APPROXIMATION OF SET-VALUED FUNCTIONS
BY CONTINUOUS FUNCTIONS

BY

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Introduction. Let \( X \) be a topological space, \( E \) a Banach space. Let \( C(X, E) \) be the space of continuous functions \( f: X \to E \). Suppose \( F \) is a map from \( X \) into subsets of \( E \). Define the distance of an \( f \in C(X, E) \) from \( F \) by the relation

\[
\varrho(f, F) = \sup_{x \in X} \sup_{y \in F(x)} \|f(x) - y\|
\]

where \( \| \| \) stands for the norm in \( E \).

The main result of this note is concerned with the existence of the best approximation of a set-valued function \( F \) by a continuous point-valued function. That is we give conditions (cf. Theorem 1, Section 2) under which there exists an \( f_0 \in C(X, E) \) such that

\[
\varrho(f_0, F) = \inf_{f \in C(X, E)} \varrho(f, F).
\]

In Section 3, we apply this result to answer the following question posed by Pełczyński [5].

Let \( X, Y \) be compact topological spaces and let \( \varphi: Y \to X \) be a continuous surjection. By \( \varphi^0: C(X, E) \to C(Y, E) \) we denote the conjugate map given by \( \varphi^0f = f \circ \varphi \) if \( f \in C(X, E) \) (\( E \) as above is a Banach space.)

QUESTION. For an arbitrary but fixed \( h \in C(Y, E) \), let

\[
d(h, \varphi^0C(X, E)) = \inf_{g \in \varphi^0C(X, E)} \|h - g\|
\]

where \( \|h - g\| = \max_{y \in Y} \|h(y) - g(y)\| \). Does there exist a \( g_0 \in \varphi^0C(X, E) \) such that

\[
d(h, \varphi^0C(X, E)) = \|h - g_0\| ?
\]

The answer to this question is affirmative if \( E \) is uniformly convex and is given by Theorem 2.
The last section concerns again the existence of the best approximation of a set-valued function \( F \) by continuous functions but (0.1) is replaced there by an essential supremum-type distance. Theorem 3 of Section 4 contains as a special case a recent result due to Holmes and Kripke [2] concerning approximations of real-valued bounded functions by continuous functions.

The proof of our main result strongly depends upon a theorem of E. Michael on the existence of continuous selections. This theorem along with some basic definitions is provided, for the convenience of the reader, in Section 1.

It is a pleasant duty for the author to thank Professor Z. Semadeni for calling the author's attention to Pełczyński's problem and Professor A. Pełczyński for a stimulating discussion and, in particular, for supplying a list of references connected with his problem.

I. Notation and definitions. Throughout this note \( X, Y \) will denote topological spaces, \( E \) a uniformly convex Banach space. Let us recall that a Banach space \( E \) is uniformly convex (Clarkson [1]) if for any \( \varepsilon > 0 \) there is a \( \delta = \delta(\varepsilon) > 0 \) such that if \( \|x\| = \|y\| = 1 \) and \( \|x - y\| \geq \varepsilon(x, y \in E) \), then \( \|(x + y)/2\| \leq 1 - \delta \). Without any loss of generality we may assume that \( \delta(\varepsilon) \) is non-decreasing and, manifestly, that \( \delta(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \).

By \( 2^E, K(E) \) and \( C(E) \) we denote the set of all subsets of \( E \), closed subsets of \( E \) and closed convex subsets of \( E \), respectively.

Let \( F \) be a map of \( X \) into \( 2^E \). The map \( F \) is upper semicontinuous (u.s.c.) if the set \( \{x | F(x) \subset G\} \) is open in \( X \) for each open \( G \subset E \). Similarly, \( F \) is lower semicontinuous (l.s.c.) if the set \( \{x | F(x) \cap G \neq \emptyset\} \) is open in \( X \) for each open \( G \subset E \).

Put
\[
(1.1) \quad r(x, F) = \inf_{y \in E} \sup_{z \in F(x)} \|y - z\|
\]
and
\[
(1.2) \quad r(F) = \sup_{x \in X} r(x, F).
\]

For an arbitrary \( F, r(x, F) \) for some \( x \) or \( r(F) \) may be infinite. Since for each \( f \in C(X, E) \) we have the inequality
\[
\sup_{y \in F(x)} \|f(x) - y\| \geq r(x, F) \quad \text{for each} \quad x \in X,
\]
then by (0.1) and (1.2) we get
\[
(1.3) \quad \rho(f, F) \geq r(F) \quad \text{for each} \quad f \in C(X, E).
\]

Hence, also,
\[
(1.4) \quad \rho(F) = \inf_{f \in C(X, E)} \rho(f, F) \geq r(F).
\]
By $B(x, r), x \in E, r \geq 0$, we denote the open ball centered at $x$ of radius $r$, and by $\bar{B}(x, r)$ the closed ball.

