POZNAN UNIVERSITY OF TECHNOLOGY ACADEMIC JOURNALS
No 93 Electrical Engineering 2018

DOI 10.21008/j.1897-0737.2018.93.0025

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POSITIVE CONTINUOUS – TIME LINEAR ELECTRICAL CIRCUIT

The positive linear continuous-time systems by the use of the Caputo and conformable fractional calculus are analyzed. The conditions for internally positive systems are presented. For the established electrical circuit, a solution fractional order state-space equation was developed for system with Caputo and conformable fractional derivative (CFD) definition.

KEYWORDS: positive, non-positive, electrical circuit, linear.

1. INTRODUCTION

Positive dynamical system is this type of the systems where variables and output take a non-negative values in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs [4, 11].

The positive electrical circuits have been described in many papers and books [6-10, 12-13, 15]. The constructability and observability of standard and positive electrical circuits has been addressed in [7], the decoupling zeros in [8] and minimal-phase positive electrical circuits in [9]. Discovered by Kaczorek a new class of continuous linear systems, class of positive 1D and 2D systems, has been introduced in [11]. Positive fractional linear electrical circuits have been investigated positive linear systems in [14, 15].

The paper is organized as follows. First in section 1 fractional state-space equation and definition Caputo and conformable fractional derivative have been characterized. In section 2 positivity of the fractional systems have been recalled. The electrical circuit and general description of the problem analyzed in section 3. Concluding remark are given in section 4. The following notation will be used: \( \mathbb{R} \) - the set of real numbers, \( \mathbb{R}^{n\times m} \) - the set of \( n \times m \) real matrices, \( \mathbb{R}^{nc} \) - the set of \( n \times m \) real matrices with nonnegative entries and \( \mathbb{R}^n_+ = \mathbb{R}^{nc}_+ \),

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\( M_n \) - the set of \( n \times n \) Metzler matrices (real matrices with nonnegative off-diagonal entries), \( I_n \) - the \( n \times n \) identity matrix.

**1.1. Fractional order state-space equations**

We will consider the continuous-time linear system described by the state-space equations [17]

\[
D^\alpha x(t) = Ax(t) + Bu(t), \quad 0 < \alpha \leq 1 \\
y(t) = Cx(t) + Du(t)
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \), \( y(t) \in \mathbb{R}^p \) are the state, input and output vectors and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m} \) and \( D^\alpha x(t) \) is fractional order derivative of the state vector.

In next paragraphs we will use the fractional order state-space equations (1) with Caputo and conformable fractional derivative (CFD) definition of fractional order derivatives.

**1.2. The Caputo Definition**

The function defined by [17]

\[
\mathcal{C}_0^\alpha \mathcal{D}_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t f^{(n)}(\tau)(t-\tau)^{n-\alpha-1} d\tau
\]

is called the Caputo fractional derivative where \( n - 1 < \alpha < n \) and \( n \in \mathbb{N} \), \( \Gamma(x) \) is the Euler gamma function and \( f^{(n)}(t) = \frac{d^n f(t)}{dt^n} \)

The solution of the equation (1a), with Caputo definition, has the form [17]

\[
x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(\tau)Bu(t-\tau) d\tau
\]

where

\[
\Phi_0(t) = E_\alpha \left(At^\alpha\right) = \sum_{k=0}^{\infty} \frac{A^k t^{\alpha k}}{\Gamma(k\alpha + 1)}
\]

\[
\Phi(t) = \sum_{k=1}^{\infty} \frac{A^{k-1} t^{\alpha k-1}}{\Gamma(k\alpha)}
\]

and \( E_\alpha \) is the one parameter Mittag-Leffler function.
Positive continuous – time linear electrical circuits

\[ E_\alpha (\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(\alpha k + 1)} \]  

(4)

1.3. The CFD Definition

If \( n < \alpha \leq n + 1, n \in \mathbb{N}_0 \), then the conformable fractional derivative (CFD) of \( n \)-differentiable at \( t \) function \( f(t) \) (where \( t > 0 \)) is defined as [5]

\[ CFD_{\alpha}^t f(t) = \lim_{\varepsilon \to 0} \frac{f([\alpha]^{-1})(t + \varepsilon [\alpha]^{-1}) - f([\alpha]^{-1})(t)}{\varepsilon} \]  

(5)

where \( [\alpha] \) is ceiling of \( \alpha \)-the smallest integer greater than or equal to \( \alpha \).

Using the definition (5) we get a simple rule

\[ CFD_{\alpha}^t f(t) = I^{[\alpha]^{-1}} f(t) \]  

(6)

where \( f \) is \( [\alpha] \) differentiable function for \( t > 0 \).

