On the study of ion cyclotron waves in a cylindrical magnetized plasma

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Abstract. In this work, a general dispersion relation of waves in the region of ion cyclotron frequency in the cylindrical magnetized plasma is derived. The waves are assumed to be cylindrically symmetric oscillations of small amplitude. Analytical calculations are performed to find the plasma dielectric tensor for the plasma consisting of hot electron and multi-component cold ions fluid. The special case of a three component plasma with hot electrons in a strong magnetic field may be interesting, e.g., in the context of fusion plasma containing D⁺, T⁺ and He²⁺. The general dispersion relation is simplified in two solutions. Firstly, \( E_1 \) wave (\( E_2 = 0 \)) which has an electrostatic character, and secondly \( E_2 \) wave (\( E_1 = 0 \)) which has an electromagnetic character. The dispersion relations for both waves are described and identified as the ion acoustic and electrostatic ion cyclotron (EIC) waves for \( E_1 \) wave and the torsional Alfvén, i.e. ion cyclotron (IC) waves and the compressional Alfvén wave for \( E_2 \) wave. These waves are studied due to their importance in the heating of plasmas.

Key words: electrostatic ion cyclotron (EIC) waves • cylindrical magnetized plasma • ion acoustic wave • torsional and compressional Alfvén waves

Introduction

The EIC mode [10, 14, 15, 27] is one of the frequency eigenmodes of a magnetized plasma. These waves are studied due to their importance in the heating of plasmas [34]. Jehan et al. [12] have investigated the nonlinear coupled ion-acoustic and ion-cyclotron waves propagating obliquely to the external magnetic field in dense collisionless electron-positron-ion magnetoplasma using the Sagdeev potential method [2, 20]. Kaneko et al. [13] have modified EIC instabilities by the parallel and perpendicular plasma flow velocity shears. Their experiments have demonstrated that the ion-cyclotron instabilities are suppressed by the perpendicular flow velocity shear.

An instability in the ion-cyclotron range of frequencies, plays an important role in heating of ions [29]. Plasma cross-field diffusion [19, 25], and anomalous resistivity in space plasmas [35], have been investigated for the case of the inhomogeneous energy density driven (IEDD) [3] instability and it is different from the conventional ion-cyclotron instability [21, 22]. Shi et al. [28] have shown that the electrostatic density shock and its corresponding solitary electric field structure can be
developed from an ion acoustic wave or an ion cyclotron wave if the Mach number and the initial electric field satisfy some conditions. Agrimson et al. [1] have studied the effect of parallel velocity shear on the EIC instability in filamentary current channels. Koepke et al. [18] have investigated space-relevant studies of ion acoustic and ion cyclotron waves.

In the condition of \([\left[\Omega / \nu_\ast \right]^2 < 1]\), where \(\Omega\) is the ion plasma frequency, the ions in plasma can be directly heated at the frequency equal to ion cyclotron frequency. For the plasma with \([\left[\Omega / \nu_\ast \right]^2 > 1]\), however, the heating of plasma becomes less efficient, since the only ions which interact with the exciting radio-frequency (RF) field are those in the surface layer of the plasma column, due to the strong skin effect [8, 9]. To avoid this undesirable effect, the rf energy is firstly poured and stored in plasma as the wave energy and then it is converted into the ion energy by means of the ion cyclotron damping. The EIC wave is well suited to the excitation of the EIC wave in a longitudinal magnetic field.

The purpose of this paper is to study the EIC wave in a cylindrical magnetized plasma. The general dispersion equation of waves near the ion cyclotron frequency is derived and simplified in two solutions which have electrostatic and electromagnetic characters.

**Mathematical model**

**The general dispersion relation**

It is assumed that a plasma cylinder is infinitely long, surrounded by a vacuum and immersed in a uniform magnetic field. \(B_\ast\) and the behavior of plasma is a subject to the following conditions:

1. Frequencies concerned here are considerably less than the electron cyclotron frequency, \(\Omega_e\), since we are interested in the region of ion cyclotron frequency, \(\omega < \Omega_i\).
2. Collision frequency \(\nu\) is considered negligibly small, \(\omega >> \nu\).
3. The Larmor radii of the particles are small compared with the radial scale of the plasma.
4. The thermal electron velocity in the axial direction is larger than the phase velocities of waves considered, and the other pressures are neglected.

