. Let $S = [0, 1]$ and $\Gamma = \mathbb{N}$. For every $x, y \in S$ and $\gamma \in \Gamma$ we define $x\gamma y = \frac{xy}{\gamma}$. Then, for every $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$(x\alpha y)\beta z = \frac{xyz}{\alpha\beta} = x\alpha(y\beta z).$$

This means that $S$ is a $\Gamma$-semigroup.

A nonempty subset $T$ of a $\Gamma$-semigroup $S$ is said to be a $\Gamma$-subsemigroup of $S$ if $TT \subseteq T$.

Definition. A nonempty subset $I$ of $\Gamma$-semigroup $S$ is called a left (right) $\Gamma$-closed subset if $SI \subseteq I$ ($IS \subseteq I$). A $\Gamma$-semigroup $S$ is called a left (right) simple $\Gamma$-semigroup if it has no proper left (right) $\Gamma$-closed subset. Also, $S$ is called a simple $\Gamma$-semigroup if it has no proper $\Gamma$-closed subset both left and right.

3. Enumeration of $\Gamma$-groups of finite order

Definition. A $\Gamma$-semigroup $S$ is called a $\Gamma$-group if $S_\alpha$ is a group, for every $\alpha \in \Gamma$.

Example 5. Let $S = \{a, b, c, d, e, f\}$ and $\Gamma = \{\alpha, \beta\}$. Define the operations $\alpha$ and $\beta$ as the following tables

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Then $S$ is a $\Gamma$-group. One can see that $f$ and $e$ are the neutral elements of $S_\alpha$ and $S_\beta$, respectively.

Theorem 6. Let $S$ be a $\Gamma$-semigroup. Then $S$ is a simple $\Gamma$-semigroup if and only if $S_\alpha$ is a group, for every $\alpha \in \Gamma$.

Proof. Let $S$ be a simple $\Gamma$-semigroup and $\alpha \in \Gamma$, we show that $S_\alpha$ is a group. Let $I = aaS$, where $a \in S$. Then, $I$ is a right $\Gamma$-closed subset of $S$, indeed

$$I\Gamma S = (aaS)\Gamma S \subseteq aaS = I.$$
Since \( S \) has no proper right \( \Gamma \)-closed subset, we have \( I = a\alpha S = S \). Similarly, we can prove that \( S\alpha a = S \). Therefore, \( S_\alpha \) is a group.

Conversely, let \( I \neq \phi \) be a left \( \Gamma \)-closed subset of \( S \), \( s \in S \) and \( a \in I \). Since \( S_\alpha \) is a group, there exists \( t \in S \) such that \( s = t\alpha a \subseteq S\alpha I \subseteq I \). So \( S = I \). Similarly, we can prove that \( S \) has no proper right \( \Gamma \)-closed subset. Therefore, \( S \) is simple.

**Corollary 7.** Let \( S \) be a \( \Gamma \)-semigroup. If \( S_\alpha \) is a group, for some \( \alpha \in \Gamma \), then \( S_\beta \) is a group, for every \( \beta \in \Gamma \).

**Proof.** Since \( S_\alpha \) is a group, previous theorem implies that \( S \) is a simple \( \Gamma \)-group. Thus, for every \( \beta \in \Gamma \), \( S_\beta \) is a group. □

**Corollary 8.** Let \( S \) be a \( \Gamma \)-semigroup. If \( S_\alpha \) is a group, for some \( \alpha \in \Gamma \), then \( S \) is a \( \Gamma \)-group.

**Proof.** By Corollary 7, it is trivial. □

**Theorem 9.** Let \( S \) be a \( \Gamma \)-group and \( \alpha, \beta \in \Gamma \). Then there exists \( b \in S \) such that \( x\beta y = x\alpha b_\alpha y \), for every \( x, y \in S \).

**Proof.** It is sufficient to put \( b = e_\alpha \beta e_\alpha \), where \( e_\alpha \) is the neutral element of \( S_\alpha \). Then, for every \( x, y \in S \), we have

\[
x\beta y = (x\alpha e_\alpha)\beta(e_\alpha \alpha y) \\
= x\alpha (e_\alpha \beta e_\alpha) \alpha y \\
= x\alpha b_\alpha y.
\]

By the previous theorem, we conclude that every \( \Gamma \)-group is derived from a group. Therefore, if \( S \) is a \( \Gamma \)-group, then we can consider \( (S, \cdot) \) as a group and \( \Gamma \subseteq S \), so \( x\alpha y \) is a product in \( (S, \cdot) \), for every \( x, y \in S \) and \( \alpha \in \Gamma \). Also, Theorem 9 implies that the groups \( S_\alpha \) and \( S_\beta \) are isomorphic, for every \( \alpha, \beta \in \Gamma \).

