ON THE EXISTENCE AND UNIQUENESS OF SOLUTIONS OF A BOUNDARY VALUE PROBLEM FOR AN ORDINARY SECOND-ORDER DIFFERENTIAL EQUATION

BY

A. LASOTA AND Z. OPIAL (KRAKÓW)

It is well known [2] that for the linear differential equations the uniqueness of solutions of a boundary value problem implies their existence. For the non-linear differential equations this interdependence of uniqueness and existence is much more complicated and involves, in general, besides the non-linear equation under consideration, an appropriate family of linear equations [1], [3], [4].

However, as we shall show in the present paper, in the special case of the non-linear second-order differential equation it is possible to infer the existence of solutions of a boundary value problem immediately from the uniqueness of solutions of this problem for the equation itself, without recurring to a comparative family of linear equations.

In Section 1 we formulate our main theorem. Section 2 is devoted to a discussion on the assumptions of this theorem and Section 3 contains its proof. In the last section we indicate some generalizations.

1. Consider a differential equation

\[ x'' = f(t, x, x') \]

and assume that the real function \( f(t, x, u) \) defined in the strip

\[ D = (a, b) \times \mathbb{R}^2 \]

(\( \mathbb{R} \) denotes the real line) satisfies the following condition:

(C) For every point \((t_0, x_0, u_0) \in D\) there exists one and only one solution \( x(t) = x(t; t_0, x_0, u_0) \) of equation (1), defined on \((a, b)\) and such that \( x(t_0) = x_0, x'(t_0) = u_0 \).

Moreover, consider a boundary value condition

\[ x(t_1) = r_1, \ x(t_2) = r_2 \quad (a < t_1 < t_2 < b). \]
Theorem 1. If the function \( f(t, x, u) \) is continuous in the strip \( D \), satisfies condition (C) and for every pair \( (t_1, r_1), (t_2, r_2) \) of points of the set \( (a, b) \times \mathbb{R} \) \( (t_1 < t_2) \) there exists at most one solution of problem (1), (2), then for each such pair there exists one and only one solution of this problem.

2. Before proceeding to the proof of Theorem 1 let us observe that it does not hold for the closed interval \([a, b]\). In order to prove this, denote by \( \varphi(p, q) \) the solution of the equation

\[ \varphi + \frac{p}{2} \arctan \varphi = q \quad (p > -2). \]

It is easily seen that the family of all solutions of the differential equation

\[ x'' = -x + \frac{1}{2} \arctan \varphi(x \sin t, x \sin t + x' \cos t) \]

is given by the formula

\[ x(t) = A \cos t + B \sin t + \frac{1}{2} \arctan B, \]

where \( A \) and \( B \) are arbitrary constants. Thus, the boundary value condition (2) for equation (4) leads to the following system of equations:

\[ A \cos t_i + B \sin t_i + \frac{1}{2} \arctan B = r_i \quad (i = 1, 2). \]

After elimination of \( A \) we have

\[ B \sin(t_2 - t_1) + \frac{1}{2} (\cos t_1 - \cos t_2) \arctan B = r_2 \cos t_1 - r_1 \cos t_2. \]

If \( 0 \leq t_1 < t_2 < \pi \) or \( 0 < t_1 < t_2 \leq \pi \), then obviously

\[ \sin(t_2 - t_1) > 0, \quad \cos t_1 - \cos t_2 > 0. \]

Hence it follows immediately that for every pair \( r_1, r_2 \) of real numbers the problem (4), (2) has a uniquely determined solution. However, when we set \( t_1 = 0 \) and \( t_2 = \pi \), equation (5) reduces to

\[ \arctan B = r_1 + r_2. \]

As before, this assures the uniqueness of solutions of problem (4), (2) but at the same time it proves that they exist only if

\[ |r_1 + r_2| < \pi/2. \]

3. Passing now to the proof of Theorem 1, fix the points \( (t_1, r_1), (t_2, r_2) \) and, for an arbitrary \( u \in \mathbb{R} \), denote by \( x(t, u) \) the solution of (1) satisfying \( x(t_1) = r_1 \) and \( x'(t_1) = u \). From assumption (C) it follows that the mapping \( T: \mathbb{R} \to \mathbb{R} \) defined by the formula \( T(u) = x(t_2, u) \) is
continuous. Similarly, from the uniqueness of solutions of problem (1), (2) it immediately follows that $T$ is an injection. Hence its range $T(R)$ is an open and connected subset of $R$, i.e., an open (finite or infinite) interval.