Thus, let us consider cylindrically symmetric oscillations of small amplitude. A list of symbols used in this paper is given (see list of symbols and definitions) together with their definitions. With these assumptions the equation of motion for the charged particle is:

\[
\frac{d^2 \mathbf{v}_i}{dt^2} = \mathbf{e} (\hat{\mathbf{E}} + \frac{\mathbf{v}_i}{c} \times \mathbf{B}_h) \tag{1}
\]

and for the hot electron fluid the basic equations are

\[
\rho_e \frac{d\mathbf{v}_e}{dt} = -\mathbf{n}_e \mathbf{E} - \frac{d\mathbf{p}_{ei}}{dt} \tag{2}
\]

where \(\mathbf{p}_{ei} = n_e \kappa T_e\)

with

\[
\frac{\partial n_e}{\partial t} + \mathbf{v}_e \cdot \nabla n_e = 0 \tag{3}
\]

\[
p_{ei} = n_e \kappa T_e \tag{4}
\]

It is assumed that \(\mathbf{B}_h\) has the z-direction and the symbol “parallel” (in \(p_{ei}\) and \(T_e\)) refers to the direction of the magnetic field. Because of the assumption (4), let us neglect the inertia term on the left hand side of Eq. (2).

From Eqs. (1) to (4), the dielectric tensor \(\tilde{K}\) for the plasma consisting of electron and multi-component ions is obtained as follows:

\[
\tilde{K} = \begin{bmatrix}
S & -iD & 0 \\
D & S & 0 \\
0 & 0 & P
\end{bmatrix}
\]

where:

\[
R = 1 + \frac{\Omega_i^2}{\Omega_e^2} + \frac{\Omega_e^2}{\Omega_i^2} - \frac{\sum_j a_j}{\Omega_i^2 (1 + \lambda_j \Omega_i)} \tag{6}
\]

\[
L = 1 + \frac{\Omega_i^2}{\Omega_e^2} - \frac{\Omega_e^2}{\Omega_i^2} - \frac{\sum_j a_j}{\Omega_i^2 (1 - \lambda_j \Omega_i)} \tag{7}
\]

\[
S = \frac{1}{2} (R + L) = 1 - \frac{\Omega_i^2}{\Omega_e^2} + \frac{\Omega_e^2}{\Omega_i^2} - \frac{\sum_j \lambda_j a_j}{1 - \lambda_j^2 \Omega_i^2} \tag{8}
\]

\[
D = \frac{1}{2} (R - L) = 1 - \frac{\Omega_i^2}{\Omega_e^2} - \frac{\Omega_e^2}{\Omega_i^2} - \frac{\sum_j \lambda_j a_j}{1 - \lambda_j^2 \Omega_i^2} \tag{9}
\]

\[
P = 1 - \frac{\Omega_i^2}{k_e^2 v_{th}^2} + \frac{\Omega_i^2}{\Omega_e \Omega_i} \sum_j \frac{a_j}{\lambda_j} \tag{10}
\]

where:

\[
\Omega_i^2 = \frac{4 \pi n_i z_i^2 e^2}{m_i}, \quad \Omega_e = \frac{z_e B_0}{m_e c}, \quad \lambda_j = \frac{\Omega_i}{\Omega_j},
\]

\[
a_j = \frac{z_j n_j}{n_e}, \quad v_{th} = \frac{kT_{ei}}{m_e}, \quad \Omega = \frac{\omega}{\Omega_i}
\]

and \(\Omega_i\) is the cyclotron frequency of the first ion. Eqs. (6)–(10) are in agreement with that obtained by Sitenko and Malnev [30] for electron and hydrogen ion only in the region of ion cyclotron frequency.

The dispersion relation can be determined by solving Maxwell’s equations according to Chen [6] in the form:

\[
\nabla \times (\nabla \times \mathbf{E}) + \frac{\omega^2}{c^2} \mathbf{K} \cdot \mathbf{E} = 0 \tag{11}
\]

Substituting the components of dielectric tensor Eqs. (8)–(10) into Eq. (11), give:

\[
\frac{k_e^2 E_i + \frac{\partial E_i}{\partial r} = \frac{\omega^2}{c^2} S E_i - \frac{\omega^2}{c^2} D E_i} \tag{12}
\]

\[
k_e^2 E_i - \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} (r E_i) \right) = i \frac{\omega^2}{c^2} D E_i + \frac{\omega^2}{c^2} S E_i \tag{13}
\]
After several transformations of these equations, the Bessel equation can be obtained for the function $E_n = E_1 + \mu_n E_0$, in the form:

$$\frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} (r E_r) \right) + k_n^2 E_r = 0$$

where:

$$k_n^2 = \left( \frac{\omega}{c} \right)^2 P(S - N^2_i) + iD(\mu_1 S + iD) = \frac{\omega^2}{c^2} \frac{P(S - N^2_i) + iD(\mu_1 S + iD)}{S}$$

