APPLICATION OF THE SHUFFLE ALGORITHM TO ANALYSIS OF DESCRIPTOR LINEAR SYSTEMS WITH SINGULAR PENCIL

Abstract

A new method of analysis of descriptor continuous-time and discrete-time linear systems with singular pencil is proposed. The method is based on transformation of singular pencil by the use of the shuffle algorithm. Conditions for transformation of the descriptor linear systems to equivalent standard linear systems are established.

1. INTRODUCTION

Descriptor linear systems with regular pencils have been considered in many papers and books [4-12, 14-17]. The descriptor linear systems with singular pencils have been addressed in [10, 11]. The Drazin inverse of matrix to analysis of linear algebraic-differential equations has been applied in [4-7, 9, 11]. The positive descriptor linear systems with regular pencils have been analyzed in [1-3, 17]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems with regular pencil in [12].

In this paper a new method of analysis of descriptor linear systems with singular pencils will be proposed. The method is based on transformation of singular pencils by the use of the shuffle algorithm.

The paper is organized as follows. In section 2 the definitions of elementary row operations are recalled and lemma on transformation of singular pencil by the use of shuffle algorithm is presented. Main results of the paper are presented in section 3 and 4. The reduction of the descriptor linear system to equivalent standard systems is presented in section 3 for continuous-time systems and in section 4 for discrete-time systems. Concluding remarks are given in section 5.

The following notation will be used: $\mathbb{R}$ - the set of real numbers, $\mathbb{R}^{n \times m}$ - the set of $n \times m$ real matrices, $\mathbb{Z}_+$ - the set of nonnegative integers, $I_n$ - the $n \times n$ identity matrix.

2. PRELIMINARIES

The following elementary row operations will be used:
1) Multiplication of the $i$th row by a real number $c$. This operation will be denoted by $L[i \times c]$.
2) Addition to the $i$th row of the $j$th row multiplied by a real number $c$. This operation will be denoted by $L[i + j \times c]$.
3) Interchange of the \( i \)th and \( j \)th rows. This operation will be denoted by \( L[i, j] \).

In a similar way we define the elementary column operations.

The proposed new approach to analysis of descriptor linear systems with singular pencil is based on the following lemma.

**Lemma 2.1.** Let \( E, A \in \mathbb{R}^{q \times n} \) and

\[
\text{rank } E < r \leq \min(q, n). \tag{2.1}
\]

If (2.1) holds then the singular pencil \([Es - A]\) with

\[
\text{rank } [Es - A] = r \text{ for some } s \in \mathbb{C} \text{ (the field of complex numbers)} \tag{2.2}
\]

can be transformed by the use of shuffle algorithm to the equivalent pencil with the same rank and modified matrix \( \overline{E} \) satisfying the condition

\[
\text{rank } \overline{E} = r. \tag{2.3}
\]

**Proof.** It is well-known \([11, 16]\) that in the shuffle algorithm the elementary row operations and shuffles are used. The elementary row operations performed on the pencil \([Es - A]\) do not change its (normal) rank. The shuffle performed on the pencil is equivalent to multiplication of some numbers of row of the pencil by \( s \). This operation also does not change the rank of the pencil but it usually increases the rank of the modified matrix \( E \). After some finite number of shuffles we obtain

\[
\text{rank } E = \text{rank } [Es - \overline{A}] = r. \quad \square \tag{2.4}
\]

### 3. REDUCTION OF DESCRIPTOR CONTINUOUS-TIME SYSTEMS TO EQUIVALENT STANDARD SYSTEMS

Consider the descriptor continuous-time linear system

\[
\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{3.1}
\]

where \( x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m \) are the state and input vectors and \( E, A \in \mathbb{R}^{q \times n}, B \in \mathbb{R}^{q \times m} \). It is assumed that pencil \([Es - A]\) is singular and

\[
\text{rank } [Es - A] = r \leq \min(q, n) \text{ for some } s \in \mathbb{C}. \tag{3.2}
\]

In particular case if \( q = n \) and \( \det[Es - A] \neq 0 \) for some \( s \in \mathbb{C} \) then the pencil is regular.

Performing elementary row operations on the array

\[
\begin{bmatrix}
E & A & B
\end{bmatrix}
\]

or equivalently on the equation (3.1) we obtain
\[
\begin{bmatrix}
E_1 & A_1 & B_1 \\
0 & A_2 & B_2 \\
\end{bmatrix}
\] (3.4)

and

\[
E_1 \dot{x}(t) = A_1 x(t) + B_1 u(t) \\
0 = A_2 x(t) + B_2 u(t)
\] (3.5a, 3.5b)

where \( E_1 \in \mathbb{R}^{r \times n} \) has full row rank. Differentiation of (3.5b) with respect to time yields

\[
A_2 \dot{x}(t) = -B_2 \ddot{u}(t)
\] (3.6)

The equations (3.5a) and (3.6) can be written in the form

\[
\begin{bmatrix}
E_1 \\
A_2
\end{bmatrix} \dot{x}(t) = \begin{bmatrix}
A_1 \\
0
\end{bmatrix} x(t) + \begin{bmatrix}
B_1 \\
0
\end{bmatrix} u(t) + \begin{bmatrix}
0 \\
-B_2
\end{bmatrix} \ddot{u}(t)
\] (3.7)

