Reciprocal Stern Polynomials

by

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Summary. A partial answer is given to a problem of Ulas (2011), asking when the \( n \)th Stern polynomial is reciprocal.

Let \( B_n(t) \) be defined by the formulae

\[
B_1(t) = 1, \quad B_{2n}(t) = tB_n(t), \quad B_{2n+1}(t) = B_n(t) + B_{n+1}(t).
\]

Klavžar, Milutinović and Petr [2] have called \( B_n(t) \) the \( n \)th Stern polynomial and Ulas [4] asked when \( B_n(t) \) is reciprocal, i.e.

\[
(1) \quad B_n^*(t) = t^\deg B_n B_n(t^{-1}) = B_n(t).
\]

As a partial answer we shall prove

THEOREM 1. If \( n \) has binary expansion

\[
(2) \quad n = a_1 a_2 \ldots a_k \quad (k \text{ odd}, \ a_i \geq 1 \ \text{for all} \ 1 \leq i \leq k),
\]

and \( l_1, \ldots, l_j \) are the lengths of blocks of 1 occurring in the sequence \( a_2, \ldots, a_k \), then (1) holds if and only if, identically in \( t \),

\[
(3) \quad \sum_{\mu=0}^{\lfloor k/2 \rfloor} \sum' \frac{T_{a_1} \cdots T_{a_k}}{T_{a_{i_1}} \cdots T_{a_{i_\mu}}} \left( \prod_{\lambda=1}^{\mu} \frac{t^{a_{i_\lambda}}}{T_{a_{i_\lambda}+1}} - t^d \prod_{\lambda=1}^{\mu} \frac{t^{a_{i_\lambda}+1-2}}{T_{a_{i_\lambda}+1}} \right) = 0,
\]

where \( \sum' \) is taken over all integer vectors \([i_1, \ldots, i_\mu]\) such that

\[
(4) \quad 1 \leq i_1 < \cdots < i_\mu < k, \quad i_{\lambda+1} \geq i_\lambda + 2 \ (1 \leq \lambda < \mu),
\]

and where

\[
T_a = \frac{t^a - 1}{t - 1}, \quad d = \left\lfloor \frac{l_1 + 1}{2} \right\rfloor + \cdots + \left\lfloor \frac{l_j + 1}{2} \right\rfloor.
\]

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Corollary. If \( n \) has binary expansion \((2)\) and
\[
a_{i+1} = a_i + 2 \quad (1 \leq i < k),
\]
then \((1)\) holds.

Theorem 2. If \( n \) has binary expansion \((2)\) and for all pairs \( 1 \leq i < j \leq k \),
\[
a_i + a_j > \max\{a_1, \ldots, a_k\} + 2,
\]
then \((1)\) is equivalent to \((4)\).

The assumption \((6)\) in Theorem 2 is not superfluous, as the two infinite sequences of odd \( n \) satisfying \((1)\) discovered by M. Gawron \( \Box \) show, as also does the following

Theorem 3. For \( k \leq 3 \), \((1)\) holds if and only if either \( k = 1 \), or \( k = 3 \) and \((4)\) holds, or \( a_1 = a - 1, a_2 = 2a, a_3 = a + 1 \) (\( a \) an integer \( > 1 \)), or \( a_2 = 1, a_3 = 2 \), or \( a_1 = a_2 - 1 \) (\( a_2 > 1 \)), \( a_3 = 1 \).

Proof of Theorem 1. It follows from \[3, \text{Theorem 1 and Lemma 5}\] that if \((2)\) holds, then
\[
B_n(t) = T_{a_1} \cdots T_{a_k} \left( 1 + \sum_{\mu=1}^{[k/2]} \sum' \frac{1}{T_{a_1} \cdots T_{a_{\mu}}} \prod_{\lambda=1}^{\mu} \frac{t^{a_{i_{\lambda}+1}}}{t^{a_{i_{\lambda}}} - 1} \right).
\]
On the other hand, by \[3, \text{Theorem 2}\] and \((2)\),
\[
deg B_n = a_1 + \cdots + a_k - k + d = \deg T_{a_1} \cdots T_{a_k} + d.
\]
Also
\[
T_a(t^{-1}) = t^{1-a} T_a(t),
\]
hence by \((1)\),
\[
B_\ast(t) = t^d T_{a_1} \cdots T_{a_k} \left( 1 + \sum_{\mu=1}^{[k/2]} \sum' \frac{1}{T_{a_1} \cdots T_{a_{\mu}}} \prod_{\lambda=1}^{\mu} \frac{t^{a_{i_{\lambda}+1}}^{a_{i_{\lambda}+1} - 2}}{t^{a_{i_{\lambda}}} - 1} \right).
\]

Theorem 1 follows from \((7)\) and \((8)\).