The values $\mu_n = \mu_1, \mu_2$ are determined by the equation:

$$\mu_2^2 + \frac{i}{D} \mu_1 - \frac{D^2 + (N^2 - S)(P - S)}{S} \mu_1^2 + \frac{P - S}{S} = 0$$

The choice of one of the two values of $\mu_1$ determines the polarization of the waves. The general dispersion relation which leaves the boundary conditions out of consideration can be obtained by substitution $\mu_2$ from Eq. (18) into Eq. (16) in the following form:

$$k_n^2 c^2 \omega^2 = -\frac{1}{2} \left[ \frac{1}{S} \left( (P + S)(N^2_i - S) + D^2 \right) \right]$$

Ion acoustic and EIC waves

Firstly, $E_1$ wave ($E_2 = 0$), which corresponds to the upper sign in Eq. (19) is considered. This wave has an electrostatic character. Let us expand in powers of $1/P$ as $P$ is very large in the frequency range $\omega \ll \Omega$. Then:

$$k_n^2 c^2 \omega^2 \approx \frac{P}{S} (S - N^2_i) + \frac{D^2 N^2_i}{S}$$

Eq. (21) is the same as the dispersion relation for the electrostatic approximation:

$$k_n^2 K_n + 2k_n K_n + k_n^2 N_n = 0$$

obtained by Stix [32]. For the case of $k_n \approx 0$, Eq. (21) can be simplified in the form:

$$\frac{1}{\omega^2} = \left( \frac{m_1}{k_n^2 Z_i kT_w} \right)^2 \left( a_1 + \frac{a_2}{\lambda_2} + \frac{a_3}{\lambda_3} \right)^2$$

which is the dispersion relation of the ion acoustic wave [32] propagating in the direction of the magnetic field. Let us assume further that

$$\frac{\Pi_2^2}{\Omega_i \Omega_r \Omega} \left( a_1 + \frac{a_2}{\lambda_2} + \frac{a_3}{\lambda_3} \right) >> 1$$

and that the direction of propagation is nearly perpendicular to the magnetic field then, the dispersion relation becomes:

$$\frac{1}{\omega^2} = \left( \frac{m_1}{k_n^2 Z_i kT_w} \right)^2 \left( a_1 + \frac{a_2}{\lambda_2} + \frac{a_3}{\lambda_3} \right) + \frac{1}{k_n^2} \left( \frac{m_1}{Z_i kT_w} \right) = 0$$

This wave is the EIC wave [5]. In the case of oblique propagation, Eq. (21) shows that the propagation of EIC waves becomes possible at frequencies above the ion cyclotron frequency corresponding to various ion species and the ion acoustic wave [17, 26] are separated from each other by gaps, at which Eq. (21) can no longer be satisfied with real $k_n$ in the frequency spectrum.

It is impossible to make the axial wave number $k_n$ zero since a plasma in a laboratory device has a boundary and the characteristic length of exciting system of wave is also finite. Consequently, the dispersion relation Eq. (21) for the EIC wave must be used for finite $k_n$ instead of Eq. (25).

Torsional and compressional Alfvén waves

Secondly, $E_2$ wave ($E_1 = 0$), which corresponds to the lower sign in Eq. (19) is considered, and expanding in terms of $1/P$ in the frequency range $\omega \ll \Omega$, gives:

$$k_n^2 c^2 \omega^2 = \frac{D^2 (N^2_i - S)^2}{(N^2_i - S)}$$

The $E_2$ wave ($E_1 = 0$), has an electromagnetic character. Substituting $S$ and $D$ for plasma containing two types of positive ions and electrons into Eq. (26) and using the next approximations: $\omega << \Omega$, and $\left( [\Pi_1 + \Pi_2^2] / \Omega_i \Omega_r \right) >> 1$, the following dispersion relation can be obtained in the form:
where $\eta = m_1/m_e$. The plus sign in Eq. (27) corresponds to the torsional Alfvén (IC) wave [16, 24], which has a resonance at $\omega = \Omega_i$, and which can propagate only if $\omega < \Omega_i$. The minus sign gives the fast hydromagnetic mode which corresponds to the compressional Alfvén wave [7] at low frequencies.

**Conclusion**

It is evident that the resonance frequency in a plasma with ion density of $10^8$ to $10^{10}$ cm$^{-3}$ is always higher than the ion cyclotron frequency and its integer multiples. In this paper, the behavior of resonances is explained by the dispersion relation of the EIC wave.