The array

\[
\begin{bmatrix}
E_1 & A_1 & B_1 & 0 \\
A_2 & 0 & 0 & -B_2
\end{bmatrix}
\] (3.8)

can be obtained from (3.4) by performing a shuffle. If

\[
\text{rank} \begin{bmatrix}
E_1 \\
A_2
\end{bmatrix} = r
\] (3.9)

then performing elementary row operations on (3.8) (or equivalently on (3.7)) we obtain

\[
\begin{bmatrix}
E_2 & A_3 & B_{10} & B_{11} \\
0 & A_4 & B_{20} & B_{21}
\end{bmatrix}
\] (3.10)

and

\[
E_2 \dot{x}(t) = A_3 x(t) + B_{10} u(t) + B_{11} \ddot{u}(t) \\
0 = A_4 x(t) + B_{20} u(t) + B_{21} \ddot{u}(t)
\] (3.11a, 3.11b)

where \( E_2 \in \mathbb{R}^{r \times n} \) has full row rank. If \( \text{rank} E_2 = r \) then there exists a nonsingular matrix \( P \in \mathbb{R}^{r \times n} \) of elementary column operations such that

\[
E_2 P = [E_2 \quad 0], \ E_2 \in \mathbb{R}^{r \times r}, \ 0 \in \mathbb{R}^{r \times (n-r)}, \ \det \ E_2 \neq 0.
\] (3.12)

Defining the new state vector

\[
\bar{x}(t) = P^{-1} x(t) = \begin{bmatrix}
\bar{x}_1(t) \\
\bar{x}_2(t)
\end{bmatrix}, \ \bar{x}_1(t) \in \mathbb{R}^r, \ \bar{x}_2(t) \in \mathbb{R}^{n-r}
\] (3.13)

from (3.11a) we obtain
\[ E_2 P P^{-1} \dot{x}(t) = A_4 P P^{-1} x(t) + B_{10} u(t) + B_{11} \dot{u}(t) \]  

(3.14)

and

\[ \tilde{E}_2 \tilde{x}_1(t) = A_{31} \tilde{x}_1(t) + A_{32} \tilde{x}_2(t) + B_{10} u(t) + B_{11} \dot{u}(t) \]  

(3.15)

where

\[ A_4 P = [A_{31} \ A_{32}], \ A_{31} \in \mathbb{R}^{r \times r}, \ A_{32} \in \mathbb{R}^{r \times (n-r)}. \]  

(3.16)

Premultiplying (3.15) by the inverse matrix \( \tilde{E}_2^{-1} \) we obtain

\[ \tilde{x}_1(t) = \tilde{A}_{31} \tilde{x}_1(t) + \tilde{A}_{32} \tilde{x}_2(t) + \tilde{B}_{10} u(t) + \tilde{B}_{11} \dot{u}(t) \]  

(3.17a)

where

\[ \tilde{A}_{31} = \tilde{E}_2^{-1} A_{31}, \ \tilde{A}_{32} = \tilde{E}_2^{-1} A_{32}, \ \tilde{B}_{10} = \tilde{E}_2^{-1} B_{10}, \ \tilde{B}_{11} = \tilde{E}_2^{-1} B_{11}. \]  

(3.17b)

From (3.11b) and (3.13) we obtain

\[ 0 = A_4 P P^{-1} x(t) + B_{20} u(t) + B_{21} \dot{u}(t) = A_{41} \tilde{x}_1(t) + A_{42} \tilde{x}_2(t) + B_{20} u(t) + B_{21} \dot{u}(t) \]  

(3.18a)

where

\[ A_4 P = [A_{41} \ A_{42}], \ A_{41} \in \mathbb{R}^{r \times r}, \ A_{42} \in \mathbb{R}^{r \times (n-r)}. \]  

(3.18b)

If \( q = n \) and \( \det A_{42} \neq 0 \) then from (3.18a) we obtain

\[ \tilde{x}_2(t) = -\tilde{A}_{41} \tilde{x}_1(t) - \tilde{B}_{20} u(t) - \tilde{B}_{21} \dot{u}(t) \]  

(3.19a)

where

\[ \tilde{A}_{41} = A_{42}^{-1} A_{41}, \ \tilde{B}_{20} = A_{42}^{-1} B_{20}, \ \tilde{B}_{21} = A_{42}^{-1} B_{21}. \]  

(3.19b)

Substitution of (3.19a) into (3.17a) yields

\[ \tilde{x}_1(t) = \tilde{A}_1 \tilde{x}_1(t) + \tilde{B}_{10} u(t) + \tilde{B}_{11} \dot{u}(t) \]  

(3.20a)

where

\[ \tilde{A}_1 = \tilde{A}_{31} - \tilde{A}_{32} \tilde{A}_{41}, \ \tilde{B}_{10} = \tilde{B}_{10} - \tilde{A}_{32} \tilde{B}_{20}, \ \tilde{B}_{11} = \tilde{B}_{11} - \tilde{A}_{32} \tilde{B}_{21}. \]  

(3.20b)

The solution of the equation (3.20a) for given consistent initial condition \( \tilde{x}_{10} = \tilde{x}_1(0) \) and admissible input \( u(t) \) and \( \dot{u}(t) \) has the form

\[ \tilde{x}_1(t) = e^{\tilde{A}_1 t} \tilde{x}_{10} + \int_0^t e^{\tilde{A}_1 (t-\tau)} [\tilde{B}_{10} u(\tau) + \tilde{B}_{11} \dot{u}(\tau)] d\tau. \]  

(3.21)

Knowing (3.21) from (3.19a) we can find \( \tilde{x}_2(t) \). If \( q \neq n \) or \( \det A_{42} = 0 \) for \( q = n \) then we choose \( \tilde{x}_2(t) \) so that (3.18a) holds for given consistent initial conditions and admissible input \( u(t) \) and from (3.17a) we have

\[ \tilde{x}_1(t) = e^{\tilde{A}_1 t} \tilde{x}_{10} + \int_0^t e^{\tilde{A}_1 (t-\tau)} [\tilde{A}_{32} \tilde{x}_2(\tau) + \tilde{B}_{10} u(\tau) + \tilde{B}_{11} \dot{u}(\tau)] d\tau. \]  