The following two propositions describe properties of uniformly convex Banach spaces we will need later. Proposition 1 is a slightly changed lemma given in [1], p. 3, but the proof of it, which we include here for convenience of the reader, is almost the same word for word.

**Proposition 1.** Let $E$ be uniformly convex. If $\|x_1 - x_2\| > \varepsilon, x_1, x_2 \in E$, then, for any $r > 0$,

\[
B\left(\frac{1}{2} (x_1 + x_2), \left(1 - \delta(\varepsilon/r)\right)r\right) \supset B(x_1, r) \cap B(x_2, r).
\]

**Proof.** Let $y$ belong to the right-hand side of (1.5). Without any loss of generality we may assume that $y = 0$. Therefore, to prove (1.5), we have to show that

\[
\|\frac{(x_1 + x_2)/2}{2}\| < (1 - \delta(\varepsilon/r))r,
\]

if $\|x_1\| < r, \|x_2\| < r$ and $\|x_1 - x_2\| > \varepsilon$. It is easy to see, by a proper dilation or contraction, that to prove (1.6) it's enough to show that

\[
\|\frac{x_1 + x_2}{2}\| < 1 - \delta(\varepsilon) \text{ if } \|x_1\| = 1, \|x_2\| = 1, \|x_1 - x_2\| > \varepsilon.
\]

There exist $y_1, y_2$ on the unit sphere such that $x_2 = \lambda_1 y_1 + \lambda_2 y_2$, where $\lambda_1, \lambda_2 > 0, \lambda_1 + \lambda_2 = 1$ and $\|x_1 - y_1\| > \varepsilon, \|x_1 - y_2\| > \varepsilon$.

That such $y_1, y_2$ exist, follows from the existence of a supporting hyperplane to the ball $\bar{B}(x_1, \|x_1 - x_2\|)$ passing through $x_2$. By definition of uniform convexity we have

\[
\|\frac{x_1 + x_2}{2}\| \leq \lambda_1 \|\frac{x_1 + y_1}{2}\| + \lambda_2 \|\frac{x_1 + y_2}{2}\|
\]

\[
\leq \lambda_1 (1 - \delta) + \lambda_2 (1 - \delta) = 1 - \delta,
\]

which completes the proof.

**Proposition 2.** If $E$ is uniformly convex, $r > 0, x, y \in E$ fixed, then there exists a function $\eta(\varepsilon) > 0$ defined and non-decreasing for $\varepsilon > 0$ and tending to 0 as $\varepsilon \to 0$ such that

\[
\bar{B}(x, r) \cap \bar{B}(y, r + \varepsilon) \subset \bar{B}(z(\eta(\varepsilon)), r),
\]

where $z(\eta) = y + \eta(x - y)/(\|x - y\|)$.

**Proof.** Put

\[
\eta(\varepsilon) = \inf\{\eta \mid \bar{B}(x, r) \cap \bar{B}(y, r + \varepsilon) \subset \bar{B}(z(\eta), r)\}.
\]

Since $\eta = \|x - y\|$ belongs to the set in the right-hand side of (1.9), $\eta(\varepsilon)$ is well defined. It is easy to see that "inf" in (1.9) can be
replaced by "min". Therefore to prove Proposition 2 it is enough to show that $\eta(\varepsilon)$ defined by (1.9) tends to zero as $\varepsilon \to 0$. Manifestly, by (1.9), $\eta(\varepsilon)$ is non-decreasing.

Suppose that $\lim_{\varepsilon \to 0} \eta(\varepsilon) = \eta_0 > 0$. Let $\varepsilon_0 > 0$ be such that

$$0 < \eta_0 \leqslant \eta(\varepsilon) < 3\eta_0/2 \quad \text{if} \quad \varepsilon < \varepsilon_0.$$  

By (1.9), Propositions 1, 2 and (1.10) we have for each $\varepsilon < \varepsilon_0$

$$\mathcal{B}(x, r) \cap \mathcal{B}(y, r+\varepsilon) \subseteq \mathcal{B}\left(z(\eta(\varepsilon)), r+\varepsilon\right) \cap \mathcal{B}(y, r+\varepsilon)$$

$$\subseteq \mathcal{B}\left(\frac{1}{2}(z(\eta(\varepsilon))+y), (1-\delta(\eta_0(r+\varepsilon)))(r+\varepsilon)\right).$$