Thus, in order to prove that $T(R) = R$, it remains only to show that $\sup T(R) = +\infty$ and $\inf T(R) = -\infty$. Suppose that $p_0 = \sup T(R) < +\infty$ and choose an increasing sequence $\{p_n\} \subset T(R)$ converging to $p_0$. By setting $u_n = T^{-1}(p_n)$ and $x_n(t) = x(t, u_n)$ we get

\[ x_n(t_1) = r_1, \quad x_n(t_2) = p_n. \]

(6)

From the uniqueness of solutions of problem (1), (2) it follows that

\[ x_n(t) > x_1(t) \quad (t_1 < t < b, \quad n = 2, 3, \ldots). \]

(7)

For infinitely many values of $n$ we have either $x'_n(t_2) \leq 0$ or $x'_n(t_2) \geq 0$. We shall consider only the first of these cases, for the other presents no further difficulties. Passing to an appropriate subsequence, if necessary, we may assume without loss of generality that

\[ x'_n(t_2) \leq 0 \quad (n = 1, 2, \ldots). \]

(8)

Let $t_3$ be a fixed point belonging to $(t_2, b)$. From (7) it easily follows that

\[ \frac{x_n(t_3) - x_n(t_2)}{t_3 - t_2} \geq \frac{x_1(t_3) - x_n(t_2)}{t_3 - t_2} \geq K = \min \left(0, \frac{x_1(t_3) - p_0}{t_3 - t_2} \right). \]

From this inequality and from (8) it follows that for every $n = 1, 2, \ldots$ the set

\[ S_n = \{t: t_2 \leq t \leq t_3, \quad K \leq x'_n(t) \leq 0\} \]

is non-empty. Setting $s_n = \min S_n$, we have $x'_n(t) \leq 0$ for $t_2 \leq t \leq s_n$ and therefore $x_n(s_n) \leq x_n(t_2) \leq p_0$. On the other hand, by (7) we have

\[ L = \min_{[t_2, t_3]} x_1(t) \leq x_1(s_n) \leq x_n(s_n), \]

so that

\[ L \leq x_n(s_n) \leq p_0, \quad K \leq x'_n(s_n) \leq 0, \quad t_2 \leq s_n \leq t_3. \]

Replacing, if necessary, the sequence $\{s_n\}$ by an appropriate subsequence, we may assume that there exist the limits

\[ s_0 = \lim_{n \to \infty} s_n, \quad x_0 = \lim_{n \to \infty} x_n(s_n), \quad u_0 = \lim_{n \to \infty} x'_n(s_n). \]

From the continuous dependence of solutions of (1) on their initial values it follows that the sequence $\{x_n(t)\}$ converges in $(a, b)$ to a solution
\( x'_0(t) \) of equation (1) such that \( x_0(s_0) = x_0 \) and \( x'_0(s_0) = u_0 \). Moreover, by (6) we have
\[
x_0(t_1) = r_1, \quad x_0(t_2) = p_0.
\]
This means that \( p_0 = T(x'_0(t_1)) \), so that \( p_0 \leq T(R) \). But this is impossible, since \( T(R) \) is open, and therefore \( \text{sup} \ T(R) = +\infty \).

The proof that \( \inf T(R) = -\infty \) is quite similar and will be left to the reader.

4. Theorem 1 remains true if we replace the boundary value condition (2) by a more general condition
\[
(9) \quad a_0(t_1) + \beta a'_0(t_1) = r_1, \quad x(t_2) = r_2 \quad (a < t_1 \neq t_2 < b; \ a^2 + \beta^2 > 0).
\]
We have then the following

**Theorem 2.** If the function \( f(t, x, u) \) is continuous in the strip \( D \), satisfies condition (C) and for every pair of points \( (t_1, r_1) \) and \( (t_2, r_2) \) of \( (a, b) \times R \) \( (t_1 \neq t_2) \) there exists at most one solution of problem (1), (9), then for every such pair of points there exists one and only one solution of this problem.

The proof of this theorem is quite analogous to that of Theorem 1, the only difference lies in the definition of \( x(t, u) \) which denotes now the solution of equation (1) satisfying the initial conditions
\[
x(t_1) = u, \quad x'(t_1) = \frac{1}{\beta} (r_1 - au).
\]
It is worth while to notice that Theorem 2 does not hold if we replace condition (9) by a slightly more general condition
\[
a_i x(t_i) + \beta_i x'(t_i) = r_i \quad (t_1 \neq t_2, \ a_i^2 + \beta_i^2 > 0, \ i = 1, 2).
\]

In order to prove this, consider the differential equation
\[
x'' = e^t \varphi(2e^{-t}, x'e^{-t}),
\]
where \( \varphi \) denotes the function defined by (3), and the boundary value condition
\[
x'(t_1) - x(t_1) = r_1, \quad x'(t_2) - x(t_2) = r_2 \quad (t_1 \neq t_2).
\]
Since the family of solutions of (10) is given by the formula
\[
x(t) = A + Be^t + t \arctg B,
\]
in which \( A \) and \( B \) are arbitrary constants, conditions (11) lead to the system of equations
\[
(1 - t_i) \arctg B - A = r_i \quad (i = 1, 2).
\]
The elimination of $A$ yields

$$(t_1 - t_2)\arctan B = r_2 - r_1.$$  

Thus problem (10), (11) has at most one solution, but for

$$|r_2 - r_1| \geq \frac{\pi}{2} |t_2 - t_1|$$

the solution does not exist.

REFERENCES


Reçu par la Rédaction le 10. 1. 1966