Coupled static and dynamic FE analyses of a nonlinear electromechanical vibration energy harvester

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The paper presents the coupled static and dynamic mechanical–electromagnetic finite element analyses of an electromechanical vibration energy harvesting converter with permanent–magnet excitation. The system consists of a small, milliwatt power range, linear–motion generator connected to a cantilever–beam spring element. The finite element equations derived for the mechanical part of the system according to the 1–D Timoshenko beam theory are coupled strongly with those describing the 2–D distribution of magnetic field in the generator and those associated with the electric circuit. The considered electromechanical coupling allows for prediction of static and dynamic response of the system subjected to action of externally applied force. The static displacement of the moving element is computed via solution of the nonlinear system of equations and is used as an initial solution in dynamic analysis. For computation of the dynamic response of the system the time–stepping procedure based on the Crank–Nicolson discretisation schema is applied. The models are positively validated against the measurements carried out on the laboratory test–stand.

KEYWORDS: energy harvesting, coupled structural–electromagnetic analysis, finite element analysis

1. Introduction

An interest in energy harvesting systems is related with growing amount of autonomous and wireless systems, like those used in condition monitoring of industrial machinery. One group of these converters are the harvesters of mechanical vibrations’ dissipation energy [1–13]. The general operation principle of these converters relies upon matching the frequency of internal resonance of the converter with that of the dissipative system.

This work analyses the electromechanical energy harvester comprising of a cantilever beam spring element driving a linear–motion permanent–magnet generator with a single–phase cored armature winding [2, 3]. The generator produces alternating voltage with the magnitude that within certain range of the operation bandwidth is proportional to the magnitude and frequency of the externally applied excitation force. The process of energy harvesting by the considered system is much more complex than that by those with coreless generators, as the system kinematics is nonlinear due to the electromagnetic force pro-
duced by the generator [2]. This force has the two components, namely the cogging force developed by interaction of magnets' magnetic field with cored armature, and the mutual force developed by interaction of magnets' magnetic field with current of the armature winding.

![Diagram of the considered system]

Fig. 1. View of the considered system: a) overall schema, b) side view with dimensions, c) dimensions of the magnetic circuit parts

The nonlinearity is brought mainly by the cogging component. As a consequence the system becomes a nonlinear oscillator whose resonance frequency depends on the magnitude of displacement in approximately the following way

\[ \omega_r = \omega_n + \kappa A^2 \]  

(1)
where \( \omega_n \) is the natural frequency attributed to internal mechanical structure of the system, and \( \kappa \) the coefficient which, depending on the variation of the cogging force, can be positive or negative [4], and \( A \) the magnitude of displacement. This phenomenon allows for construction of the energy harvesters whose frequency bandwidth is several Hz wide [5, 6].

There are numerous publications on the analysis of nonlinear oscillators used in energy harvesting systems [7]. The most common approach to determination of the mechanical response of these systems use the lumped-parameter models based on solution of the Duffing equation [4, 8, 9]. However, main parts of the considered system (see Fig. 1), namely the cantilever–beam spring and the linear–motion generator, are the distributed–parameter systems coupled not only through the cogging force, but also through the electromagnetic force involving the armature reaction field. The latter cannot be simply accounted for in the analysis based on the lumped–parameter model due to a nonlinear ferromagnetic core existing in the system.

In the presented approach to modeling dynamics of the considered system, the equation of motion, derived via the 1–D cantilever–beam theory, is coupled with that for the magnetic field through the electromagnetic forces and mechanical displacement. The interpolation procedures and the time–discretization schema are used to account for motion of the moving regions (yokes with magnets) in the magnetic field analysis. In order to determine the initial conditions for the dynamic model excited by the externally applied force, the coupled static model is also developed.

2. Coupled models

2.1 Dynamic model

Main assumption in the considered analyses is that the 3–d effects involving a non–straight–line trajectory of motion of the mover (yokes with magnets in Fig. 1e) can be ignored in the magnetic field analysis. This assumption is, however not too far from real condition which is illustrated in Fig. 2 by the preliminary results of simulation of the real trajectory of some particular points attached to the moving element.