The solution to the equation (1a) with CFD definition of fractional order derivative (5) for \( 0 < \alpha \leq 1 \) is given by [1]

\[ x(t) = e^{A(t) - t} x_0 + e^{A(t)} \int_0^t e^{-A(t)} B(t) u(\tau) d\tau \]  

(7a)

where

\[ e^{A(t)} = \sum_{k=0}^{\infty} \frac{A(t)^k}{k!} \]  

(7b)

The proof of equation (7a)

Using formula (6) we can write a state equation in the form

\[ t^{1-\alpha} \dot{x}(t) = A x(t) + B u(t), \quad 0 < \alpha \leq 1 = [\alpha] \]  

(8)

then

\[ \dot{x}(t) = A^{1-\alpha} x(t) + B t^{\alpha-1} u(t), \quad 0 < \alpha \leq 1 \]  

(9)

Introducing matrices \( A(t) = A^{t^{\alpha-1}} \) and \( B(t) = B t^{\alpha-1} \) depending on time variable we write (9)

\[ \dot{x}(t) = A(t) x(t) + B(t) u(t) \]  

(10)

The solution of equation (10) has the form [4]

\[ x(t) = e^{f(t)} x_0 + e^{f(t)} \int_0^t e^{-f(\tau)} B(\tau) u(\tau) d\tau \]  

(11)

where
Substituting (12) and $B(t) = Bt^{\alpha-1}$ in (11), we get

$$x(t) = e^{A\alpha}x_0 + e^{A\alpha} \int_0^t e^{-A\zeta} B u(\tau) \tau^{\alpha-1} d\tau$$

(13)

### 2. POSITIVITY OF THE FRACTIONAL SYSTEMS

**Definition 1.** [2, 11, 17] The system (1) is called internally positive if and only if $x(t) \in \mathbb{R}^n_+$ and $y = y(t) \in \mathbb{R}^p_+$ for any $x_0 = x(0) \in \mathbb{R}^n_+$ and every $u(t) \in \mathbb{R}^m_+$, $t \geq 0$.

**Definition 2.** [2, 11, 17] A real square matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is called Metzler matrix if $a_{ij} \geq 0$ for $i \neq j$.

**Lemma 1.** Let $A \in \mathbb{R}^{n \times n}$ and $\alpha > 0$. Then

$$e^{A\zeta} \in \mathbb{R}^{n \times n}_+ \text{ for } \zeta \geq 0$$

(14)

if and only if $A$ is Metzler matrix.

**Proof.** From (7b) written in the form

$$e^{A\zeta} = I_n + \frac{A}{\alpha} \zeta + ...$$

(15)

it follows that $e^{A\zeta} \in \mathbb{R}^{n \times n}_+$ for small $\zeta > 0$ only if $A \in M_n$.

If $A \in M_n$ then there exists $\lambda > 0$ such that $\frac{A}{\alpha} + \lambda I_n \in \mathbb{R}^{n \times n}_+$ and

$$e^{A\zeta} = e^{(\frac{A}{\alpha} + \lambda I_n)\zeta} e^{-\lambda I_n \zeta} \in \mathbb{R}^{n \times n}_+,$$  

(16)

since $e^{(\frac{A}{\alpha} + \lambda I_n)\zeta} \in \mathbb{R}^{n \times n}_+$ and $e^{-\lambda I_n \zeta} \in \mathbb{R}^{n \times n}_+$, $\zeta \geq 0$.

**Theorem 1** [2, 11, 17] The system (1) is internally positive for Caputo and CFD definition if and only if

$$A \in M_n, \quad B \in \mathbb{R}^{n \times m}_+, \quad C \in \mathbb{R}^{p \times n}_+, \quad D \in \mathbb{R}^{p \times m}_+$$

(17)
where \( M_n \) is the set of \( n \times n \) Metzler matrices. Proof for Caputo definition is given in [17].

Proof (CFD). Sufficiency. By solution of equation (1), if \( A \in M_n \), \( B \in \mathbb{R}^{n \times m} \), \( x_0 \in \mathbb{R}^n_+ \) and \( u(\tau) \in \mathbb{R}^m_+ \), \( \tau \geq 0 \), then
\[
\frac{\mathcal{D}_t^\alpha}{e^{\alpha \tau}} x_0 \in \mathbb{R}^n_+
\]
and
\[
\frac{\mathcal{D}_t^{(\alpha, \tau)}}{e^{\alpha \tau}} Bu(\tau)e^{\alpha \tau-1} \in \mathbb{R}^n_+ \text{ for } t \geq \tau \geq 0
\]
and from (7a) we obtain \( x(t) \in \mathbb{R}^n_+ \) for \( t \geq 0 \).

