Let us recall the notion of an $A^k$-algebra, introduced in [2].

Definition. An $A^k$-algebra is an ordered pair $\langle A, \circ \rangle$, where $A$ is a non-empty set of arbitrary elements and $\circ$ is a binary operation $\circ: A^2 \to A$ fulfilling the following system of $k$ axioms:

(W_0) $x \circ x = x$,
(W_i) $x^i[y] = y^{\varphi(i)}[x]$ for $i = 1, 2, \ldots, k - 2$,
(W_{k-1}) $x^{k-1}[y] = y$,

where $x^i[y] = x \circ y$, $x^{n+1}[y] = x \circ (x^n[y])$ and $\varphi: \{1, 2, \ldots, k - 2\} \to \{1, 2, \ldots, k - 2\}$ is a fixed function.

It has been proved in [2] that every $A^k$-algebra $\mathfrak{A} = \langle A, \circ \rangle$ which satisfies axioms (W_0)-(W_{k-1}) is a quasi-group and that each pair of different elements of $A$ generates a subset of $A$ containing exactly $k$ elements. Moreover, the existence of a non-contradictory system of axioms (W_0)-(W_{k-1}) is ensured for each $k = p^m$ if $p$ is a prime number.

We say that the elements generated by a pair $a, b$ ($a \neq b$) are collinear. An $A^k$-algebra $\mathfrak{A}$ for which every triple of non-collinear elements of $A$ generates exactly $k^2$ different elements (the affine essential plane as in [1]) is called an $A^2_kA^k$-algebra. Each $A^2_A^k$-algebra which is an essential plane is an affine plane ($A^k_{k^2}$-algebra as in [1]).

In this paper some properties of $A^2_k$-algebras are considered and some new problems concerning such algebras are formulated.

1. Theorem 1. If an $A^k$-algebra fulfills the condition $x \circ (y \circ z) = z \circ (x \circ y)$, then $\varphi(1) = 2$ (1).

To prove Theorem 1 it is enough to put $x = y$ and to apply the condition (W_0).

I am going to prove

Theorem 2. Each $A^k$-algebra $\mathfrak{A}$ containing at least $k + 1$ elements for which $\varphi(1) = 2$ and $x \circ (y \circ z) = z \circ (y \circ x)$ is an $A^2_kA^k$-algebra.

(1) Algebras $A^4$, $A^{5''}$ and $A^{9'}$ in notation of [2] may serve as examples of $A^k$-algebras.
Proof. By the suppositions of theorem, $\mathcal{A}$ fulfills the following axioms:

\begin{align*}
(W_0) & \quad x[x] = x, \\
(W_1) & \quad x[y] = y^2[x], \\
(W_i) & \quad x^i[y] = y^{\varphi(i)}[x] \text{ for } i = 3, 4, \ldots, k - 2; \varphi(i) \in \{3, 4, \ldots, k - 2\}, \\
(W_{k-1}) & \quad x^{k-1}[y] = y, \\
(W_k) & \quad x \circ (y \circ z) = z \circ (y \circ x).
\end{align*}

Let $a_1, a_2 \in A$ and $a_1 \neq a_2$. Then $a_1, a_2$ generate a $k$-elements set \{\{a_1, \ldots, a_k\}\}. Let $b_0 \in A$ and $b_0 \notin \{a_1, \ldots, a_k\}$. Denote by $b_{i,0} = a_i \circ b_0$, $b_{i,1} = a_i \circ b_{i,0}$, $b_{i,2} = a_i \circ b_{i,1}$, \ldots, $b_{i,k-2} = a_i \circ b_{i,k-3}$.

It is evident that $b_{1,k-2} = a_1 \circ b_{1,k-3} = a_1 \circ (a_1 \circ b_{1,k-4}) = \ldots = a_1^{k-1}[b_0] = b_0$.

Let us take $\bar{A} = \{a_1, \ldots, a_k, b_0, b_{1,0}, b_{1,1}, \ldots, b_{1,k-2}, \ldots, b_{k,k-3}, b_{2,k-2}, \ldots, b_{k,k-2}\}$.

We shall prove first that the product of each pair of elements of $A$ belongs to $\bar{A}$.

Let $a_i \circ a_j = a_i \circ a_m$ and $a_i \circ a_k = a_k$. Then $a_r \circ b_{i,j} = a_r \circ (a_i \circ b_{1,j-1})$.

We have $a_r \circ b_{1,k-2} = a_r \circ (a_i \circ b_{1,k-3}) = a_r \circ (a_i \circ b_{1,k-2}) = a_k \circ b_{1,k-2} = a_k \circ b_0 = b_0$.

We are going to prove that $b_0 \circ b_{i,j} = b_0 \circ (a_i \circ b_{1,j-1}) = \ldots = b_0 \circ (a_i^{i+1}[b_0]) = b_0 \circ (b_0^{(i+1)+1}[a_1]) = b_0 \circ (b_0^{(i+1)+1}[a_1]) = b_0 \circ (b_0^{(i+1)+1}[b_0]) = b_0 \circ \bar{A}$.

Then $b_{i,j} \circ a_k = a_k^2[b_{i,j}] = \bar{A}$, and $b_{1,i} \circ b_0 = b_0[b_{1,i}] = b_0 \circ (b_0 \circ b_{i,i})$.

On the other hand, we have $b_0 \circ b_{i,j} = b_0 \circ (a_i \circ b_{j,j-1}) = b_0 \circ (b_{j,j-2} \circ (a_i \circ a_i)) = b_0 \circ (b_{j,j-2} \circ a_i)$.