(3.22)
If rank \( E < r \) then we repeat the procedure for array (3.8) and by Lemma 2.1 after \( \mu \) steps (shuffles) we obtain

\[
E_{\mu+1} \hat{x}(t) = A_{\mu+2} x(t) + B_{10} u(t) + B_{11} \hat{u}(t) + ... + B_{1,\mu} u^{(\mu)}(t) \tag{3.23a}
\]

\[
0 = A_{\mu+3} x(t) + B_{20} u(t) + B_{21} \hat{u}(t) + ... + B_{2,\mu} u^{(\mu)}(t) \tag{3.23b}
\]

where \( E_{\mu+1} \in \mathbb{R}^{r \times n} \) has full row rank and \( u^{(\mu)}(t) = \frac{d^\mu u(t)}{dt^\mu} \). If rank \( E_{\mu+1} = r \) then there exists a nonsingular matrix \( P \in \mathbb{R}^{n \times n} \) of elementary column operations such that

\[
E_{\mu+1} P = [\bar{E}_{\mu+1} \ 0], \quad \bar{E}_{\mu+1} \in \mathbb{R}^{r \times r}, \quad 0 \in \mathbb{R}^{r \times (n-r)}, \quad \text{det} \bar{E}_{\mu+1} \neq 0. \tag{3.24}
\]

Defining the new state vector (3.13) from (3.23a) we obtain

\[
\bar{E}_{\mu+1} \hat{x}_1(t) = A_{\mu+2,1} x_1(t) + A_{\mu+2,2} x_2(t) + B_{10} u(t) + B_{11} \hat{u}(t) + ... + B_{1,\mu} u^{(\mu)}(t) \tag{3.25}
\]

where

\[
A_{\mu+2} = [A_{\mu+2,1} \ A_{\mu+2,2}], \quad A_{\mu+2,1} \in \mathbb{R}^{r \times r}, \quad A_{\mu+2,2} \in \mathbb{R}^{r \times (n-r)}. \tag{3.26}
\]

Premultiplying (3.25) by the matrix \( \bar{E}_{\mu+1}^{-1} \) we obtain

\[
\hat{x}_1(t) = \bar{A}_{\mu+2,1} x_1(t) + \bar{A}_{\mu+2,2} x_2(t) + \bar{B}_{10} u(t) + \bar{B}_{11} \hat{u}(t) + ... + \bar{B}_{1,\mu} u^{(\mu)}(t) \tag{3.27a}
\]

where

\[
\bar{A}_{\mu+2,1} = \bar{A}_{\mu+1,1}, \quad \bar{A}_{\mu+2,2} = \bar{A}_{\mu+1,2}, \quad \bar{B}_{1,k} = \bar{E}_{\mu+1} B_{1,k}, \quad k = 0,1,\ldots,\mu. \tag{3.27b}
\]

From (3.23b), (3.13) and (3.24) we obtain

\[
0 = A_{\mu+3,1} x_1(t) + A_{\mu+3,2} x_2(t) + B_{20} u(t) + B_{21} \hat{u}(t) + ... + B_{2,\mu} u^{(\mu)}(t) \tag{3.28a}
\]

where

\[
A_{\mu+3} = [A_{\mu+3,1} \ A_{\mu+3,2}], \quad A_{\mu+3,1} \in \mathbb{R}^{r \times r}, \quad A_{\mu+3,2} \in \mathbb{R}^{r \times (n-r)}. \tag{3.28b}
\]

If \( q = n \) and \( \text{det} A_{\mu+3,2} \neq 0 \) then from (3.28a) we obtain

\[
x_2(t) = -\overline{A}_{\mu+3,1} x_1(t) - \overline{B}_{20} u(t) - \overline{B}_{21} \hat{u}(t) - ... - \overline{B}_{2,\mu} u^{(\mu)}(t) \tag{3.29a}
\]

where

\[
\overline{A}_{\mu+3,1} = A_{\mu+3,2}^{-1} A_{\mu+3,1}, \quad \overline{B}_{2,k} = A_{\mu+3,2}^{-1} B_{2,k}, \quad k = 0,1,\ldots,\mu. \tag{3.29b}
\]

Substitution of (3.29a) into (3.27a) yields

\[
\hat{x}_1(t) = \bar{A}_{\mu+1} \hat{x}_1(t) + \bar{B}_{10} u(t) + \bar{B}_{11} \hat{u}(t) + ... + \bar{B}_{1,\mu} u^{(\mu)}(t) \tag{3.30a}
\]

where

\[
\bar{A}_1 = \bar{A}_{\mu+2,1} - \bar{A}_{\mu+2,2} \overline{A}_{\mu+3,1}, \quad \bar{B}_{1,k} = \bar{B}_{1,k} - \bar{A}_{\mu+2,2} \overline{B}_{2,k}, \quad k = 0,1,\ldots,\mu. \tag{3.30b}
\]
The solution of the equation (3.30a) for a given consistent initial condition \( x_0 = x(t) \) and input \( u^{(k)}(t) \), \( k = 0, 1, \ldots, \mu \) has the form

\[
\bar{x}_1(t) = e^{\bar{A}t}x_0 + \int_0^t e^{\bar{A}(t-\tau)}[\bar{B}_0u(\tau) + \bar{B}_1\dot{u}(\tau) + \cdots + \bar{B}_{1,\mu}u^{(\mu)}(\tau)]d\tau. \tag{3.31}
\]

Knowing (3.31) from (3.29a) we can find \( x_2(t) \) and from (3.13) we have

\[
x(t) = P \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{3.32}
\]