As shown in figures for the magnitude of displacement being equal to a few (2 to 3) millimeters, the deviation of the trajectories from a straight line does not exceed six per cent of the translational magnitude. This is when the shaded rectangular areas that represent permanent magnets in Fig. 2a, intersect the shaded square area representing the core. In such the case results of the magnetic field analysis will not be significantly affected by the 3–d effects.
Fig. 2. Illustration of motion trajectory of moving part of the generator: a) simulated deflection of spring showing relative positions of mover with respect to the armature, b) trajectories of characteristics points attached to corners of permanent magnets in Fig. a

The finite element equation of motion of the distributed mass of spring, fixed at one end, loaded by the lumped mass (mover) derived from the Timoshenko beam theory is considered in form [14, 15]

\[
(M + M_m) \frac{d^2 u}{dt^2} - D \frac{du}{dr} + Ku - F_m(u, \varphi, i) = F_{ext}(t) + F_g
\]

(2)

where: \( u = [u_{y1}, u_{y2}, \psi_{y1}, \psi_{y2}]^T \) is the vector of displacements containing the two translational components along the \( y \) axis, \( u_{y1} \) and \( u_{y2} \), and the two
rotating components, \( \psi_{r1} \) and \( \psi_{r2} \) around the x axis attached to element nodes [14]. \( \mathbf{M}, \mathbf{D} \) and \( \mathbf{K} \) are the distributed mass, damping and stiffness matrix, respectively, while \( \mathbf{M}_m \) is the diagonal matrix corresponding with the lumped mass of mover. In the considered approach the damping matrix is expressed via the Rayleigh damping model \( \mathbf{D} = a \mathbf{M} + b \mathbf{K} \) with constants \( a, b \) to be identified. The vector \( \mathbf{F}_m \) describes the electromagnetic force developed by the generator, dependent on the vector of displacement \( \mathbf{u} \), the vector of the magnetic potential \( \varphi \) and current \( i \) through the generator armature winding, while \( \mathbf{F}_{\text{ext}} \) the externally applied mechanical force and \( \mathbf{F}_g \) the gravitation force.

Solution for the distribution of the magnetic potential \( \varphi \) is carried out assuming two–dimensionality of the magnetic field distribution. The finite element equation governing this distribution over plane defined by the cross–section of the generator in Fig. 1c, is considered in form [16, 17]

\[
\mathbf{S} \varphi = \mathbf{K}_w^T \varphi + \mathbf{F}_\mu
\]

(3)

where \( \mathbf{S} \) is the reluctance matrix, \( \mathbf{K}_w \) the winding function vector that distributes the current density uniformly over the coils' cross–section and takes account for sense of turns, and \( \mathbf{F}_\mu \) the vector of magnetomotive force due to permanent magnets [16]. The current through the armature winding is calculated from equation

\[
\ell_{\text{eff}} \mathbf{K}_w^T \frac{d\varphi}{dt} + (R + R_{\text{load}})i + L_e \frac{di}{dt} = 0
\]

(4)

with \( \ell_{\text{eff}} \) being the effective turn length, \( R \) the resistance of the winding, the load resistance, and \( L_e \) the leakage inductance of the winding end connections. In order to model motion of the generator mover the interpolation sliding interface technique is used [18]. In such the case the linear constraints in the form of

\[
a_{ij}(\mathbf{u}) \varphi_j = 0
\]

(5)

with \( a_{ij} \) being coefficients of the first–order Lagrange interpolation polynomial, are imposed on the magnetic potentials attached to air–gap sliding interface. In such the case the equations (3) and (4) modify to

\[
\mathbf{Q}(\mathbf{u}) \mathbf{S} \varphi \mathbf{Q}(\mathbf{u})^T \varphi' = \mathbf{Q}(\mathbf{u}) \mathbf{K}_w^T \varphi + \mathbf{Q}(\mathbf{u})^T \mathbf{F}_\mu
\]

(6)

\[
\ell_{\text{eff}} \mathbf{K}_w^T \mathbf{Q}(\mathbf{u}) \frac{d\varphi}{dt} + (R + R_{\text{load}})i + L_e \frac{di}{dt} = 0
\]

(7)

with \( \mathbf{Q}(\mathbf{u}) \) being the transformation matrix that enforces constraints (4) on the vector \( \varphi \) such that \( \mathbf{Q}(\mathbf{u}) \varphi = \varphi \) with \( \varphi \) being the vector of nodal potentials of the constrained system.

