Approximation of positive stable continuous-time linear systems by positive stable discrete-time systems

Tadeusz Kaczorek
Faculty of Electrical Engineering, Białystok University of Technology, Poland

Abstract: The positive asymptotically stable continuous-time linear systems are approximated by positive asymptotically stable discrete-time linear systems by the use of Padé type approximation. It is shown that the approximation preserves the positivity and asymptotic stability of the systems. The stabilization problem of positive unstable continuous-time and corresponding discrete-time linear systems by state-feedbacks is also addressed.

Keywords: approximation, continuous-time, discrete-time, linear positive system, stability

1. Introduction

In positive systems inputs, state variables and outputs take only non-negative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear behavior can be found in engineering, management science, economics, social sciences, biology and medicine, etc. Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of state of the art in positive systems theory is given in the monographs [5, 7].

Stability of positive linear systems has been investigated in [5, 7] and of fractional linear systems in [2-4, 10]. The problem of preservation of positivity by approximation the continuous-time linear systems by corresponding discrete-time linear systems has been addressed in [8].

In this paper it will be shown that using Padé type approximation of the exponential matrix the positive asymptotically stable continuous-time linear systems can be approximated by corresponding positive asymptotically stable discrete-time linear systems.

The paper is organized as follows. In section 2 basic definitions and theorems concerning positive continuous-time and discrete-time linear systems are recalled. The positivity of the linear systems are considered in section 3 and the asymptotic stability of the systems in section 4. The stabilization problem by state-feedbacks of the positive systems is addressed in section 5. Concluding remarks are given in section 6.

The following notation will be used: \( \mathbb{R} \) - the set of real numbers, \( \mathbb{R}^{m \times n}_{\text{non-negative}} \) - the set of real matrices with nonnegative entries and \( \mathbb{R}^+ = \mathbb{R}^{m \times m}_{\text{non-negative}} \). \( M_\infty \) - the set of Metzler matrices (real matrices with nonnegative off-diagonal entries), \( M_m \) - the set of asymptotically stable Metzler matrices, \( \mathbb{R}^{m \times m}_{\text{asymptotically stable}} \) - the set of asymptotically stable positive matrices, \( I_n \) - the identity matrix.

2. Preliminaries and the problem formulation

Consider the continuous-time linear system

\[
\dot{x}(t) = A_c x(t) + B_c u(t), \quad x(0) = x_0 \quad (2.1)
\]

where \( x(t) \in \mathbb{R}^n \), \( u(t) \in \mathbb{R}^m \) are the state and input vectors and \( A_c \in \mathbb{R}^{m \times n} \), \( B_c \in \mathbb{R}^{m \times n} \).

Definition 2.1. [5, 7] The system (2.1) is called (internally) positive if \( x(t) \in \mathbb{R}^+ \), \( t \geq 0 \) for any initial conditions \( x(0) = x_0 \in \mathbb{R}^+ \) and all inputs \( u(t) \in \mathbb{R}^+ \), \( t \geq 0 \).

Theorem 2.1. [5, 7] The positive system (2.1) is asymptotically stable if and only if all coefficients of the polynomial

\[
\det[I - A_c] = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0 \quad (2.4)
\]

are positive, i.e. \( a_i > 0 \) for \( i = 0, 1, \ldots, n-1 \).

Now let us consider the discrete-time linear system

\[
x_{i+1} = A_d x_i + B_d u_i, \quad i \in \mathbb{Z}^+
\]
where \( x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^n \) are the state and input vectors and \( A_d \in \mathbb{R}^{n \times n}, B_d \in \mathbb{R}^{n \times m} \).

**Definition 2.3.** [5, 7] The system (2.5) is called (internally) positive if \( x_i \in \mathbb{R}_+^n, i \in Z \) for any initial conditions \( x_0 \in \mathbb{R}_+^n \) and all inputs \( u_i \in \mathbb{R}_+^n, i \in Z \).