**Definition.** Let \( S \) be a \( \Gamma \)-group and \( S' \) be a \( \Gamma' \)-group. If there exist mappings \( \varphi_\gamma : S \longrightarrow S' \), for every \( \gamma \in \Gamma \), and \( f : \Gamma \longrightarrow \Gamma' \) such that

\[\varphi_\gamma(x\gamma y) = \varphi_\gamma(x)f(\gamma)\varphi_\gamma(y),\]

for all \( x, y \in S \), then we say \((\{\varphi_\gamma\}_{\gamma \in \Gamma}, f)\) is a homomorphism between \( S \) and \( S' \). Also, if \( f \) and \( \varphi_\gamma \), for every \( \gamma \in \Gamma \), are bijections, then \((\{\varphi_\gamma\}_{\gamma \in \Gamma}, f)\) is called an isomorphism, and \( S \) and \( S' \) are called isomorphic.

**Lemma 10.** Let \( S \) be a \( \Gamma \)-group and \( S' \) be a \( \Gamma' \)-group. Then \( S \) and \( S' \) are isomorphic if and only if \( S \) and \( S' \) are isomorphic group and \(|\Gamma| = |\Gamma'|\).
Proof. If $S$ and $S'$ are isomorphic, then by the previous definition, for every $\alpha \in \Gamma$, the groups $S_\alpha$ and $S'_\alpha$ are isomorphic where $f : S \to S'$ is a bijection and $f(\alpha) = \alpha'$.

Theorem 11. The number of $\Gamma$-groups of order $n$ is $nk$, up to isomorphism, where $k$ is the number of isomorphism classes of groups of order $n$.

Proof. Suppose that $(S, \cdot)$ is a group and $\Gamma$ and $\Gamma'$ be two subsets of $S$ such that $|\Gamma| = |\Gamma'|$. Then by previous lemma, there exists only one $\Gamma$-group derived from $(S, \cdot)$, up to isomorphism. So, for every $m \leq n$ there exists only one $\Gamma$-group, where $\Gamma$ is a subset of $S$ such that $|\Gamma| = m$. Thus, the number of $\Gamma$-groups derived from $(S, \cdot)$ is $n$, up to isomorphism. Therefore, if there exist $k$ groups of order $n$, then we have $nk$ $\Gamma$-groups of order $n$, up to isomorphism.

Corollary 12. Suppose that $n > 1$ is an integer with decomposition into primes as $n = p_1^{e_1}p_2^{e_2} \cdots p_r^{e_r}$. If $n$ is prime to

$$\prod_{j=1}^{r}(p_j^{e_j} - 1)$$

and $e_j \leq 2$, then the number of $\Gamma$-groups of order $n$ is $n2^m$, where $m$ is the number of $j$'s with $e_j = 2$.

Proof. By a result of Rédei [12], all such groups of order $n$ are abelian. Thus, the number of isomorphism types of abelian groups of order $n$ is given by

$$\prod_{j=1}^{r} p(e_j) = 2^m,$$

where $p(e_j)$ is the number of partitions of $e_j \leq 2$ and $p(1) = 1, p(2) = 2$. The proof is completed by applying Theorem 11.

The case $m = 0$ of the Corollary 12 was studied by Szele [18]. In connection with this, Erdős [7] showed that the number of $n \leq x$ such that $(n, \varphi(n)) = 1$ ($\varphi(n)$ is Euler’s phi function) is asymptotic to

$$\frac{e^{-\gamma}x}{\log\log\log x}$$

where $\gamma$ is Euler’s constant. For additional results on the asymptotic of $n \leq x$ satisfying Rédei’s condition and asymptotic enumeration of finite abelian groups see [1, 11, 19].

In the following table we give the number of $\Gamma$-groups of order less than 30.
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References


[4] A. Cayley, *On the theory of groups, as depending on the symbolic equation $\theta^n = 1$*, Phil. Mag. 7 (1854) 40–47.


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