EQUIVALENT DESCRIPTIONS OF A DISCRETE–TIME FRACTIONAL–ORDER LINEAR SYSTEM AND ITS STABILITY DOMAINS

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Two description forms of a linear fractional-order discrete system are considered. The first one is by a fractional-order difference equation, whereas the second by a fractional-order state-space equation. In relation to the two above-mentioned description forms, stability domains are evaluated. Several simulations of stable, marginally stable and unstable unit step responses of fractional-order systems due to different values of system parameters are presented.

Keywords: fractional calculus, linear discrete-time system, stability domain.

1. Introduction

Fractional calculus (Oldham and Spanier, 1974; Miller and Ross, 1993; Samko and Marichev, 1993; Oustaloup, 1995; Podlubny, 1999; Ostalczyk, 2008; Kaczorek, 2011) has become a recognized mathematical tool in many scientific areas. One can mention some successful applications in dynamic system identification (Ostalczyk, 2008) and the synthesis of PID (Podlubny, 1999; Valério and Costa, 2006) or CRONE (Oustaloup, 1991; 1995; 1999) controllers in closed-loop dynamical systems. A main advantage of such controllers is that they have additional parameters, i.e., differentiation and integration orders, to reshape the transient characteristics of the designed closed-loop system. The closed-loop system stability is the first requirement of a synthesis (Dzieliński and Sierociuk, 2008; Guermah et al., 2010). Thus a simple and readable criterion may be helpful.

There are equivalent (under some assumptions) definitions of the Fractional-Order (FO) derivative. The so-called Riemann–Liouville left-sided derivative of order of a real function having continuous derivatives for is defined as the following integral:

\[ t_0 D_t^{(\alpha)} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{y(\tau)}{(t-\tau)^{n+1}} d\tau, \quad (1) \]

where \( n = [\alpha] + 1 \), \([\alpha]\) is the integer part of \( \alpha \), \([t_0, t]\) is the differentiation range, \( \Gamma \) is the Euler gamma function.

One can prove that (1) is equivalent to the Grünwald–Letnikov form

\[ t_0 D_t^{(\alpha)} y(t) = \lim_{h \to 0^+} \frac{k_0 \Delta_k^{(\alpha)} f(kh)}{h^\alpha}, \quad (2) \]

where

\[ k_0 \Delta_k^{(\alpha)} f(kh) = \sum_{i=k_0}^{k} d_i^{(\alpha)} y(kh + k_0 h - ih) \quad (3) \]

is the Grünwald–Letnikov backward difference, \([t_0, t] = [k_0 h, kh]\) is the differentiation range, \( h \) is the differentiation step.

In the numerical evaluation of the FO derivative (2), \( h \) is finite and constant. To simplify the notation, one can assume \( h = 1 \) and omit it in the formula. Here one should emphasise that left (\( k_0 \)) and right (\( k \)) subscripts in the difference sign \( \Delta \) denote a differentiation range (a fixed summation range), whereas (\( k \)) in the function \( y \) denotes its
discrete variable. Here, one should care about notation because, in general,
\[
\kappa_0 \Delta_k^{(\alpha)} y(kh) \neq \kappa_0 \Delta_{k+1}^{(\alpha)} y(kh) \quad (5)
\]
and all the fractional orders considered are rational numbers, i.e., they can be expressed as a ratio of positive integers
\[
\alpha = \frac{1}{d} n = \nu n \quad (d, n \in \mathbb{Z}_+, \quad \alpha, \nu \in \mathbb{R}_+), \quad (7)
\]
with
\[
0 < \frac{1}{d} = \nu < 1. \quad (8)
\]
and Greek letters are reserved for non-integer numbers.

2. Equivalent descriptions of the FO linear dynamical system

In this section the FO commensurate state-space description of the FO linear single-input single-output discrete-time system is discussed. A relationship between this description and the FO difference equation is established.

2.1. Commensurate FO state-space description. Any linear time-invariant FO Differential Equation (FODE) with orders satisfying the condition \( (7) \) can be represented by the commensurate state-space equations
\[
0 \Delta_{k+1}^{(\nu)} x[(k + 1)h] = A x(kh) + b u(kh), \quad (9)
\]
and
\[
y(kh) = c x(kh), \quad (10)
\]
where
\[
0 \Delta_{k+1}^{(\nu)} x[(k + 1)h] = \begin{bmatrix}
0 \Delta_{k+1}^{(\nu)} x_1[(k + 1)h] \\
0 \Delta_{k+1}^{(\nu)} x_2[(k + 1)h] \\
\vdots \\
0 \Delta_{k+1}^{(\nu)} x_n[(k + 1)h]
\end{bmatrix}. \quad (11)
\]
It is well-known that there exists a similarity transformation matrix \( T_F \) transforming the state matrix \( A \) in \( (9) \) to the Frobenius canonical form \( A_F \) (Kailath, 1980),
\[
A_F = T_F A (T_F)^{-1} = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
& & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & 1 \\
-A_0 & -A_1 & -A_2 & \cdots & -A_{n-1}
\end{bmatrix}, \quad (12)
\]
where
\[
b_F = \begin{bmatrix}
0 \\
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}, \quad c_F = [B_0 \cdots B_n] = 0. \quad (13)
\]
Another similarity transformation of the state vector \( (11) \) represented by a matrix \( T_D \) transforms a state matrix in the formula \( (9) \) to the diagonal form \( A_D \),
\[
0 \Delta_{k+1}^{(\nu)} x[(k + 1)h] = A_D x(kh) + B_D u(kh), \quad (14)
\]
and
\[
y(kh) = c_D x(kh), \quad (15)
\]
where
\[
A_D = \begin{bmatrix}
\bar{p}_1 & 0 & \cdots & 0 \\
0 & \bar{p}_2 & \cdots & 0 \\
& & \ddots & \ddots \\
0 & 0 & \cdots & \bar{p}_n
\end{bmatrix}. \quad (16)
\]
Here, without loss of generality, one may assume that all eigenvalues are distinct. Because the Jordan canonical forms of the state matrix of the system \( (9) \) with different configurations of Jordan blocks of one multiple eigenvalue lead to the same characteristic polynomial, in the system stability analysis the multieigenvalue case may be considered similarly.