2.2. Static model

Solution of the initial–value problem defined by the equations (1), (3) and (5) must be preceded by determination of the initial conditions at time \( t = 0 \). Given the external force vector \( \mathbf{F}_{\text{ext}} \) involving the distributed mechanical force applied
to the beam nodes corresponding with length of the mover at \( t = 0 \), these conditions are found assuming that current \( i = 0 \) at \( t = 0 \) and solving the following system of nonlinear equations describing the steady–state of the considered system

\[
\begin{bmatrix}
  K & -C_f(u, \varphi, i)Q(u) \\
  0 & Q^T(u)S(\varphi)Q(u)
\end{bmatrix}
\begin{bmatrix}
  u_0 \\
  \varphi_0
\end{bmatrix} = \begin{bmatrix}
  F_{ext0} \\
  F_{\mu0}
\end{bmatrix}
\]

(8)

in which the matrix \( C^T \varphi_0 \) should be understood as to express the electromagnetic force vector \( F_{m0} \) via the discrete formula describing the Maxwell stress tensor [16]. The above system of equations is solved using the fixed–point technique.

2.3. Discretisation of time in dynamic model

The initial value/algebraic problem described by equations (2)–(7) is solved considering solution of (8) by substituting in (2) \( du/dt \) with the vector of velocity \( \nu \) and applying the Crank–Nicolson discretisation schema [16]. The obtained system of coupled difference equations has form

\[
\begin{bmatrix}
  \frac{E}{\Delta t} & \frac{E}{2} & 0 & 0 \\
  -K & M & -C_f \frac{\Delta t}{2} & 0 \\
  2 & 2 & \frac{Q_n^T S Q_n}{2} & 0 \\
  0 & 0 & \frac{Q_n^T K_{E0} Q_n}{\Delta t} & \frac{R + R_{load}}{2} + \frac{L_e}{\Delta t}
\end{bmatrix}
\begin{bmatrix}
  u_n \\
  v_n \\
  \varphi_n \\
  i_n
\end{bmatrix} = \begin{bmatrix}
  f_{ext(n)} + f_{ext(n-1)} + F_{\nu} \\
  0 \\
  0 \\
  0
\end{bmatrix}
\]

(9)

where \( \Delta t \) is the time–step length, and \( n \) is used as the index of a time–instant for the variables and matrices with entries that change with time. The system of equations (9) is solved starting from the initial point \( [u_0, 0, \varphi_0, 0]^T \). Note that unlike the rotating converters where lengths of the vectors \( \varphi_n \) and \( \varphi_{n-1} \) are always the same, application of the sliding interface technique in the linear motion converters results in different length of these vectors at the two consecutive instants of time. In order to keep the dimensions of the left– and right–hand–side constant for the transition between two time–instant, the system of equations (9) takes into account that \( Q_n \varphi_{n-1} = \varphi_{n-1} \), \( Q_n \varphi_n = \varphi_n \), as well as that \( Q_n^T \varphi_n = \varphi_n \).
The nonlinearity in the above system of equations exists due to action of the implicit time–discretisation formula on the electromagnetic force vector \( \mathbf{F} \) as well as due to inherent nonlinearity related with properties of the magnetic materials. The solution procedure for the considered nonlinear problem given by the equations (8) and (9) using the fixed–point technique is outlined in diagram in Fig. 3.

Fig. 3. Flowchart diagram of the algorithm of solution of the considered coupled problem, \( \alpha \in (0,1) \) is the relaxation factor adapted by the program at each iteration

### 3. Computations

The considered system has the specifications summarized in Table 1. Fig. 5 displays the manufactured prototype mounted on the laboratory test–stand.