**Theorem 2.3.** [5, 7] The system (2.5) is positive if and only if
\[
A_d \in \mathbb{R}^{n \times n}_{\text{pos}}, \quad B_d \in \mathbb{R}^{n \times m}_{\text{pos}}.
\] (2.6)

**Definition 2.4.** [5, 7] The positive system (2.5) is called asymptotically stable if for \( u_i = 0, i \in Z \)
\[
\lim_{t \to \infty} x_i = 0 \text{ for all } x_0 \in \mathbb{R}_+^n.
\] (2.7)

**Theorem 2.4.** [5, 7] The positive system (2.5) is asymptotically stable if and only if all coefficients of the polynomial
\[
\det[I_n(z+1) - A_d] = z^n + a_{n-1}z^{n-1} + \ldots + a_0 = 0
\] (2.8)
\[\text{are positive, i.e. } a_i > 0 \text{ for } i = 0, 1, \ldots, n-1.\]

It is well-known that if the sampling is applied to the continuous-time system (2.1) then the corresponding discrete-time system (2.5) has the matrices
\[
A_d = e^{A_h}, \quad B_d = e^A \int_0^h B_d dt
\] (2.9)
where \( h > 0 \) is the sampling time.

In this paper the following approximation of the matrix (2.9) will be applied
\[
A_d = (A_d + I_n \alpha)(I_n - A_d)^{-1}
\] (2.10)
where the coefficients \( \alpha = \alpha(h) = \frac{\alpha}{h} > 0 \) is chosen so that \( [A_d + I_n \alpha] \in \mathbb{R}^{n \times n}_{\text{pos}} \). It is well-known [1] that if \( A_d \in M_\infty \) then \( \det[I_n - A_d] \neq 0 \) for any \( \alpha > 0 \).

In the next sections it will be shown that the approximation (2.10) preserves:
1. the positivity, i.e. if \( A_d \in M_\infty \) then \( A_d \in \mathbb{R}^{n \times n}_{\text{pos}} \).
2. the asymptotic stability, i.e. if \( A_d \in M_\infty \) then \( A_d \in \mathbb{R}^{n \times n}_{\text{pos}} \).

### 3. Positivity of the systems

In what follows the following lemma will be used.

**Lemma 3.1.** If \( A_d \in M_\infty \) then
\[
-A_d^{-1} \in \mathbb{R}^{n \times n}_{\text{pos}}.
\] (3.1)

**Proof.** The proof will be accomplished by induction. For \( n = 1 \) the hypothesis is evident. The hypothesis is true for \( n = 2 \) since
\[
-A_d^{-1} = \begin{bmatrix}
1 & a_{12} \\
-a_{21} & a_{22}
\end{bmatrix}^{-1} \in \mathbb{R}^{2 \times 2}
\] (3.2)
for \( a_{i,j} \geq 0 \); \( i, j = 1, 2 \).

Assuming that the hypothesis is true for \( k \geq 1 \) we shall show that it is also valid for \( k+1 \). Let \( A_{d+1} \in M_{d+1} \),
\[
det A_{d+1} \neq 0
\]
then it is well-known [9] that
\[
-A_{d+1}^{-1} = \begin{bmatrix}
a_{11} & -a_{12} & \ldots & -a_{1,k+1} \\
-a_{21} & a_{22} & \ldots & -a_{2,k+1} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{k+1,1} & -a_{k+1,2} & \ldots & a_{k+1,k}
\end{bmatrix}
\] (3.3)
\[\text{then } A_{d+1} = \begin{bmatrix}
a_{11} & -a_{12} & \ldots & -a_{1,k} \\
-a_{21} & a_{22} & \ldots & -a_{2,k} \\
\vdots & \vdots & \ddots & \vdots \\
-a_{k,1} & -a_{k,2} & \ldots & a_{k,k}
\end{bmatrix} \in M_{k},
\] (3.4)
\[A_{k+1}^{-1} = \begin{bmatrix}
v_k A_k^{-1} & A_{k+1}^{-1} & 1 \\
A_k^{-1} & -1 & a_{k+1} \\
-v_k A_k^{-1} & A_k & a_{k+1}
\end{bmatrix}
\] where
\[
- A_{d+1}^{-1} \in \mathbb{R}^{k+1 \times (k+1)} \quad \text{and} \quad -v_k \in \mathbb{R}_+^k, \quad -v_k \in \mathbb{R}_+^k
\]
\[a_{k+1} > 0. \quad \text{Hence from (3.4) we have } - A_{d+1}^{-1} \in \mathbb{R}^{k+2 \times (k+2)}.
\]
This completes the proof. □

**Theorem 3.1.** If the continuous-time system (2.1) is positive and asymptotically stable then the discrete-time system (2.5) with the matrix (2.10) is also positive for any sampling time \( h > 0 \).