If \( q \neq n \) or \( \text{det} A_{\mu+3,2} = 0 \) for \( q = n \) then we choose \( x_2(t) \) so that (3.28a) holds for given consistent initial conditions and admissible input \( u(t) \) and from (3.27a) we have

\[
\bar{x}_1(t) = e^{\bar{A}_{\mu+3,2}t}x_0 + \int_0^t e^{\bar{A}_{\mu+3,2}(t-\tau)}[\bar{A}_{\mu+2,3}\bar{x}_2(\tau) + \bar{B}_{0}u(\tau) + \bar{B}_{1}\dot{u}(\tau) + \cdots + \bar{B}_{1,\mu}u^{(\mu)}(\tau)]d\tau. \tag{3.33}
\]

**Theorem 3.1.** If \( q = n \) and \( \text{det} A_{\mu+3,2} \neq 0 \) then the equation (3.1) with singular pencil satisfying (3.2) can be reduced by the use of shuffle algorithm to the equation (3.30a). The solution \( x_1(t) \) of the equation (3.30a) for a given consistent initial conditions and admissible input \( u^{(k)}(t) \), \( k = 0, 1, \ldots, \mu \) is given by (3.31). If \( q \neq n \) or \( \text{det} A_{\mu+3,2} = 0 \) for \( q = n \) then \( x_2(t) \) is chosen so that the equality (3.28a) is satisfied for given consistent initial conditions and admissible input \( u^{(k)}(t) \), \( k = 0, 1, \ldots, \mu \). The solution \( \bar{x}_1(t) \) of the equation (3.27a) is given by (3.33).

**Remark 3.1.** In particular case when \( q > n, n = r \) and \( A_{\mu+3} = 0 \in \mathbb{R}^{(q-n)\times n}, B_{2,k} = 0 \in \mathbb{R}^{(q-n)\times n} \) for \( k = 0, 1, \ldots, \mu \); then from (3.23a) we have

\[
\dot{x}(t) = E_{\mu+1}^{-1}A_{\mu+2}x(t) + E_{\mu+1}^{-1}B_{10}u(t) + E_{\mu+1}^{-1}B_{1}\dot{u}(t) + \cdots + E_{\mu+1}^{-1}B_{1,\mu}u^{(\mu)}(t). \tag{3.34}
\]

**Example 3.1.** Consider the equation (3.1) with the matrices

\[
E = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}. \tag{3.35}
\]

In this case \( q = 4, n = 3, m = 1, \) rank \( E = 2 \) and

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\[ \text{rank} [Es - A] = \text{rank} \begin{bmatrix} s & -1 & s \\ s & -1 & s \\ 2s & -1 & 2s \\ s & -1 & s \end{bmatrix} = 3 \text{ for some } s \in \mathbb{C}. \quad (3.36) \]

Performing on the array
\[
E \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 & 1 & 0 & 1 & -1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}
\]

the elementary row operations \( L[3+1\times(-2)] , \ L[4+2\times(-1)] \) we obtain
\[
E_i \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

and after a shuffle
\[
E_i \begin{bmatrix} A_i \\ B_i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}
\]

The matrix
\[
E_2 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}
\]

is nonsingular and from (3.39) and (3.34) we obtain the standard equation
\[
\dot{x}(t) = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix} \dot{u}(t)
\]
\[
= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & -1 \end{bmatrix} u(t) + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \dot{u}(t).
\quad (3.40)
\]

**Example 3.2.** Consider the equation (3.1) with the matrices
\[
E = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} , \quad A = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \quad B = \begin{bmatrix} 1 \end{bmatrix}
\]

In this case \( q = n = 3, \ m = 1 \), \( \text{rank} \ E = 1 \) and
\[ \text{rank} [E_s - A] = \text{rank} \begin{bmatrix} s & 2 & s \\ 0 & 1 & 0 \\ -s & 0 & -s \end{bmatrix} = 2 \text{ for some } s \in C. \quad (3.42) \]

Performing on the array
\[
E \quad A \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 & 0 & 1 \\ -1 & 0 & -1 & 0 & 0 & 0 & 1 \end{bmatrix}
\]
the elementary row operation \( L[3+1\times 1] \) we obtain
\[
E_1 \quad A_1 \quad B_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{bmatrix} \quad (3.44)
\]
and after a shuffle
\[
E_1 \quad A_1 \quad B_1 = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.45)
\]
Performing on the array (3.45) the elementary row operation \( L[3+2\times(-2)] \) we get
\[
E_2 \quad A_3 \quad B_{10} \quad B_{11} = \begin{bmatrix} 1 & 0 & 1 & 0 & -2 & 0 & 1 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad (3.46)
\]
and
\[
E_2 \dot{x}(t) = A_4 x(t) + B_{10} u(t) + B_{11} \dot{u}(t) \]
\[ 0 = A_4 x(t) + B_{20} u(t) + B_{21} \ddot{u}(t) \quad (3.47a) \]
where
\[
E_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_{10} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0 \end{bmatrix}, \quad A_4 = [0 \ 0 \ 0], \quad B_{20} = [0], \quad B_{21} = [-1]. \quad (3.47b)
\]

From (3.47b) it follows that the set of admissible inputs is defined by \( \dot{u}(t) = 0 \), it is the set of constant inputs \( u(t) = u \) (constant). In this case the matrix \( P \) and \( E_2 \) have the forms
\[
P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \bar{E}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad (3.48)
\]

since

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From (3.13), (3.47a) and (3.48) we have

\[
\bar{x}(t) = P^{-1}x(t) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) + x_3(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix},
\]

(3.50)

\[
\begin{align*}
\bar{x}_1(t) &= \begin{bmatrix} x_1(t) + x_3(t) \\ x_2(t) \end{bmatrix}, & \bar{x}_2(t) &= x_3(t)
\end{align*}
\]

and

\[
\dot{\bar{x}}_i(t) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \bar{x}_i(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t).
\]

(3.51)

The equation (3.51) has solution for any initial condition \( \bar{x}_i(0) = \bar{x}_{i0} \). Note that \( \bar{x}_2(t) = x_3(t) \) can be arbitrary.