Necessity Let \( u(t) = 0 \) for \( t \geq 0 \) and \( x_0 = e_i \) is \( i \)-th column of the identity matrix \( n \times n \). The trajectory dose not leave the orthant \( \mathbb{R}^n_+ \) only if the derivative of order \( \alpha \), \( \frac{\mathcal{D}_0^\alpha}{e^{\alpha \tau}} x(0) = Ae_i \in \mathbb{R}^n_+ \), what implies \( a_{ij} \geq 0 \) for \( i \neq j \), what mean that matrix \( A \) is a Metzler matrix. From the same reason for \( x_0 = 0 \) we have \( \frac{\mathcal{D}_0^\alpha}{e^{\alpha \tau}} x(0) = Bu(0) \in \mathbb{R}^n_+ \), what implies \( B \in \mathbb{R}^{n \times m} \), since \( u(0) \in \mathbb{R}^m_+ \) can be arbitrary \((i \)-th column of the identity matrix \( m \times m \)). From (1b) \( u(t) = 0 \) for \( t \geq 0 \) we have \( y(0) = Cx_0 \in \mathbb{R}^p_+ \) and \( C \in \mathbb{R}^{p \times m} \), since \( x_0 \in \mathbb{R}^n_+ \) can be arbitrary. In a similar way assuming \( x_0 = 0 \), we obtain \( y(0) = Du(0) \in \mathbb{R}^p_+ \) and \( D \in \mathbb{R}^{p \times m} \) since \( u(0) \in \mathbb{R}^m_+ \) is arbitrary.

3. ELECTRICAL CIRCUIT AND GENERAL DESCRIPTION OF THE PROBLEM

In this section the solutions of the positive fractional linear electrical circuits based on the Caputo and conformable definitions of the fractional derivatives will be presented and compared. Consider the electrical circuit shown on Figure 1 with given resistances \( R_1, R_2, R_3 \), inductances \( L_1, L_2, L_3 \) and source voltages \( e_1, e_2 \). Denote by \( i_1, i_2 \) the mesh currents.
The voltage $u_L(t)$ on the coil (inductor) with the inductance $L$ be the $\beta$-order derivative of its magnetic flux $\Psi(t) = Li_L(t)$

$$u_L(t) = L \frac{d^\beta i_L(t)}{dt^\beta}$$ (19)

where $i_L(t)$ is the current in the coil. Using the equation (19) and Kirchhoff’s laws we may describe the transient states in the electrical circuit by the fractional differential equation

$$\frac{d^\beta \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}}{dt^\beta} = A \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B \begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$ (20)

where

$$A = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} -L_2(R_1 + R_3) - L_3R_1 & L_2R_3 - L_3R_2 \\ L_1R_3 - L_3R_1 & -L_1(R_2 + R_3) - L_3R_2 \end{bmatrix}$$ (21a)

and

$$B = \frac{1}{L_1(L_2 + L_3) + L_2L_3} \begin{bmatrix} L_2 + L_3 & L_3 \\ L_3 & L_1 + L_3 \end{bmatrix}.$$ (21b)

Matrix $A$ is a Metzler matrix if and only if

$$L_2R_3 \geq L_3R_2 \text{ and } L_1R_3 \geq L_3R_1.$$ (22)

From (21) and theorem 1 it follows that the electrical circuit is positive when $A$ is 2x2 square matrix, what implies (22).

**Example 1** For further analysis we assume to equations (3a), (7a) with $\alpha = 0.5$. Let resistances $R_1 = 10 \ \Omega$, $R_2 = 5 \ \Omega$, $R_3 = 20 \ \Omega$ and three identity coils of
inductances $L_1 = L_2 = L_3 = 1 \text{H}$ have been applied. The constant input was established as $e_1 = 0.5 \text{ V}, e_2 = 0.1 \text{ V}$. The initial conditions given as $i_1(0) = 0.0 \text{ A}$, $i_2(0) = 0.2 \text{ A}$, state vector $x(t) = [i_1(t) \ i_2(t)]^T$, $x_0 = [i_1(0) \ i_2(0)]^T$ and input vector $u(t) = [e_1 \ e_2]^T = U$. Matrix $A$

$$A = \begin{bmatrix} -13.333 & 5.0 \\ 3.333 & -10.0 \end{bmatrix} \in M_2 \text{ and } B = \begin{bmatrix} 0.667 & 0.333 \\ 0.333 & 0.667 \end{bmatrix}$$

are non-negative.

