ALMOST PERIODIC EXTENSIONS OF FUNCTIONS, III

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We recall the definition of \( I \)-sets and \( I_\theta \)-sets (see [1] and [2]) in locally compact abelian groups (LCA groups):

\( A \) is an \( I \)-set if every bounded real or complex valued function on \( A \) which is uniformly continuous on \( A \) with respect to the uniform group structure in \( G \) can be extended to an almost periodic function over \( G \).

To define the (stronger) property \( I_\theta \) we simply omit the assumption of uniform continuity of the function which has to be extended.

In [2], we proved that if \( E \) is an \( I_\theta \)-set in a separable non-discrete LCA group \( G \) and \( \tilde{E} \) means the (weak) closure of \( E \) in the Bohr compactification \( \tilde{G} \) of \( G \), then \( \mu(\tilde{E}) = 0 \), \( \mu \) denoting the Haar measure in \( \tilde{G} \). This was an answer to the first question in problem P 452 which was raised in [1] for \( G = \mathbb{R} \) (the real line) and reformulated in [2] for arbitrary LCA groups. This result was extended and strengthened by Kahane [3], in particular, we know at present that \( \mu(\tilde{E}) = 0 \) for \( E \in I_\theta \), whatever be the LCA group \( G \). Here we intend to answer the second question in P 452 for \( G = \mathbb{R} \) by proving the following

**Theorem.** If \( E \subset \mathbb{R} \) is an \( I \)-set, then \( \mu(\tilde{E}) = 0 \).

Denote by \( G_d \) a group \( G \) with discrete topology, and by \( \hat{G} \) or \( G^\wedge \) the character group of \( G \).

**Lemma.** If \( H \) is a dense subgroup of \( \mathbb{R} \) and \( \varphi \) an isomorphic continuous imbedding of \( \mathbb{R} \) into \( \tilde{R} = (R_d)^\wedge \), then \( \varphi(R) \cap (R_d/H_d)^\wedge \) = (0).

In fact, \( (R_d/H_d)^\wedge \) is the annihilator of \( H_d \) and so, in view of the density of \( H \), it does not contain any non-trivial continuous character of \( R \), thus any non-zero element of \( \varphi(R) \).

Let us observe that \( \tilde{R} \) is the cartesian product of \( 2^{\mathbb{N}_0} \) copies of the (solenoidal) group \( \hat{S} \), \( S = S_d \) denoting the group of rationals. We therefore can regard \( \tilde{R} \) as the product of two compact groups: the metric group \( \hat{S} \) (one copy) and a non-metric complementary factor \( T \) (isomorphic to the group \( \tilde{R} \) itself). Whatever be such splitting, the lemma gives
\( \varphi(R) \cap \hat{S} = (0) \). To see this we may take for \( H \) a summand complementary to \( S \), so that \( R_d = S_d + H_d \) and \( S_d = R_d | H_d \).

We now proceed to the proof of the Theorem. It is obvious that there is an \( I_0 \)-set \( A \subset E \) and a compact set (closed interval) \( K \) such that \( A + K \supset E \). We shall prove that \( \mu((A + K)^c) = \mu(\hat{A} + K) = 0 \). (Here we identify every subset of \( R \) with its \( \varphi \)-image.) Let \( \mu_1 \) be the Haar measure in \( \hat{S} \) and \( \mu_2 \) that in \( T \). Then \( \mu = \mu_1 \times \mu_2 \) is the Haar measure in \( \hat{R} \). It is enough to prove that for each \( y \in T \) we have \( \mu_1(\{x : (x, y) \in \hat{A} + K\}) = 0 \) and to apply Fubini's theorem. Actually, we will show that for each \( y \in T \) there is only a finite number of \( x \)'s such that \( (x, y) \in \hat{A} + K \). In fact, all such \( x \)'s make a compact metric set in \( \hat{S} \), hence, if there were infinitely many of them for some \( y \), there would be a convergent sequence \( \{(x_n, y)\} \) of distinct elements in \( \hat{A} + K \). Putting \( x_n = \xi_n' + \xi_n'' \) and \( y = \eta_n' + \eta_n'' \) with \( (\xi_n', \eta_n') \in \hat{A} \) and \( (\xi_n'', \eta_n'') \in K \) we could select a convergent subsequence \( \{(\xi_{n_k}', \eta_{n_k})\} \), owing to the compactness and metrizability of \( K \). Then \( (\xi_{n_k}', \eta_{n_k}) \) would equally converge. But since \( \hat{A} \) is homeomorphic to the Čech compactification \( \beta(N) \) of the set of integers (see [1]), it does not contain any non-trivial convergent sequence. So we would have \( \xi_{n_k}' = \text{const} \), \( \eta_{n_k}' = \text{const} \) and \( \eta_{n_k}'' = y - \eta_{n_k}' = \text{const} \) for \( k > k_0 \). Hence, for those \( k \), \( \{(\xi_{n_k}'', \eta)\} \) would consist of distinct elements of \( K \). This, however, is impossible, because the "axis" \( \hat{S} \) having no non-zero element in common with \( R \supset K \), no set \( \{(x, y)\} \) with a fixed \( y \) contains more than one point from \( K \). The proof is thus complete.

The Theorem can be reformulated in the intrinsic language of \( R \) as follows:

Is \( E \) an \( I \)-set in \( R \), then, for every \( \varepsilon > 0 \), there is an almost periodic function on \( R \), equal 1 on \( E \), non-negative and of mean value less than \( \varepsilon \).

REFERENCES


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