Choose $\varepsilon_1 > 0$ such that $(1-\delta(\eta_0/r+\varepsilon_0))(r+\varepsilon_1) \leqslant r$, and note that $(z(\eta(\varepsilon))+y)/2 = z(\eta(\varepsilon)/2)$. This and the last inclusion prove that $\eta(\varepsilon_1)/2$ belongs to the set in the right-hand side of (1.9). But $\eta(\varepsilon_1) > 0$, thus a contradiction with (1.9), and hence $\eta_0 = 0$, which was to be proved.

Finally, let us state a theorem due to Michael [3] to be used in the next section. Before, let us recall that $F: X \to 2^E$ admits a (continuous) selection if there is an $f \in C(X, E)$ such that $f(x) \in F(x)$ for each $x \in X$.

**Theorem of Michael** [3]. The following properties of $T_1$-spaces are equivalent:

(a) $X$ is paracompact.

(b) If $E$ is a Banach space, then every l.s.c. $F$ of $X$ into $C(E)$ admits a selection.

2. The main result. We will now prove the following

**Theorem 1.** Suppose that $X$ is paracompact and $E$ is uniformly convex Banach space. For each u.s.c. map $F: X \to K(E)$ there exists a best approximation by functions from $C(X, E)$; that is, there exists an $f_0 \in C(X, E)$ such that

$$\varrho(f_0, F) = \inf_{f \in C(X, E)} \varrho(f, F).$$

Moreover, for each such $f_0$ we have the equality $\varrho(f_0, F) = r(F)$.

**Proof.** Note that if $r(F) = \infty$, then by (1.3) and (1.4) $\varrho(f, F) = +\infty$ for each $f \in C(X, E)$ and the Theorem is trivial. Thus the only interesting case is if $0 < r(F) < +\infty$.

Define

$$H(x) = \{p \in E \mid F(x) \subseteq \mathcal{B}(p, r(F))\}, \quad x \in X.$$  

We shall prove first that $H(x)$ is not empty closed and convex for each $x \in X$ and that the map $H: X \to C(E)$ is l.s.c. The closedness of $H(x)$
follows from closedness of \( F(x) \) and (2.1). Suppose that \( y_1, y_2 \in H(x), \lambda_1, \lambda_2 \geq 0, \lambda_1 + \lambda_2 = 1 \). Then by (2.1) we have

\[
||y - \lambda_1 y_1 - \lambda_2 y_2|| \leq \lambda_1 ||y - y_1|| + \lambda_2 ||y - y_2|| \leq r(F) \quad \text{for each } y \in F(x),
\]

thus \( H(x) \) is convex. If \( r(x, F) < r(F) \), then clearly \( H(x) \neq \emptyset \). Suppose then that \( r(x_0, F) = r(F) \) for an \( x_0 \in X \). Put

\[
H_\gamma = \{ p \mid F(x_0) \subset B(p, r(F) + \gamma) \}, \quad \gamma > 0.
\]

The set \( H_\gamma \) is not empty for each \( \gamma > 0 \). Let \( \varepsilon > 0 \) be arbitrary and choose \( \gamma \) such that

\[
r_1 = \left(1 - \delta \left(\frac{\varepsilon}{(r(F) + \gamma)}\right)\right)(r(F) + \gamma) < r(F).
\]

Since \( \delta \left(\frac{\varepsilon}{r(F) + \gamma}\right) \geq \delta \left(\frac{\varepsilon}{r(F) + 1}\right) \) if \( \gamma \leq 1 \), there is a \( \gamma \) with \( 0 < \gamma < 1 \) satisfying (2.3). If (2.3) holds true and \( p_1, p_2 \in H_\gamma \), then \( ||p_1 - p_2|| < \varepsilon \). Indeed, suppose the contrary. Then by Proposition 1, (2.2) and (2.3) we have \( F(x_0) \subset B((p_1 + p_2)/2, r_1) \). Thus \( r(x_0, F) \leq r_1 < r(F) \), which contradicts the assumption that \( r(x_0, F) = r(F) \). We have proved that the diameter of \( H_\gamma \) is small if \( \gamma \) is small. Since \( H_\gamma \subset H_\delta \) if \( \gamma < \delta \), the intersection \( \bigcap_{\gamma > 0} H_\gamma \) is not empty and reduces to a single point \( p_0 \). It is obvious that \( F(x_0) \subset B(p_0, r(F)) \) and that \( H(x_0) = \{p_0\} \). Thus \( H(x) \) is not empty for each \( x \in X \).