Proof of Corollary. If \( a_{i+1} - a_i = 2 \ (1 \leq i < k) \), then \( d = 0 \) and the Corollary follows from Theorem 1.

For the proof of Theorem 2 we need two lemmas.

Lemma 1. If \( k \geq 2 \), \( a_i \ (1 \leq i \leq k) \) is a sequence of positive integers and
\[
\sum_{i=1}^{k-1} \left( \frac{1}{t^{a_i}} - \frac{1}{t^{a_{i+1}} - 2} \right) = 0,
\]
identically in $t$, then

$${a_2, \ldots, a_{k-1}} = \{a_1 + 2, \ldots, a_1 + 2(k-2)\},$$

(10) $a_k = a_1 + 2(k-1).

Proof. Differentiating (9) and substituting afterwards $t = 1$ we obtain

$$\sum_{i=1}^{k-1} (a_i - a_{i+1} + 2) = 0,$$

thus (11) holds. Substituting in (9) we obtain

$$k-1 \sum_{i=1}^{a_i} - 1 = \sum_{i=1}^{a_i} \frac{1}{t a_i} - \frac{1}{a_k}$$

$$= \frac{t^{2(k-1)} - 1}{t a_1 + 2(k-1)} = (t^2 - 1)(t^{2(k-2)} + t^{2(k-3)} + \cdots + 1),$$

and on dividing both sides by $t^2 - 1$,

$$\sum_{i=1}^{k-1} \frac{1}{t a_i} = \sum_{i=1}^{k-1} \frac{1}{t a_1 + 2i}.$$

Substituting $t = u^{-1}$ we obtain an identity for polynomials which implies (10). \[ \Box \]

Lemma 2. If (6) holds, then for any $2 \leq \mu < k$ integers $i_\lambda$ $(1 \leq \lambda \leq \mu)$ satisfying (4) we have

$$a_{i_1} + \cdots + a_{i_\mu} \geq \frac{\mu}{2} (\max\{a_1, \ldots, a_k\} + 3).$$

(12) $a_{i_1} + \cdots + a_{i_\mu} \geq \frac{\mu}{2} (\max\{a_1, \ldots, a_k\} + 3)$.

Proof. By (6) for any positive integers $\lambda < \nu \leq \mu$ we have

$$a_{i_\lambda} + a_{i_\nu} \geq \max\{a_1, \ldots, a_k\} + 3.$$

Summing over all pairs $\lambda, \nu$ in question we obtain

$$(\mu - 1)(a_{i_1} + \cdots + a_{i_\mu}) \geq \left(\frac{\mu}{2}\right) (\max\{a_1, \ldots, a_k\} + 3),$$

which implies (12). \[ \Box \]

Proof of Theorem 2. Let us write the sum $S$ occurring in (3) in the form

$$S = \sum_{\mu=0}^{[k/2]} S_\mu.$$

If (6) holds, we have $a_i > 2$ for all $i \leq k$, thus $d = 0$, $S_0 = 0$ and by
Theorem 1 and Lemma 2 for all \( \mu \geq 2 \),

\[
\deg(t-1)^{k-2}S_\mu \leq a_1 + \cdots + a_k - 2 + 2\mu - \min^* \min \left(-2\mu + \sum_{\lambda=1}^{\mu} a_{i_{\lambda+1}}, \sum_{\lambda=1}^{\mu} a_{i_\lambda}\right)
\]

\[
\leq \max \left(a_1 + \cdots + a_k - 2 + 2\mu - \frac{\mu}{2} \left(\max\{a_1, \ldots, a_k\} - 3\right),
    a_1 + \cdots + a_k - 2 - \max\{a_1, \ldots, a_k\} - 3\right)
\]

\[
< a_1 + \cdots + a_k - \max\{a_1, \ldots, a_k\},
\]

where \( \min^* \) is taken over all integer vectors \([i_1, \ldots, i_\mu]\) satisfying (4).