4. REDUCTION OF DESCRIPTOR DISCRETE-TIME LINEAR SYSTEMS TO EQUIVALENT STANDARD SYSTEMS

Consider the descriptor discrete-time linear system

\[
E_{i+1}x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = \{0,1,...\}
\]

(4.1)

where \( x_i \in \mathbb{R}^n \), \( u_i \in \mathbb{R}^m \) are the state and input vectors and \( E, A \in \mathbb{R}^{q \times q} \), \( B \in \mathbb{R}^{q \times m} \). It is assumed that pencil \([ Ez - A ]\) is singular and

\[
\text{rank} [ Ez - A ] = r \leq \text{min}(q,n) \quad \text{for some} \quad z \in \mathbb{C}.
\]

(4.2)

In particular case if \( q = n \) and \( \det[Ez - A] \neq 0\) for some \( s \in \mathbb{C} \) then the pencil is regular. Performing elementary row operations on the array

\[
\begin{bmatrix} E & A & B \end{bmatrix}
\]

(4.3)

or equivalently on the equation (4.1) we obtain

\[
\begin{bmatrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix}
\]

(4.4)

and

\[
E_i x_{i+1} = A_ix_i + B_i u_i
\]

(4.5a)

\[
0 = A_2x_i + B_2u_i.
\]

(4.5b)

Substitution in (4.5b) \( i \) by \( i+1 \) yields
The equations (4.5a) and (4.6) can be written in the form

\[ A_2 y_{i+1} = -B_2 u_{i+1} \]  

(4.6)

The equations (4.5a) and (4.6) can be written in the form

\[ \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} y_{i+1} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} y_i + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ -B_2 \end{bmatrix} u_{i+1} \]  

(4.7)

The array

\[
\begin{bmatrix}
E_1 & A_1 & B_1 & 0 \\
A_2 & 0 & 0 & -B_2
\end{bmatrix}
\]  

(4.8)

can be obtained from (4.4) by performing a shuffle. If

\[
\text{rank} \begin{bmatrix} E_1 \\ A_2 \end{bmatrix} = r
\]  

(4.9)

then performing elementary row operations on (4.8) (or equivalently on (4.7)) we obtain

\[
\begin{bmatrix} E_2 & A_3 & B_{10} & B_{11} \\
0 & A_4 & B_{20} & B_{21}
\end{bmatrix}
\]  

(4.10)

and

\[
\begin{align*}
E_2 y_{i+1} & = A_{31} x_i + B_{10} u_i + B_{11} u_{i+1} \\
0 & = A_{41} x_i + B_{20} u_i + B_{21} u_{i+1}
\end{align*}
\]  

(4.11a)

(4.11b)

where \( E_2 \in \mathbb{R}^{r \times r} \) has full row rank. If \( \text{rank} E_2 = r \) then there exists a nonsingular matrix \( P \in \mathbb{R}^{r \times r} \) of elementary column operations such that

\[
E_2 P = [\begin{bmatrix} \bar{E}_2 & 0 \end{bmatrix}, \bar{E}_2 \in \mathbb{R}^{r \times r}, 0 \in \mathbb{R}^{r \times (n-r)}, \det \bar{E}_2 \neq 0].
\]  

(4.12)

Defining the new state vector

\[
\bar{x}_i = P^{-1} x_i = \begin{bmatrix} \bar{x}_{1j} \\ \bar{x}_{2j} \end{bmatrix}, \quad \bar{x}_{1j} \in \mathbb{R}^r, \quad \bar{x}_{2j} \in \mathbb{R}^{r-r}
\]  

(4.13)

from (4.11a) we obtain

\[
E_2 PP^{-1} x_{i+1} = A_{31} \bar{x}_{1j} + B_{10} u_i + B_{11} u_{i+1}
\]  

(4.14)

and

\[
\bar{E}_2 \bar{x}_{i+1} = A_{32} \bar{x}_{2j} + B_{10} u_i + B_{11} u_{i+1}
\]  

(4.15)

where

\[
A_{31} P = [A_{31} \quad A_{32}], \quad A_{11} \in \mathbb{R}^{r \times r}, \quad A_{32} \in \mathbb{R}^{r \times (n-r)}.
\]  

(4.16)

Premultiplying (4.15) by the matrix \( \bar{E}_2^{-1} \) we obtain
\[
\bar{x}_{i,j+1} = \frac{1}{E} A_{31} \bar{x}_{i,j} + \frac{1}{E} A_{32} \bar{x}_{i,j} + \frac{B_1 B_2}{E} u_i + \frac{B_2}{E} u_{i+1}
\tag{4.17a}
\]

where \[
\frac{1}{E} A_{31} = E^{-2} A_{31}, \quad \frac{1}{E} A_{32} = E^{-2} A_{32}, \quad \frac{B_1 B_2}{E} = E^{-2} B_1, \quad \frac{B_2}{E} = E^{-2} B_2.
\tag{4.17b}
\]