To prove that \( H: X \to C(E) \) is l.s.c. consider the set

\[
A = \{ x \in X \mid H(x) \cap G \neq \emptyset \},
\]

where \( G \subset E \) is a fixed open set. Let \( x_0 \in A \) and \( p_0 \in H(x_0) \cap G \). Since \( F \) is u.s.c., there exists, for each \( \varepsilon > 0 \), a neighborhood \( N(\varepsilon) \) of \( x_0 \) such that

\[
F(x) \subset B(p_0, r(F) + \varepsilon) \quad \text{if } x \in N.
\]

Let \( \varepsilon \) be such that \( \eta(\varepsilon) \) of Proposition 2 is smaller than \( \eta_0 \), where \( B(p_0, \eta_0) \subset G \). By Proposition 2 and (2.5) we have

\[
F(x) \subset B(p_1, r(F)) \cap B(p_0, r(F) + \varepsilon) \subset B\left(\frac{\eta(\varepsilon)}{\varepsilon}, r(F)\right),
\]

where \( p_1 \in H(x) \) and \( \eta(\varepsilon) \) is \( B(p_0, \eta_0) \). By (2.6), \( \eta(\varepsilon) \) is \( H(x) \), too, whence \( B(p_0, \eta_0) \cap H(x) \subset G \cap H(x) \neq \emptyset \). Since \( x \) is an arbitrary point of \( N \), this shows that \( N \subset A \). Hence \( A \) is open and \( H \) is l.s.c.

We can now apply Michael's Theorem, by which there is an \( f_0 \in C(X, E) \) such that

\[
f_0(x) \in H(x) \quad \text{for each } x \in X.
\]
This, (2.1) and (0.1) imply that 
\( \varrho(f_0, F) \leq r(F) \), which in turn together with (1.3) shows that 
\( \varrho(f_0, F) = r(F) \). This and (1.4) proves that 
\( f_0 \) is the best approximation as well as that

\[ r(F) = \inf_{f \in C(X, E)} \varrho(f, F). \]

This completes the proof of Theorem 1.

3. An application. We will now apply Theorem 1 to answer Pelczyński’s question stated in the introduction. In this section, \( X \) and \( Y \) are compact, \( \varphi: Y \to X \) is a continuous surjection. By \( \varphi^o: C(X, E) \to C(Y, E) \) we denote the conjugate map given by \( \varphi^o f = f \circ \varphi \) if \( f \in C(X, E) \) (\( E \), as above, is a uniformly convex Banach space).

Note that each \( g \in \varphi^o C(X, E) \) is constant on \( \varphi^{-1}(x) = \{y \in Y \mid \varphi(y) = x\} \) for every \( x \in X \). We have for each \( h \in C(Y, E) \) the inequality

\[ s(h) = \sup_{x \in X} \inf_{z \in E} \sup_{y \in \varphi^{-1}(x)} \|h(y) - z\| \leq d(h, \varphi^o(X, E)), \tag{3.1} \]

where \( d \) is given by (0.3). Indeed,

\[ \|h - g\| = \sup_{x \in X} \sup_{y \in \varphi^{-1}(x)} \|h(y) - g(y)\| \geq \sup_{x \in X} \inf_{z \in E} \sup_{y \in \varphi^{-1}(x)} \|h(y) - z\| = s(h), \]

thus (3.1) follows from (0.3).

Theorem 2. For each \( h \in C(Y, E) \) there exists the best approximation \( g_h \) of \( h \) by functions from \( \varphi^o C(X, E) \); that is, \( g_h \in \varphi^o C(X, E) \) and is such that

\[ \|h - g_h\| = d(h, \varphi^o C(X, E)). \]

Moreover, each such \( g_h \) satisfies the equality

\[ \|h - g_h\| = s(h). \]

In the case that \( E \) is the real line, the second part of Theorem 2 was given by Pelczyński [5] and a proof of the first part due to S. Mazur can be found in [7], p. 20 (cf. also [2]). In the case that \( E \) is the complex plane, the second part of Theorem 2 was obtained by Pelczyński [6].