If we decrease the coefficients $i, j$ repeatedly as above, we come to the conclusion that $b_{i,j} \circ b_{i,j} = a_i^{i+1}[b_0 \circ b_{i,i}] = \bar{A}$.

Furthermore, $b_{1,r} \circ b_{i,j} = b_{1,r} \circ (a_i \circ b_{1,j-1}) = b_{1,r} \circ (a_i \circ (a_i \circ b_{1,j-2})) = b_{1,r} \circ (b_{1,j-2} \circ a_i) = a_s \circ (b_{1,j-2} \circ b_{1,r})$.

Finally, $b_r \circ b_{i,j} = b_r \circ (a_i \circ b_{1,j}) = b_r \circ (a_i \circ (a_i \circ b_{1,j-2})) = b_r \circ (b_{1,j-2} \circ a_i)$.

We have thus proved that the elements generated by any non-collinear elements $a_1, a_2, b_0$ of $\bar{A}$ belong to $\bar{A}$. Now we shall prove that $\bar{A}$ contains exactly $k^2$ elements or that all elements of the set $\bar{A}$ are different.

We shall prove first

11. $b_{i,s} \neq a_j$.

Let us suppose that $b_{i,s} = a_j$. Then $a_i \circ (a_i^t[b_0]) = a_j$, $a_i^t[b_0] = a_i^{k-2}[a_j]$, or $b_0 = a_i^{k-1-s}[a_i^{k-2}[a_j]] = a_i$, which contradicts the supposition $b_0 \notin \{a_1, \ldots, a_k\}$.
L2. For each $b_{m,n}$ and for each $t$ there exists an $s$ such that $a_s \circ b_{m,n} = b_{t,n+1}$.

Let $a_s = a_{m}^{k-2} [a_1 \circ a_t]$, whence $a_m \circ a_s = a_1 \circ a_t$. Then $a_s \circ b_{m,n} = a_s \circ (a_m \circ b_{1,n}) = b_{1,n} \circ (a_m \circ a_s) = b_{1,n} \circ (a_1 \circ a_t) = a_t \circ (a_1 \circ b_{1,n}) = b_{t,n+1}$.

L3. Let $a_1 \circ a_t = a_p$, whence $a_t \circ a_1 = a_1 \circ a_p$. Then $a_1 \circ b_{i,j} = a_1 \circ (a_t \circ a_1) = b_{1,j-1} \circ (a_t \circ a_1) = b_{1,j-1} \circ (a_1 \circ a_t) = a_p \circ (a_1 \circ b_{1,j-1}) = a_p \circ b_{1,j} = b_{p,j+1}$.

From L2 and L3 we immediately obtain

L4. For each $b_{i,j}$ and $b_{m,n}$ ($j < n$) there exists an $s$ such that $b_{i,j} = a_i^{n-j+1} [a_s \circ b_{m,n}]$.

Now we shall prove that

(*) For every $b_{i,j}$ and $b_{m,n}$ ($j < n$) (it is clear that $n - j \leq k - 2$) we have $b_{i,j} \neq b_{m,n}$.

Suppose that $b_{i,j} = b_{m,n}$. Taking L4 into account we could obtain $a_i^{n-j+1} [a_s \circ b_{m,n}] = b_{m,n}$, then $a_s \circ b_{m,n} = a_i^{k-n+j-2} [b_{m,n}]$ and then $b_{m,n}^{2} [a_i^{k-n+j-2-2}] \circ [a_1]$. The above equation gives $a_s = b_{m,n}^{2} [a_i^{k-n+j-2-2}] \circ [a_1] = a_i^{2} [a_i^{k-n+j-2-2}] \circ [b_{m,n}]$, which leads to a contradiction (see L1 and L3).

We shall prove also that

(**) $b_{i,s} \neq b_{j,s}$ for $i \neq j$.

Let us suppose that $b_{i,s} = b_{j,s}$ ($i \neq j$). We have $a_i \circ b_{1,s-1} = a_j \circ b_{1,s-1}$, whence $a_i = a_j$ which leads to a contradiction.

From L1, (*) and (**) it immediately results that the set $\mathcal{A}$ contains exactly $k^2$ elements.

2. It is easy to prove that

1. $\mathcal{A}^3$-algebras fulfilling the axiom

$$x \circ (y \circ z) = z \circ (y \circ (z \circ x))$$

and containing at least 4 elements are $\mathcal{A}_2 \mathcal{A}^3$-algebras and

2. $\mathcal{A}^5$-algebras fulfilling the axiom

$$x \circ (y \circ z) = z \circ (y \circ (x \circ (x \circ z)))$$

and containing at least 6 elements are $\mathcal{A}_2 \mathcal{A}^5$-algebras (see [2]).

PROBLEM 1. Is it true that each $\mathcal{A}^k$-algebra fulfilling the axiom

$$x \circ (y \circ z) = z \circ (y \circ (z^{k-3+\varphi(1)} [a]))$$

and containing at least $k+1$ elements is an $\mathcal{A}_2 \mathcal{A}^k$-algebra? (P 597)

It is seen from (1), (2) and Theorem 2 of §1 that the answer is in the affirmative for all $\mathcal{A}^k$-algebras for which $k < 7$ or for which $\varphi(1) = 2$.

The author does not know the answer to the following question:
PROBLEM 2. Is it true that each subset of an $A_2 A^k$-algebra $\mathcal{A}$ containing $m$ elements which is not generated by any proper subset generates a subset of $A$ containing exactly $k^{m-1}$ elements? (P 598)

REFERENCES


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