From (4.11) and (4.13) we obtain
\[
0 = A_{4i} P P^{-1} x_i + B_{2i} u_i + B_{2i} u_{i+1} = A_{4i} \bar{x}_{i,j} + A_{42} \bar{x}_{j,j} + B_{2i} u_i + B_{2i} u_{i+1}
\tag{4.18a}
\]

where
\[
A_{4i} P = [A_{4i} \quad A_{42}], \quad A_{4i} \in \mathbb{R}^{r \times r}, \quad A_{42} \in \mathbb{R}^{r \times r \times (n-r)}.
\tag{4.18b}
\]

If \(q = n \) and \( \det A_{42} \neq 0 \) then from (4.18a) we obtain
\[
\bar{x}_{i,j} = \frac{1}{A_{4i}} \bar{x}_{i,j} - \frac{1}{B_{2i}} u_i - \frac{1}{B_{2i}} u_{i+1}
\tag{4.19a}
\]

where
\[
\frac{1}{A_{4i}} = A_{4i}^{-1} A_{4i}, \quad \frac{1}{B_{2i}} = A_{42} B_{20}, \quad \frac{1}{B_{2i}} = A_{42} B_{21}.
\tag{4.19b}
\]

Substitution of (3.19a) into (3.17a) yields
\[
\bar{x}_{i,j+1} = \bar{A} \bar{x}_{i,j} + \bar{B}_0 u_i + \bar{B}_1 u_{i+1}
\tag{4.20a}
\]

where
\[
\bar{A} = \bar{A} - \bar{A}_{32} \bar{A}_{41}, \quad \bar{B}_0 = \bar{B}_0 - \bar{A}_{32} \bar{B}_{20}, \quad \bar{B}_1 = \bar{B}_1 - \bar{A}_{32} \bar{B}_{21}.
\tag{4.20b}
\]

The solution of the equation (4.20a) for given consistent initial condition \( \bar{x}_{i0} \) and admissible input \( u_i \) has the form
\[
\bar{x}_{i,j} = \bar{A}\bar{x}_{i0} + \sum_{k=0}^{i-1} \bar{A}^{-k-1} (\hat{B}_0 u_k + \hat{B}_1 u_{k+1}), \quad i \in Z_+.
\tag{4.21}
\]

Knowing (4.21) from (4.19a) we can find \( \bar{x}_{i,j} \). If \( q \neq n \) or \( \det A_{42} = 0 \) for \( q = n \) then we choose \( \bar{x}_{i,j} \) so that (4.18a) holds for given consistent initial conditions and admissible input \( u_i \) and from (4.17a) we have
\[
\bar{x}_{i,j} = \bar{A}\bar{x}_{i0} + \sum_{k=0}^{i-1} \bar{A}^{-k-1}_{31} (\bar{A}_{32} \bar{x}_{2,k} + \bar{B}_0 u_k + \bar{B}_1 u_{k+1}), \quad i \in Z_+.
\tag{4.22}
\]

If rank \( E < r \) then we repeat the procedure for array (4.8) and by Lemma 2.1 after \( \mu \) steps (shuffles) we obtain
\[
E_{\mu+1} x_{i+1} = A_{\mu+1} x_i + B_{10} u_i + B_{11} u_{i+1} + \ldots + B_{1,\mu} u_{i+\mu}
\tag{4.23a}
\]
\[
0 = A_{\mu+1} x_i + B_{20} u_i + B_{21} u_{i+1} + \ldots + B_{2,\mu} u_{i+\mu}
\tag{4.23b}
\]
where $E_{\mu+1} \in \mathbb{R}^{r \times r}$ has full row rank. If rank $E_{\mu+1} = r$ then there exists a nonsingular matrix $P \in \mathbb{R}^{r \times r}$ of elementary column operations such that

$$E_{\mu+1} P = [E_{\mu+1} \ 0], \ E_{\mu+1} \in \mathbb{R}^{r \times r}, \ 0 \in \mathbb{R}^{r \times (n-r)}, \ \det E_{\mu+1} \neq 0.$$  \hspace{1cm} (4.24)

Defining the new state vector (4.13) from (4.23a) we obtain

$$\bar{x}_{i,j+1} = A_{\mu+2,1} x_{i,j} + A_{\mu+2,2} x_{i,j} + B_{10} u_i + B_{11} u_{i+1} + ... + B_{1,\mu} u_{i+\mu}$$  \hspace{1cm} (4.25)

where

$$A_{\mu+1} P = [A_{\mu+2,1} \ A_{\mu+2,2}], \ A_{\mu+2,1} \in \mathbb{R}^{r \times r}, \ A_{\mu+2,2} \in \mathbb{R}^{r \times (n-r)}.$$  \hspace{1cm} (4.26)

Premultiplying (4.25) by the matrix $E_{\mu+1}^{-1}$ we obtain

$$\bar{x}_{i,j+1} = \bar{A}_{\mu+2,1} x_{i,j} + \bar{A}_{\mu+2,2} x_{i,j} + \bar{B}_{10} u_i + \bar{B}_{11} u_{i+1} + ... + \bar{B}_{1,\mu} u_{i+\mu}, \ i \in Z_+$$  \hspace{1cm} (4.27a)

where

$$\bar{A}_{\mu+2,1} = E_{\mu+1}^{-1} A_{\mu+2,1}, \ \bar{A}_{\mu+2,2} = E_{\mu+1}^{-1} A_{\mu+2,2}, \ \bar{B}_{1,k} = E_{\mu+1}^{-1} B_{1,k}, \ k = 0,1, ..., \mu.$$  \hspace{1cm} (4.27b)