**Proof.** If the continuous-time system (2.1) is positive and asymptotically stable then \( A_d \in M_\infty \) and there exists such \( \alpha > 0 \) that \( [A_d + I_n \alpha] \in \mathbb{R}^{n \times n}_{\text{pos}} \). If \( A_d \in M_\infty \) then \( \det[I_n \alpha - A_d] \neq 0 \) for any \( \alpha > 0 \) and \( [I_n \alpha - A_d]^{-1} \in \mathbb{R}^{n \times n}_{\text{pos}} \).

In this case \( A_d = [A_d + I_n \alpha][I_n \alpha - A_d]^{-1} \in \mathbb{R}^{n \times n}_{\text{pos}} \) and the discrete-time system (2.5) by Theorem 2.3 is positive. □
4. Asymptotic stability of the system

**Lemma 4.1.** If \( s_k \), \( k = 1, 2, ..., n \) are eigenvalues of the matrix \( A_k \in M_n \) then the eigenvalues \( z_k, k = 1, 2, ..., n \) of the matrix \( A_k \in \mathbb{R}_+^{m \times m} \) are given by

\[
z_k = \frac{s_k + \alpha}{\alpha - s_k} \quad \text{for} \ k = 1, 2, ..., n
\]

**Proof.** If \( A_k \in M_n, \ \alpha > 0 \) is chosen so that \( [A_k + I_\alpha]s_k \in \mathbb{R}_+^{m \times m} \) and \( \alpha \neq s_k \) then the function

\[
f(s_k) = \frac{s_k + \alpha}{\alpha - s_k}
\]

is well defined on the spectrum \( s_k, k = 1, 2, ..., n \) of the matrix \( A_k \). In this case it is well-known [6, 9] that equality (4.1) holds. \( \square \)

**Theorem 4.1.** If the positive continuous-time system (2.1) is asymptotically stable then the corresponding positive discrete-time system (2.5) is also asymptotically stable.

**Proof.** If the positive continuous-time system (2.1) is asymptotically stable then the real parts \( \alpha \) of its eigenvalues \( s_k = -\alpha_k \pm j\beta_k, \ k = 1, 2, ..., n \) are negative. In this case using (4.1) we obtain

\[
|z_k| = \left| \frac{\alpha - \alpha_k \pm j\beta_k}{\alpha + \alpha_k \pm j\beta_k} \right| \leq 1
\]

and the discrete-time system (2.5) is also asymptotically stable. \( \square \)

5. Stabilization of the system

Consider the positive continuous-time linear system (2.1) and the corresponding positive discrete-time linear system (2.5). It is assumed that

\[
\det A_k \neq 0 \quad \text{and} \quad \text{rank } B_k = m.
\]

If \( \det A_k \neq 0 \) then from (2.9) we have

\[
B_k = A_k^{-1}(e^{A_k} - I_k)B_c
\]

and

\[
\text{rank } B_k = m
\]

since \( \det(e^{A_k} - I_k) \neq 0 \) and \( \text{rank } B_c = m \).

If the positive system (2.1) is unstable then applying a suitable state-feedback with a matrix \( K_c \in \mathbb{R}_+^{m \times m} \) we may stabilize the system, i.e.,

\[
\overline{A}_k = A_k + B_cK_c \in M_n.
\]

The corresponding matrix of the discrete-time close-loop system

\[
\overline{A}_d = [\overline{A}_k + I_\alpha][I_\alpha - \overline{A}_k]^{-1} \in \mathbb{R}_+^{m \times m}
\]

is nonnegative and asymptotically stable.

By Theorem 4.1 if \( s_k, k = 1, 2, ..., n \) are the eigenvalues of \( \overline{A}_d \) located in the open left half of the complex plane, then the eigenvalues \( z_k, k = 1, 2, ..., n \) of \( \overline{A}_d \) are given by (4.1) and are located in the unit circle of the complex plane. Therefore, the asymptotic stability of the continuous-time system with \( \overline{A}_d \) implies the asymptotic stability of the discrete-time system with \( \overline{A}_d \) defined by (5.5).