Proof of Theorem 2. Because of (3.1) it is enough to prove the existence of a \( g \in \varphi^o C(X, E) \) such that

\[ \|h - g\| = s(h). \tag{3.2} \]

Put

\[ F(x) = \{z \in E \mid z = h(y), y \in \varphi^{-1}(x)\}. \tag{3.3} \]

Since \( \varphi \) is continuous and \( Y \) is compact, \( \varphi^{-1}(x) \) is also compact for each \( x \in X \), and so is \( F(x) \). Hence (3.3) defines a map \( F: X \to K(E) \).

Suppose now that \( f_0 \in C(X, E) \) is such that

\[ \varrho(f_0, F) = r(F). \tag{3.4} \]
where \(g\) and \(r\) are given by (0.1) and (1.2), respectively. Put \(g_0 = q^0 f_0 \circ \varphi = f_0 \circ \varphi\). Then by (0.1) and (3.1) we get

\[
g(f_0, F) = \sup_{x \in X} \sup_{x \in F(x)} \|z - f_0(x)\| = \sup_{x \in X} \max_{y \in \varphi^{-1}(x)} \|h(y) - g_0(y)\| = \|h - g_0\|.
\]

On the other hand, by (1.1) and (3.1) we have

\[
r(F) = \sup_{x \in X} \inf_{y \in F(x)} \max_{x \in E} \|y - z\| = \sup_{x \in X} \inf_{y \in F(x)} \max_{x \in E} \|h(y) - z\| = s(h).
\]

Thus we see that if \(f_0 \in C(X, E)\) satisfies (3.4), then \(g_0 = f_0 \circ \varphi\) satisfies (3.2). Hence to complete the proof it is enough to check, because of Theorem 1, that \(F\) defined by (3.3) is u.s.c. To prove this let us take an open subset \(G \subset E\). By (3.3), \(F(x) \subset G\) if and only if \(\varphi^{-1}(x) \subset h^{-1}(G)\). Since \(h\) is continuous, \(h^{-1}(G)\) is an open subset of \(Y\). Now, it is easy to check that

\[
A = \{x \in X \mid \varphi^{-1}(x) \subset h^{-1}(G)\} = X \setminus \varphi(Y \setminus h^{-1}(G)).
\]

Since \(h^{-1}(G)\) is open and \(Y\) is compact, \(Y \setminus h^{-1}(G)\) is also compact and so is \(\varphi(Y \setminus h^{-1}(G))\), because \(\varphi\) is continuous. Hence the set \(A\) given by (3.5) is open. But \(A = \{x \in X \mid F(x) \subset G\}\). Therefore \(F\) is u.s.c. and Theorem 1 completes the proof of Theorem 2.

4. Approximation of bounded functions. In this section, \(E\) is a Euclidean space, \(X\) is paracompact. Let \(\mu\) be a measure defined for all open subsets of \(X\) and such that \(\mu(U) > 0\) for each open \(U \subset X\). We denote by \(\mathcal{M}\) the family of all \(\mu\)-null subsets of \(X\). Consider a map \(F : X \to 2^E\). We say that \(F\) is locally \(\mu\)-essentially bounded if for each \(x \in X\) there is an open set \(U \subset X\) and a \(\mu\)-null set \(N\) such that \(x \in U\) and \(F|_{U \setminus N}\) is bounded (\(F(x)\) is contained in a ball for each \(x \in U \setminus N\)).

Let \(f \in C(X, E)\). Put

\[
q^*_f(f, F) = \text{ess sup}_{x \in X, y \in F(x)} \|f(x) - y\|.
\]

By the latter we mean, as usual,

\[
\inf_{N \in \mathcal{M}} \sup_{x \in X \setminus N, y \in F(x)} \|f(x) - y\|.
\]

Now we define a distance of \(F\) from \(C(X, E)\) by

\[
\text{dist}(F, C(X, E)) = \inf_{f \in C(X, E)} q^*_f(f, F).
\]
In this section we are interested in the following question: if \( \text{dist} (F, C(X, E)) \) is finite, does there exist an \( f_0 \) for which the infimum in (4.3) is attained? A particular case of this question has been recently answered in the affirmative by Holmes and Kripke [2], namely the case when \( F \) is a bounded function of \( X \) into \( E \) and \( E \) is one-dimensional. The theorem which follows gives an answer to the question in a more general case.

**Theorem 3.** Let \( X, E \) and \( \mu \) be as described above. Suppose that \( F \) is a map of \( X \) into \( 2^E \) and assume it is locally \( \mu \)-essentially bounded.