From (4.23a), (4.13) and (4.24) we obtain

$$0 = A_{\mu+3,3} x_{i,j} + A_{\mu+3,2} x_{i,j} + B_{20} u_i + B_{21} u_{i+1} + ... + B_{2,\mu} u_{i+\mu}$$  \hspace{1cm} (4.28a)

where

$$A_{\mu+3} P = [A_{\mu+3,1} \ A_{\mu+3,2}], \ A_{\mu+3,1} \in \mathbb{R}^{r \times r}, \ A_{\mu+3,2} \in \mathbb{R}^{r \times (n-r)}.$$  \hspace{1cm} (4.28b)

If $q = n$ and $\det A_{\mu+3,2} \neq 0$ then from (4.28a) we obtain

$$x_{i,j+1} = -A_{\mu+3,3} x_{i,j} - B_{20} u_i - B_{21} u_{i+1} - ... - B_{2,\mu} u_{i+\mu}, \ i \in Z_+$$  \hspace{1cm} (4.29a)

where

$$A_{\mu+3,1} = A_{\mu+3,2}^{-1} A_{\mu+3,1}, \ \bar{B}_{2,k} = A_{\mu+3,2}^{-1} B_{2,k}, \ k = 0,1, ..., \mu.$$  \hspace{1cm} (4.29b)

Substitution of (4.29a) into (4.27a) yields

$$x_{i,j+1} = \bar{A}_i x_{i,j} + \hat{B}_{i0} u_i + \hat{B}_{i1} u_{i+1} + ... + \hat{B}_{i,\mu} u_{i+\mu}, \ i \in Z_+$$  \hspace{1cm} (4.30a)

where

$$\bar{A}_i = \bar{A}_{\mu+2,1} - A_{\mu+3,2} \bar{A}_{\mu+3,1}, \ \hat{B}_{i,k} = \bar{B}_{i,k} - A_{\mu+3,2} \bar{B}_{2,k}, \ k = 0,1, ..., \mu.$$  \hspace{1cm} (4.30b)

The solution of the equation (4.30a) for a given consistent initial condition $x_{10}$ and admissible input $u_i$ is given by

$$x_{i,j} = \bar{A}_i x_{i,j} + \sum_{k=0}^{i-1} \bar{A}_i^{-k-1} (\hat{B}_{i0} u_k + \hat{B}_{i1} u_{k+1} + ... + \hat{B}_{i,\mu} u_{k+\mu}), \ i \in Z_+.$$  \hspace{1cm} (4.31)
Knowing \( \bar{x}_{i,j} \) from (4.29a) we can find \( \bar{x}_{2,j} \) and from (4.13) we have

\[
\bar{x}_i = P \left[ \begin{array}{c} \bar{x}_{i,j} \\ \bar{x}_{2,j} \end{array} \right], \; i \in \mathbf{Z}_+. \tag{4.32}
\]

If \( q \neq n \) or \( \det A_{\mu+3,2} = 0 \) for \( q = n \) then we choose \( \bar{x}_{2,j} \) so that (4.28a) holds for given consistent initial conditions and admissible input \( u_i \) and from (4.27a) we have

\[
\bar{x}_{i,j} = \bar{A}_{\mu+2,2} \bar{x}_{i,0} + \sum_{k=0}^{j-1} \bar{A}_{\mu+2,2} (A_{\mu+2,2} \bar{x}_{k,j} + B_{i_0} u_{i_0} + B_{i_1} u_{i_1} + \ldots + B_{i_{\mu+n}} u_{i_{\mu+n}}), \; i \in \mathbf{Z}_+. \tag{4.33}
\]

Therefore, the following theorem has been proved.

**Theorem 4.1.** If \( q = n \) and \( \det A_{\mu+3,2} \neq 0 \) for \( q = n \) then the equation (4.1) with singular pencil satisfying (4.2) can be reduced by the use of shuffle algorithm to the equation (4.30a). The solution \( \bar{x}_{i,j} \) of the equation (4.30a) for a given consistent initial conditions and admissible input \( u_i \) is given by (4.31). If \( q \neq n \) or \( \det A_{\mu+3,2} = 0 \) for \( q = n \) then \( \bar{x}_{2,j} \) is chosen so that the equality (4.28a) is satisfied for given consistent initial conditions and admissible input \( u_i \). The solution \( \bar{x}_{i,j} \) of the equation (4.27a) is given by (4.33).

**Remark 4.1.** In particular case when \( q > n \), \( n = r \) and \( A_{\mu+3} = 0 \in \mathbb{R}^{(q-n)\times n} \), \( B_{2,k} = 0 \in \mathbb{R}^{(q-n)\times m} \) for \( k = 0,1,\ldots,\mu \); then from (4.23a) we have

\[
x_{i+1} = E^{-1} A_{\mu+3} x_i + E^{-1} B_{i} u_i + E^{-1} B_{i_1} u_{i_1} + \ldots + E^{-1} B_{i_{\mu+n}} u_{i_{\mu+n}}. \tag{4.34}
\]

**Example 4.1.** Consider the equation (4.1) with the matrices

\[
E = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \; A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.35}
\]

In this case \( q = 3 \), \( n = 2 \), \( m = 1 \), \( \text{rank } E = 1 \) and

\[
\text{rank } [Ez - A] = \text{rank } \begin{bmatrix} -2 & z \\ 0 & -1 \\ 1 & -z \end{bmatrix} = 2 \text{ for some } z \in \mathbb{C}. \tag{4.36}
\]

Performing on the array

\[
\begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 0 & -1 & -1 & 0 & 0 \end{bmatrix}
\]

the elementary row operation \( L[3+1x1] \) we obtain
and after a shuffle

\[
E_1 A_1 B_1 = \begin{bmatrix}
0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 1 & -1
\end{bmatrix}
\]

Performing on the array (4.39) the elementary row operations \( L[2+1x(-1)] , \ L[2,3] \) we get

\[
E_2 A_3 B_1 = \begin{bmatrix}
0 & 1 & 2 & 0 & 1 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