Let the discrete-time system with a matrix \( A_k \) be unstable. We are looking for a state-feedback matrix \( K_d \in \mathbb{R}_+^{m \times m} \) such that the close-loop system is positive and asymptotically stable with the matrix \( \overline{A}_d \), i.e.,

\[
\overline{A}_d = A_d + B_dK_d \in \mathbb{R}_+^{m \times m}.
\]

Solving the equation (5.6) with respect to \( K_d \) for given \( \overline{A}_d, A_d \) and \( B_d \) we obtain

\[
K_d = [B_d^T B_d]^{-1}B_d^T(\overline{A}_d - A_d).
\]

The matrix (5.7) is the solution of (5.6) if and only if

\[
B_d[B_d^T B_d]^{-1}B_d^T(\overline{A}_d - A_d) = \overline{A}_d - A_d.
\]

Therefore, the following theorem has been proved.

**Theorem 5.1.** There exists a state-feedback gain matrix (5.7) of the positive and asymptotically stable discrete-time close-loop system if the condition (5.8) is met.

**Remark 5.1.** The state-feedback gain matrix \( K_c \) and \( K_d \) stabilizing the systems are in general case different and are related by

\[
[A_k + I_\alpha + B_cK_c][I_\alpha - A_k - B_cK_c]^{-1} = [A_k + I_\alpha][I_\alpha - A_k]^{-1} + A_k^{-1}[A_k + I_\alpha][I_\alpha - A_k]^{-1} - I_\alpha B_cK_c.
\]

This equality follows immediately from (5.5), (5.4), (5.2) and (2.10).

**Example 5.1.** Given the positive unstable continuous-time system (2.1) with the matrices

\[
A_k = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Pomiary Automatyka Robotyka nr 2/2013 361
Find a state-feedback gain matrix $K_s \in \mathbb{R}^{1 \times 2}$ which preserve the positivity and stabilize the system. Let the close-loop matrix has the form

$$
\tilde{A}_c = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}.
$$

(5.11)

In this case the state-feedback gain matrix has the form

$$
K_s = \begin{bmatrix} -1 & -4 \end{bmatrix}
$$

(5.12)

since

$$
\tilde{A}_c = A_c + B_s K_s = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}.
$$

(5.13)

Using (2.10) and (5.2) we can compute the matrices $A_d$ and $B_d$ of the corresponding discrete-time system (2.5) for $h = 1$ and $\alpha = 4$

$$
A_d = [A_c + I_s \alpha][I_s \alpha - A_c]^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ -1 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 7 \\ 17 & 8 \\ 31 \end{bmatrix}
$$

(5.14)

and

$$
B_d = A_d^{-1}[A_c - I_s]B_c = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -10 & 8 \\ 8 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 17 & 12 \end{bmatrix}
$$

(5.15)

By Theorem 2.3 the discrete-time system is positive since the matrices (5.14) and (5.15) have positive entries but the system is unstable. The polynomial (2.8) for the matrix (5.14) has the form

$$
\text{det}(I_s(z+1) - A_d) = \begin{vmatrix} z + 10 & -8 \\ -8 & z + 14 \end{vmatrix} = z^2 - \frac{4}{17} z - \frac{204}{289}
$$

(5.16)

By Theorem 2.4 the discrete-time system is unstable since two coefficients of the polynomial (5.16) are negative.

Using (4.1) and taking into account that the matrix $\tilde{A}_d$ has the eigenvalues $s_1 = -2$, $s_2 = -3$ we obtain

$$
z_1 = \frac{s_1 + \alpha}{\alpha - s_1} = \frac{-2 + 4}{4 + 2} = \frac{1}{3},
$$

$$
z_2 = \frac{s_2 + \alpha}{\alpha - s_2} = \frac{-3 + 4}{4 + 3} = \frac{1}{7}
$$

(5.17)

Therefore, the corresponding close-loop discrete-time system is also asymptotically stable.