On the other hand, by (6), the sum of all terms of \((t-1)^{k-2}S_1\) of degree \( \geq a_1 + \cdots + a_k - \max\{a_1, \ldots, a_k\}\) equals

\[
\sum_{i=1}^{k-1} t^{a_1+\cdots+a_k-a_i+1} - \sum_{i=1}^{k-1} t^{a_1+\cdots+a_k-2-a_i}.
\]

Substituting \( t = 1 \) leads by Theorem 1 to the conclusion that for all \( i < k \) we have \( a_i + 2 \leq \max\{a_1, \ldots, a_k\} \), and that

\[
\sum_{i=1}^{k-1} \left(\frac{1}{t^{a_i+1}} - \frac{1}{t^{a_i+2}}\right) = 0.
\]

By Lemma 1 we obtain

\[
a_k = a_1 + 2(k-1),
\]

\[
\{a_2, \ldots, a_{k-1}\} = \{a_1 + 2, \ldots, a_1 + 2(k-2)\},
\]

\[
s = \sum_{i=1}^{k} a_i = k(a_1 + k - 1).
\]

Therefore, we have

\[
(t-1)^{k-2}S_1 = -\sum_{i=1}^{k-1} \sum_{h=1}^{k} t^{s-a_i+1-a_h} - \sum_{i=1}^{k-1} \sum_{h=1 \neq i,i+1}^{k} t^{s-2-a_i-a_h} + O(t^{s-3a_1-6}),
\]

\[
(t-1)^{k-2}S_2 = (t-1)^{2} \left(\sum_{1<i+1<h<k} t^{s-a_i+1-a_{i+1}+1} - \sum_{1<i+1<h<k} t^{s-4-a_h-a_k}\right) + O(t^{s-3a_1+4}),
\]

\[
(t-1)^{k-2}S_\mu = O(t^{s-3a_1-8}) \quad (\mu \geq 3).
\]
We shall show by induction on \( i < k \) that
\[
(13) \quad a_{i+1} = a_1 + 2i.
\]
Suppose that
\[
a_j = a_i + 2, \quad j > 2.
\]
Then \((t-1)^{k-2}S_1\) contains the term \(-ts^{-2ai-2}\) (for \( i = 1, h = i - 1 \)) which does not cancel with any other term of \((t-1)^{k-2}S_1\) since \(2 + a_i + a_h \leq 2a_i + 2\) is impossible for \( i \neq h \). Thus
\[
a_2 = a_1 + 2.
\]
Assume now that \( a_{i+1} = a_i + 2i \) for \( i \leq l \leq k - 3, \ l \geq 1 \). Then
\[
(t-1)^{k-2}S_1 = - \sum_{i=l+1}^{k-1} \sum_{\begin{subarray}{c}h=1 \\ h \neq i,i+1 \end{subarray}}^{k} ts^{-a_{i+1}-a_h} + \sum_{i=l+1}^{k-1} \sum_{\begin{subarray}{c}h=1 \\ h \neq i,i+1 \end{subarray}}^{k} ts^{-2a_i-a_h} + O(t^{s-3a_1-6}),
\]
\[
(t-1)^{k-2}S_2 = (t-1)^2 \left( \sum_{1<i+1<h<k \atop h>l} ts^{-a_{i+1}-a_h+1} - \sum_{1<i+1<h<k \atop h>l} ts^{-4-a_i-a_h} \right) + O(t^{s-3a_1+4}).
\]
Suppose that
\[
a_j = a_1 + 2l, \quad j > l + 1.
\]
Then \((t-1)^{k-2}S_2\) contains the term \(-2ts^{-(2a_1+2l+1)}\) (for \( i = 1, h = j - 1 \)), which does not cancel any other term of \((t-1)^{k-2}S\). Indeed, we have \(2a_1 + 2l + 1 < 3a_1 + 2\) and the terms of \((t-1)^{k-2}S_1\) of degree \( \geq s-(3a_1+2) \) are of the form \(ts^{-2m}, m \) integer. So are also the terms of \((t-1)^{k-2}S_2\) of degree \( \geq s-(3a_1+2) \) except the terms of
\[
-2t \left( \sum_{1<i+1<h<k \atop h>l} ts^{-a_{i+1}-a_h+1} - \sum_{1<i+1<h<k \atop h>l} ts^{-4-a_i-a_h} \right).
\]
However, for \( h > l \) we have
\[
3 + a_i + a_h > 2a_1 + 2l + 1.
\]
This proves \((13)\), and \((4)\) follows.