Then there exists an \( f_0 \in C(X, E) \) such that

\[
\varrho_* (f_0, F) = \text{dist}(F, C(X, E)).
\]

(4.4)

Theorem 3 is a consequence of two lemmas given below and of Theorem 1.

**Lemma 1.** Define

\[
F_*(x) = \bigcap_{U \in \mathcal{B}(x)} \bigcap_{N \in \mathcal{N}} \bigcup_{y \in U \setminus N} F(y),
\]

where \( \mathcal{B}(x) \) stands for a neighborhood base at \( x \), \( \mathcal{N} \) is the family of \( \mu \)-null subsets of \( X \) and the bar indicates the closure.

Then \( F_* \) is a u.s.c. map of \( X \) into \( K(E) \).

**Proof.** Consider the family (for an \( x \in X \) fixed)

\[
\{ F_{U,N} \} = \left\{ \bigcup_{y \in U \setminus N} F(y) \right\}, \quad \text{if} \quad U \in \mathcal{B}(x) \text{ and } N \in \mathcal{N}.
\]

(4.6)

Family (4.6) has the finite intersection property, that is, any finite subfamily has a non-empty intersection. Since \( F \) is assumed to be locally essentially bounded and since \( E \) is finite-dimensional, there is a member of (4.6) which is compact, and without any loss of generality we may assume that all members of (4.6) are compact and contained in a fixed compact ball. Then by the finite intersection property, family (4.6) has a non-empty intersection which is exactly the set \( F_*(x) \) given by (4.5). Hence (4.5) defines a map of \( X \) into \( K(E) \). Let us now take an open subset \( G \) of \( E \) and suppose \( F_*(x) \subset G \) for an \( x \in X \). Again by the finite intersection property there is an \( F_{U,N} \supseteq F_*(x) \) and \( F_{U,N} \subset G \). The latter together with (4.5) implies that \( F_*(x) \subset G \) for each \( x \in U \). Thus we have proved that if an \( x_0 \) belongs to the set \( A = \{ x \mid F(x) \subset G \} \), where \( G \subset E \) is open, then there is an open \( U \subset X \) such that \( x_0 \in U \subset A \), whence \( A \) is open and \( F_* \) is u.s.c., which was to be proved.

**Lemma 2.** If \( F \) is locally essentially bounded and \( F_* \) is defined by (4.5), then we have the inequality

\[
\varrho(f, F_*) \geq \text{dist}(F, C(X, E)) \equiv r(F_*) \quad \text{for each } f \in C(X, E),
\]

(4.7)

where \( r \) is defined by (1.2).
Proof. Let us fix \( f \in C(x, E), x_0 \in X \) and \( \varepsilon > 0 \). Put \( \varrho(f, F_\ast) = \varrho_0 \). By (0.1) and (4.5) there is an \( U_0 \in \mathcal{A}(x_0) \) and \( N_0 \in \mathcal{N} \) such that

\[
F_\ast(x) \subseteq F_{U_0 \setminus N_0} \subseteq \bar{B}(f(x), \varrho_0 + \varepsilon) \quad \text{if} \quad x \in U_0.
\]

It follows from formula (4.8) that \( \sup \|f(x) - y\| \leq \varrho_0 + \varepsilon \), where the supremum is taken for \( x \in U_0 \setminus N_0 \) and \( y \in F(x) \), which in turn implies that \( \varrho_\ast(f, F) \leq \varrho_0 + \varepsilon \) (cf. (4.1) and (4.2)). Since \( \varepsilon \) is arbitrary, we have \( \varrho(f, F_\ast) \geq \varrho_\ast(f, F) \) and the first part of inequality (4.7) follows.

On the other hand, by (4.8) and (1.1) it is easy to see that

\[
\sup_{x \in U_0 \setminus N_0 \setminus y \in F(x)} \|f(x) - y\| \geq r(x, F_\ast) \quad \text{if} \quad x \in U_0.
\]

Therefore by (4.1) and (4.2) we get

\[
\varrho_\ast(f, F) \geq \sup_{x \in X} r(x, F_\ast) = r(F_\ast).
\]

Hence, by (4.3), \( \text{dist}(F, C(F, X)) \geq r(F_\ast) \) and the proof of Lemma 2 is completed.

Proof of Theorem 3. It follows from Lemma 1 and Theorem 1 that there exists an \( f_0 \in C(X, E) \) such that \( \varrho(f_0, F_\ast) = r(F_\ast) \). Using inequality (4.7) of Lemma 2 we see that the same \( f_0 \) satisfies (4.4), which was to be proved.

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