**Solution 1** Using the Caputo definition of fractional derivative [17] we obtain the following solution to the equation (3a)

$$x_c(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t)BUd\tau = \sum_{k=0}^{\infty} \frac{A^kx_0}{k!(\kappa+1)} + \int_0^t \sum_{k=1}^{\infty} \frac{A^{k-1}B^{k\alpha}}{k!(\kappa+1)}Ud\tau = \sum_{k=0}^{\infty} \frac{A^kx_0}{k!(\kappa+1)} + \sum_{k=1}^{\infty} \frac{A^{k-1}B^{k\alpha}}{k!(\kappa+1)}Ux_0 + \sum_{k=1}^{\infty} \frac{A^{k-1}B^{k\alpha}}{k!(\kappa+1)}(Ax_0 + BU)$$

Substituting in to (23a) numerical data we obtain

$$x_c(t) = \begin{bmatrix} 0.70 \\ 0.2 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{t^k}{k!(\kappa+1)} \begin{bmatrix} -13.333 & 5.0 \\ 3.333 & -10.0 \end{bmatrix} = \begin{bmatrix} 0.70 \\ 0.2 \end{bmatrix} + \sum_{k=1}^{\infty} \frac{t^k}{k!(\kappa+1)} \begin{bmatrix} -13.333 & 5.0 \\ 3.333 & -10.0 \end{bmatrix} = \begin{bmatrix} 1.367 \\ 0.112 \end{bmatrix}$$

**Solution 2** Using the CFD definition of fractional derivative [1] we obtain the following solution to the equation (7)

$$x_c(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu d\tau = x_c(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu d\tau = x_c(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu d\tau = x_c(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu d\tau = x_c(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu d\tau = x_c(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu d\tau = x_c(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu d\tau$$

Substituting in to (24a) numerical data we obtain
The solutions using the Caputo definition (23b) and CFD (24b) for currents across coils $L_1$ and $L_2$ for fractional order are shown in Fig. 2 and Fig. 3.

Fig. 2. Solutions using the Caputo and CFD definition for the first coil where $\alpha = 0.5$

Fig. 3. Solutions using the Caputo and CFD definition for the second coil where $\alpha = 0.5$
Example 2 For further analysis we assume (3a), (7a) \( \alpha = 0.5 \), \( R_1 = 50 \Omega \), \( R_2 = 20 \Omega \), \( R_3 = 10 \Omega \) and three identity coils of inductances \( L_1, L_2, L_3 = 1 \) H. The constant input given as \( e_1 = 0.5 \) V, \( e_2 = 0.1 \) V. The initial conditions was established as \( i_1(0) = 0.05 A \), \( i_2(0) = 1.0 A \). Matrix

\[
A = \begin{bmatrix}
-36.667 & -3.333 \\
-13.333 & -16.667 \\
\end{bmatrix} \neq M_2 \text{ and } B = \begin{bmatrix}
0.667 & 0.333 \\
0.333 & 0.667 \\
\end{bmatrix}
\]

are non-negative.

Solution 3 Using the Caputo definition of fractional derivative we obtain the following solution to the equation (23a)

\[
x(t) = \begin{bmatrix}
0.05 \\
1.0 \\
\end{bmatrix} + \sum_{k=1}^{\infty} \frac{t^k}{k!(k+1)} \begin{bmatrix}
-36.667 & -3.333 \\
-13.333 & -16.667 \\
\end{bmatrix} \Gamma (k+1, 1) \begin{bmatrix}
0.05 \\
1.0 \\
\end{bmatrix} + \begin{bmatrix}
0.667 & 0.333 \\
0.333 & 0.667 \\
\end{bmatrix} \begin{bmatrix}
0.5 \\
0.1 \\
\end{bmatrix} = \begin{bmatrix}
4.8 \\
17.1 \\
\end{bmatrix} = (25)
\]

Solution 4 Using the CFD definition of fractional derivative [1] we obtain the following solution to the equation (24a)

\[
x_{CFD}(t) = \begin{bmatrix}
0.05 \\
1.0 \\
\end{bmatrix} + \sum_{k=1}^{\infty} \frac{t^k}{k!(k+1)} \begin{bmatrix}
-36.667 & -3.333 \\
-13.333 & -16.667 \\
\end{bmatrix} \Gamma (k+1, 1) \begin{bmatrix}
0.05 \\
1.0 \\
\end{bmatrix} + \begin{bmatrix}
0.667 & 0.333 \\
0.333 & 0.667 \\
\end{bmatrix} \begin{bmatrix}
0.5 \\
0.1 \\
\end{bmatrix} = \begin{bmatrix}
4.8 \\
17.1 \\
\end{bmatrix} = (26)
\]

The solutions using the Caputo definition (25) and CFD (26) for currents across coils \( L_1 \) and \( L_2 \) for fractional order are shown in Fig. 4 and Fig. 5.
Both Caputo and CFD definition was used to solve equation fractional order state-space equation. Necessary and sufficient conditions for the positivity of the fractional linear systems have been established.

In this paper electrical circuit in Example 1 is positive. When parameters of the electrical circuit in Example 2 making change, it is non-positive.

This work was supported by Ministry of Science and Higher Education in Poland under work No. MB/WE/3/2017
REFERENCES


(Received: 18.01.2018, revised: 03.03.2018)