The general dispersion Eq. (19) is derived for an infinitely long infinity plasma cylinder, surrounded by a vacuum and immersed in a uniform magnetic field, $B_0$. From Eq. (19), the following two waves can be found as:

(A) $E_1$ wave ($E_2 = 0$), which corresponds to the upper sign in Eq. (19), and

(B) $E_2$ wave ($E_1 = 0$), which corresponds to the lower sign in Eq. (19).

(A) For $E_1$ wave ($E_2 = 0$), which has an electrostatic character, we derive the following dispersion equations: (A.1) Eq. (23) of the ion acoustic wave propagating in the direction of magnetic field, and (A.2) Eq. (25) of the EIC wave. In the case of oblique propagation, Eq. (21) shows that the propagation of EIC becomes possible at frequencies above the ion cyclotron frequency corresponding to various ion species and the ion acoustic waves are separated from each other by gaps, at which Eq. (21) can no longer be satisfied with real $k$ in the frequency spectrum.

(B) For $E_2$ wave ($E_1 = 0$), which has an electromagnetic character, the dispersion Eqs. (27) are derived. The following dispersion equations can be obtained:

(B.1) the plus sign in Eq. (27) corresponds to the torsional Alfvén (IC) wave, which has a resonance at $\omega = \Omega_i$, and which can propagate only if $\omega < \Omega_i$, and

(B.2) the minus sign gives the fast hydromagnetic mode which corresponds to the compressional Alfvén wave at low frequencies. These waves are studied due to their importance in the heating of plasmas.

The dispersion relation Eq. (21) for the EIC wave must be used for the finite $k_z$ instead of Eq. (25). The dispersion relation of EIC wave, Eq. (21) is calculated numerically under the experimental conditions [23]. The dispersion relation was calculated from Eq. (21) for $k_z = 0.2095$, $T_e = 5 \times 10$ deg. K for H$^+$. The special case of a three component plasma with hot electrons in a strong magnetic field may be interesting, e.g., in the context of fusion plasma [4, 11] containing D$^+$, T$^+$ and He$^{2+}$.

**List of symbols and definitions**

$B$ – magnetic field,

$B_0$ – static magnetic field,
\( D = \frac{1}{2}(R - L) \)

\( E \) – electric field,  
\( E_0 = E + \nu_k E_o \) – electric charge,  
\( I_n \) – \( n \)-th order modified Bessel function of the 1st kind,  
\( J_n \) – \( n \)-th order Bessel function of the 1st kind,  
\( J' \) – surface current density,  
\( K \) – dielectric tensor,  
\( K_n \) – \( n \)-th order modified Bessel function of the 2nd kind,  
\( k \) – wave number vector,  
\( L \) – Eq. (7),  
\( m_i \) – mass of the \( k \)-th ion,  
\( N_i = k_c c/\Omega_i \),  
\( N_{1} = k_c c/\Omega_{1} \),  
\( n_i \) – number of the \( k \)-th particle per unit volume,  
\( P \) – Eq. (10),  
\( p_k \) – pressure of the \( k \)-th particle,  
\( R \) – Eq. (6),  
\( S = \frac{1}{2}(R + L) \),  
\( T_k \) – temperature of the \( k \)-th particle,  
\( T_{\perp} \) – parallel temperature,  
\( T_{\perp} \) – perpendicular temperature,  
\( v_k \) – axial drift velocity,  
\( v_{\text{Th}} \) – thermal velocity of the \( k \)-th particle,  
\( \lambda \) – wave length,  
\( \nu_k \) – velocity of the \( k \)-th particle,  
\( \alpha = \frac{Z_j n_j}{n_s} \),  
\( \eta = \frac{\Omega_i}{\Omega_{1}} \),  
\( \kappa \) – Boltzmann’s constant,  
\( \varepsilon_k \) – sign of charge, \( \pm 1 \), for the \( k \)-th particle,  
\( \nu \) – collision frequency,  
\( z_k \) – charge of the \( k \)-th ion, in units of the proton charge,  
\( \lambda_{\gamma} = \frac{\Omega_i}{\Omega_{1}} \),  
\( k_{x}^{2} = \frac{\mu_{e}^{2}}{\omega_{e}^{2}} m_{e} \),  
\( \mu_e \) – Eq. (18),  
\( \Omega = \omega/\Omega_{e} \),  
\( \Omega_{1} \) – plasma frequency of the \( k \)-th particle,  
\( \Omega_{0} = \omega/\Omega_{1} \),  
\( \Omega_{1} \) – cyclotron frequency of the first ion,  
\( \Omega_{k} \) – cyclotron frequency of the \( k \)-th ion.

References

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