Table 1. Technical specifications of the considered system

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Permanent magnets</td>
<td>NdFeB with remanent magnetic flux density equal to 1.1 T.</td>
</tr>
<tr>
<td>Armature winding</td>
<td>100 turns, ( R = 5.7 , \Omega ), ( I_{\text{eq}} = 0.01 , \text{mH} )</td>
</tr>
<tr>
<td>Spring</td>
<td>Material: Aluminum, Young modulus = 96 MPa, Poisson ratio = 0.33 MPa, Mass density = 2800 kg/m³, Damping constants (determined experimentally) ( \alpha = 1.34 \times 10^{-5} , \text{s}, b = 2.75 , \text{1/S} )</td>
</tr>
</tbody>
</table>
The core of the generator armature was made of the soft magnetic powder material, while the moving yokes from the solid mild steel. The magnetisation characteristics of both materials are plotted in Fig. 4.

![Magnetisation characteristics of magnetic materials used](image)

**Fig. 4. Magnetisation characteristics of magnetic materials used**

Figure 6 presents the results of static analysis carried out on the test stand in Fig. 5 by increasing span of the link connecting mover with the support via the dynamometer. The displacement of the center of mass of the mover was measured as a function of external force applied to the mover.

![Considered energy harvester installed on the laboratory test-stand](image)

**Fig. 5. Considered energy harvester installed on the laboratory test-stand**
In order to validate the proposed dynamic model the laboratory experiment was carried out in the following way. The system was mounted on a fixed base as shown in Fig. 5, while the spring was stretched applying a constant force equal to 30 N directly to the mover. The impulse force test was carried out releasing the spring immediately while the armature winding was loaded by the resistance equal to twice the resistance of the armature winding ($R_{\text{load}} = 11.4 \, \Omega$). Next, for the same conditions the computer simulation was carried out using the time step–size $\Delta t = 0.33$ ms. The comparison of measured and computed transient waveforms of current through the armature winding is shown in Fig. 7. Figure 8 displays sample distributions of the magnetic field at different positions of mover.

![Graph showing comparison of measured and computed values of displacement of yokes](image1)

**Fig. 6.** Comparison of measured and computed values of displacement of yokes

![Graph showing comparison of measured and computed transient waveforms of current through the armature winding](image2)

**Fig. 7.** Comparison of computed and measured transient waveforms of current through the armature winding
One can deduce from Figs. 6 and 7 the proposed models provides satisfactory predictions of the static dynamic performances. The negative displacement in static analysis is due to the gravitational force. The time–constant and the fundamental frequencies of the computed and measured waveforms in Fig. 7 are very close, though the magnitudes are slightly different. It should be noted that the considered harvester is highly nonlinear system with the kinematics similar to that of the externally driven physical pendulum [14]. In this type of systems the nonlinearity brought by the periodic force dependent on the position of mover (cogging force) causes parametric dependence of the oscillation frequency on the magnitude of oscillations. As one can see in Fig. 7, the measured and computed waveforms show the same feature, namely a decrease of frequency with magnitude. Owing to the considerations carried out in the introductory sections including equation (1), it can be deduced that the considered system has negative coefficient η. This is the result of internal design of the system, particularly the ratio of magnet–to–core width and the air–gap width used.
The simulation executed on the standard desktop computer for the time-range shown in Fig. 7 took approximately two hours, which makes the proposed model applicable in further research.

4. Conclusion

The proposed approaches can be used in simulation of the static response and the dynamic performance of the considered system when subjected to arbitrary excitations. The static analysis is particularly useful for designing of the spring element. The presented dynamic analysis can avoid the need for seeking the equivalent parameters for the lumped-parameter models. Here, the authors presented the results for which the high-order mechanical vibration modes are not manifested, the same model can be used in case when these modes are important, and creation and identification of the lumped-parameter models in such the case becomes a challenge. It can also be used in simulation of the system response to electric loading and prediction of core losses.

The presented result show some mismatch involving smaller computed magnitudes than measured. In opinion of the authors the main source of this disagreement is an imperfect identification of material characteristics and damping constants.

References


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