Using (2.10) we may compute the matrix $\tilde{A}_d$ of the close-loop system

$$
\tilde{A}_d = [\tilde{A}_c + I_s \alpha][I_s \alpha - \tilde{A}_d]^{-1} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 7 \\ 21 & 0 \\ 3 \end{bmatrix}
$$

(5.18a)

and

$$
\tilde{B}_d = \tilde{A}_d^{-1}[\tilde{A}_d - I_s]B_c = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 17 & 8 \\ 31 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 21 & 6 \end{bmatrix}
$$

(5.18b)

Figure 1 presents step response of the continuous-time system with matrices $\tilde{A}_d$ and $B_c$ and its discrete-time approximation with matrices (5.18).

Next from (5.7) the state-feedback gain matrix

$$
K_d = [B_d^T B_c]^{-1} B_d^T [\tilde{A}_d - A_d]
$$

$$
= \begin{bmatrix} 1 & 289 \\ 12 & 12 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 289 & 12 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 7 \\ 21 & 0 \\ 3 \end{bmatrix}
$$

(5.19)

Note that the matrix (5.19) is different then the matrix (5.12).

Using (2.9) we may compute the matrix $\tilde{A}_d$ of the close-loop system

$$
\tilde{A}_d = e^{T_s h} = \begin{bmatrix} 0.1353 & 0.0855 \\ 0 & 0.0498 \end{bmatrix}
$$

(5.20a)
\[
\tilde{B}_e = \tilde{A}_e^{-1} [e^{\tilde{A}_e} - I] B_e = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}^i \\
\times \begin{bmatrix} 0.1353 & 0.0855 \\ 0 & 0.0498 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.1156 \\ 0.3167 \end{bmatrix} \tag{5.20b}
\]

In figure 2 we have the same step response of the continuous-time system but with discrete-time system given by the matrices (5.20).

Fig. 2. Step response of the continuous-time system and its discrete representation (5.20)

Rys. 2. Odpowiedź skokowa układu z czasem ciągłym i jej dyskretna reprezentacja (5.20)

6. Concluding remarks

The approximation of positive asymptotically stable continuous-time linear system by the use of Pade type approximation of the exponential matrix has been addressed. It has been shown that the approximation preserves the positivity and asymptotic stability of the systems. The stabilization problem of unstable positive linear system by state-feedback has been analyzed. Sufficient conditions for the stabilization of discrete-time linear systems by state-feedbacks have been established. The considerations have been illustrated by numerical example. The presented approach can be extended for fractional linear systems [10].

Acknowledgment

I wish to thank very much Professor M. Busłowicz for his constructive comments and remarks. This work was supported under work S/WE/1/11.

References

Tadeusz Kaczorek, PhD Eng, DSc

Born 27.04.1932 in Poland, received the MSc, PhD and DSc degrees from Electrical Engineering of Warsaw University of Technology in 1956, 1962 and 1964, respectively. In the period 1968–69 he was the dean of Electrical Engineering Faculty and in the period 1970–73 he was the prorector of Warsaw University of Technology. Since 1971 he has been professor and since 1974 full professor at Warsaw University of Technology. In 1986 he was elected a corresponding member and in 1996 full member of Polish Academy of Sciences. In the period 1988–1991 he was the director of the Research Centre of Polish Academy of Sciences in Rome. In June 1999 he was elected the full member of the Academy of Engineering in Poland. In May 2004 he was elected the honorary member of the Hungarian Academy of Sciences. He was awarded by the University of Zielona Góra (2002) by the title doctor honoris causa, the Technical University of Lublin (2004), the Technical University of Szczecin (2004), Warsaw University of Technology (2004), Białystok University of Technology (2008), Łódź University of Technology (2009), Opole University of Technology (2009) and Poznań University of Technology (2011).

His research interests cover the theory of systems and the automatic control systems theory, specially, singular multidimensional systems, positive multidimensional systems and singular positive 1D and 2D systems. He has initiated the research in the field of singular 2D, positive 2D linear systems and positive fractional 1D and 2D systems. He has published 24 books (7 in English) and over 1000 scientific papers.

He supervised 69 PhD theses. More than 20 of this PhD students became professors in USA, UK and Japan. He is editor-in-chief of Bulletin of the Polish Academy of Sciences, Techn. Sciences and editorial member of about ten international journals.

e-mail: kaczorek@isep.pw.edu.pl