From (4.40) we have

\[
0 = [-2 \ 0]x_i - u_j + u_{j+1}
\]  \hspace{1cm} (4.41a)

\[
x_{i+1} = \begin{bmatrix}
0 & 0 \\
2 & 0
\end{bmatrix} x_i + \begin{bmatrix}
0 \\
1
\end{bmatrix} u_i + \begin{bmatrix}
0 \\
-1
\end{bmatrix} u_{i+1}
\]  \hspace{1cm} (4.41b)

From (4.41) it follows that

\[
x_{i+1} = \begin{bmatrix}
0 & 0 \\
2 & 0
\end{bmatrix} x_i + \begin{bmatrix}
0 \\
1
\end{bmatrix} u_i + \begin{bmatrix}
0 \\
-1
\end{bmatrix} u_{i+1}
\]  \hspace{1cm} (4.42a)

and

\[
x_{i,j} = -0.5u_i + 0.5u_{i+1} .
\]  \hspace{1cm} (4.42b)

The equation (4.42) has the solution

\[
x_j = \begin{bmatrix}
0 & 0 \\
2 & 0
\end{bmatrix} x_0 + \sum_{k=0}^{j-1} \begin{bmatrix}
0 & 0 \\
2 & 0
\end{bmatrix} \begin{bmatrix}
0 \\
1
\end{bmatrix} u_k + \begin{bmatrix}
0 \\
-1
\end{bmatrix} u_{k+1}
\]  \hspace{1cm} (4.42a)

for the consistent initial conditions \( x_{i0} = -0.5u_0 + 0.5u_i \), arbitrary \( x_{20} \) and admissible input satisfying (4.42b).

\[ \text{Example 4.2.} \] Consider the equation (4.1) with the matrices

\[
E = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} , \ A = \begin{bmatrix}
1 & 0 & 1 \\
0 & -2 & 1
\end{bmatrix} , \ B = \begin{bmatrix}
1 \\
-1
\end{bmatrix}.
\]  \hspace{1cm} (4.43)

In this case \( q = 2, \ n = 3, \ m = 1, \ \text{rank} \ E = 2 \) and

\[
\text{rank} [ Ez - A ] = \text{rank} \begin{bmatrix}
-1 & z & -1 \\
-1 & z & -1
\end{bmatrix} = 2 \text{ for some } z \in C .
\]  \hspace{1cm} (4.44)

For the matrix
\[
P = \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix}
\]  
we have

\[
EP = \begin{bmatrix}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix},
\]  

\[
AP = \begin{bmatrix}
1 & 0 & 1 \\
0 & -2 & 1 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 0 \\
0 & 0 & -1
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
-2 & 0 & -1
\end{bmatrix}
\]  
and

\[
\bar{x}_i = \begin{bmatrix}
\bar{x}_{1,i} \\
\bar{x}_{2,i} \\
\bar{x}_{3,i}
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & -1
\end{bmatrix} \begin{bmatrix}
x_{1,i} \\
x_{2,j} \\
x_{3,j}
\end{bmatrix} = \begin{bmatrix}
x_{1,i} \\
x_{2,i} + x_{3,i} \\
-x_{3,i}
\end{bmatrix}.
\]  

From (4.1) with (4.43) and (4.46) we have

\[
\begin{bmatrix}
\bar{x}_{1,i} \\
\bar{x}_{2,i}
\end{bmatrix} = \begin{bmatrix}
0 & 1 \\
-2 & 0
\end{bmatrix} \begin{bmatrix}
x_{1,i} \\
x_{2,i}
\end{bmatrix} + \begin{bmatrix}
0 \\
-1
\end{bmatrix} x_{3,i} + \begin{bmatrix}
1 \\
-1
\end{bmatrix} u_i.
\]  

Note that the equation (4.47) has the solution for any given \(\bar{x}_{3,i} = -x_{3,i}, \ i \in Z_+,\) arbitrary initial condition \(\bar{x}_{10} = x_{10}, \ \bar{x}_{20} = x_{20} + x_{30}\) and arbitrary input \(u_i, \ i \in Z_+\).

5. CONCLUDING REMARKS

The shuffle algorithm has been applied to analysis of descriptor continuous-time and discrete-time linear systems with singular pencils. It has been shown that if \(q = n\) and \(\det A_{\mu+3,2} \neq 0\) then the equation (3.1) with singular pencil satisfying (3.2) can be reduced by the use of the shuffle algorithm to the equation (3.30a) (Theorem 3.1). If \(q \neq n\) or \(\det A_{\mu+3,2} = 0\) for \(q = n\) then \(\bar{x}(t)\) is chosen so that (3.28a) is satisfied for given consistent initial conditions and admissible inputs and the solution \(\bar{x}(t)\) of (3.27a) is given by (3.33). Similar results have been obtained for descriptor discrete-time linear systems with singular pencils (Theorem 4.1). the considerations can be extended to positive descriptor linear systems with singular pencils. An open problem is an extension of these considerations for fractional descriptor linear systems [13].

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ZASTOSOWANIE ALGORYTMU PRZESUWANIA DO ANALIZY DESKRYPTOWYCH UKŁADÓW Z PĘKIEM SYNGULARNYM

Streszczenie
Podano nową metodę analizy deskryptowych liniowych układów ciągłych i dyskretnych z pękami singularnymi. Proponowana metoda jest oparta na redukcji deskryptowych układów o pękach singularnych do równoważnych układów standardowych za pomocą algorytmu przesuswania. Podano warunki istnienia tej redukcji oraz ilustrowano proponowaną metodę przykładem numerycznym.

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