For the proof of Theorem 3 we need

**Lemma 3.** If \( T_\alpha T_\beta = T_\gamma T_\delta \), where \( \alpha, \beta, \gamma, \delta \in \mathbb{N} \setminus \{0\}, \ \alpha \leq \beta, \ \gamma \leq \delta \), then
\[
\alpha = \gamma, \quad \beta = \delta.
\]
Proof. Assume that $\beta > \delta$. Then $T_\beta(\zeta_\beta) = 0$, where $\beta$ is a positive $\beta$th root of unity, but $T_\gamma(\zeta_\beta) \neq 0$, $T_\delta(\zeta_\beta) \neq 0$, a contradiction. Thus $\beta \leq \delta$ and by symmetry $\beta = \delta$. Hence $T_\alpha = T_\gamma$ and $\alpha = \gamma$. 

Proof of Theorem 3. (1) holds obviously for $k = 1$. For $k = 3$ we shall consider successively the following cases:

A. $a_2 \geq 2$, $a_3 \geq 2$,
B. $a_2 = 1$, $a_3 \geq 2$,
C. $a_2 = 2$, $a_3 = 1$,
D. $a_2 = a_3 = 1$.

A. Here we have $d = 0$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_3}(t^{a_1} - t^{a_2-2}) + T_{a_1}(t^{a_2} - t^{a_3-2}) = 0,$$

thus to

$$\varepsilon t^{\min\{a_1,a_2-2\}} T_{a_3} T_{|a_1-a_2+2|} + \eta t^{\min\{a_2,a_3-2\}} T_{a_1} T_{|a_2-a_3+2|} = 0,$$

where

$$\varepsilon = \text{sgn}(a_1 - a_2 + 2), \quad \eta = \text{sgn}(a_2 - a_3 + 2).$$

Hence, there are the following possibilities: either

1. $\varepsilon = \eta = 0$, so $a_2 = a_1 + 2$, $a_3 = a_2 + 2$, and (1) holds; or
2. $\min\{a_1,a_2-2\} = \min\{a_2,a_3-2\}$,

$$\varepsilon = -\eta \neq 0,$$

and by Lemma 3 either

$$a_3 = a_1, \quad |a_1 - a_2 + 2| = |a_3 - a_2 + 2|,$$

or

$$a_3 = |a_2 - a_3 + 2|, \quad a_1 = |a_1 - a_2 + 2|.$$

The formulae (16) give $a_1 = a_2 = a_3$, contrary to (14) and (15). We cannot have $a_3 = a_3 - a_2 - 2$, thus from (16) we obtain $a_3 = a_2 - a_3 + 2$, from (14) $\eta = 1$, from (15) $\varepsilon = -1$ and from (16) $a_1 = a_2 - a_1 - 2$. Taking $a_2 = 2a$, we obtain $a_1 = a - 1$, $a_3 = a + 1$ (with integer $a > 1$).

B. Here we have $d = 1$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_1} T_{a_3}(1 - t) + T_{a_3}(t^{a_1} - 1) + T_{a_1}(t - t^{a_3-1}) = 0,$$

hence

$$T_{a_1}(t - t^{a_3-1}) = 0, \quad a_3 = 2.$$
C. Here we have $d = 1$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_1}T_{a_2}(1-t) + t^{a_1} - t^{a_2-1} + T_{a_1}(t^{a_2} - 1) = 0,$$

hence

$$t^{a_1} - t^{a_2-1} = 0, \quad a_1 = a_2 - 1.$$

D. Here we have $d = 1$. By Theorem 1 the identity (1) is equivalent to the identity

$$T_{a_1}(1-t) + t^{a_1} - 1 + T_{a_1}(t-1) = 0,$$

hence

$$t^{a_1} = 1, \quad \text{impossible.} \quad \blacksquare$$

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