2.2. FO difference equation description. From equations derived from the state-space description \( (9) \) with \( (12) \) we obtain
\[
0 \Delta_{k+1}^{(\nu)} x_{i-1}[(k + 1)h] = x_i(kh) \quad \text{for} \quad i = 2, \ldots, n, \quad (17)
\]
and
\[
0 \Delta_{k+1}^{(\nu)} x_{i-1}[(k + 1)h] = 0 \Delta_{k+1}^{(\nu)} x_i[(k + 1)h] \quad \text{for} \quad i = 2, \ldots, n, \quad (18)
\]
and, after simple rearrangements,
\[
\sum_{i=0}^{n} A_i z^i (1 - z^{-1})^\nu Y(z) = \sum_{i=0}^{m} B_i z^i (1 - z^{-1})^\nu U(z).
\]
(20)

From the state-space equations and the Z-transform, we obtain
\[
\prod_{i=0}^{n} z (1 - z^{-1})^\nu - \bar{\nu}_i Y(z) = B_m \prod_{i=0}^{m} (z (1 - z^{-1})^\nu - r_i) U(z).
\]
(21)

An analogous procedure performed on Eqn. (14) gives
\[
X(z) = \text{diag} \left\{ \frac{1}{z (1 - z^{-1})^\nu - \bar{\nu}_i} \right\} U(z).
\]
(22)

The equalities (21) and (22) form the system characteristic polynomial containing information about the system stability. Thus
\[
z (1 - z^{-1})^\nu - \bar{\nu}_i = 0
\]
(23)

may be expressed as
\[
z^{1-\nu} (z - 1)^\nu - \bar{\nu}_i = 0
\]
(24)

and further
\[
\prod_{j=1}^{\infty} (z - b_j) = 0.
\]
(25)

The characteristic polynomial (26) is stable if and only if \( b_j \) are settled in the interior of a unit circle defined by \(|z| = 1\) (Ogata, 1987).

3.2. Stability domains of the system described by FO state-space equations. Defining the one-to-one transformation
\[
p_d(\theta) = e^{j\theta} \left(1 - e^{-j\theta}\right)^2 \quad \text{for} \quad \theta \in [0, 2\pi),
\]
(26)

we obtain system stability regions in the space of parameters \( \bar{\nu}_j \). For different orders (8) defined by the integer \( d = 1, 2, \ldots, 10 \), the corresponding stability domains are plotted in Fig. 1. As \( d \to \infty \), stability domains tend to a unit circle except for a real positive axis. This domain (evaluated for \( d = 100 \)) is presented in Fig 2.

On the other hand, for \( \nu = 1 \) (the case of a classical integer order system) from (23) we get
\[
z - (1 + \bar{\nu}_i) = 0,
\]
(27)

which explains the left shift of the unit circle present in Fig. 1. The formula (26) can be also expressed in the form
\[
p_d(\theta)
\]
(28)

Then
\[
\lim_{d \to \infty} p_d(\theta) = e^{j\theta}.
\]
(29)
4. FO discrete linear system unit step response

Now several unit step responses of FO discrete linear systems are numerically evaluated. All the presented FO systems are characterised by the same FO $\nu = 0.5$. First, the system

$$\begin{align*}
\varrho \Delta_{k+2} y(k+2) + A_1 \varrho \Delta_{k+1} y(k+1) + A_0 y(k) &= A_0 1(k),
\end{align*}$$

where $1(k)$ denotes a discrete unit step function, and the condition $B_0 = A_0$ preserve the steady-state response level equal to 1. In the following figures black dots indicate response values which are connected by thin lines to provide better clarity of the response shape. In Figs. 3 and 4, critically stable responses are presented. The first one is characterised by

$$\bar{p}_1 = \bar{p}_2 = -\sqrt{2}$$

or, equivalently, by

$$A_1 = 2\sqrt{2}, \quad A_0 = 4.$$  \hspace{1cm} (32)

The second system is characterised by

$$\bar{p}_1 = j, \quad \bar{p}_2 = -j$$

or

$$A_1 = 0, \quad A_1 = 2.$$  \hspace{1cm} (34)

One should note that in both the cases considered the poles 41 and 33 lie precisely on boundary of the stability domain. Next, the unit step responses for two asymptotically stable systems are presented. In Figs. 3 and 4 responses related to the coefficients

$$\bar{p}_1 = -1.4 + j0.1, \quad \bar{p}_2 = -1.4 - j0.1$$

are displayed, respectively.

Next, we consider again Eqn. (15) with $n = 4$, $m = 0$, $\nu = 0.5$ and $B_0 = A_0$. All parameters $\bar{p}_j$, $j = 1, 2, 3, 4$ are on boundary of the stability domain,

$$\begin{align*}
\bar{p}_1 &= -1.284110014049142 + j0.5318957833982609, \\
\bar{p}_2 &= \bar{p}_1^*, \\
\bar{p}_3 &= -1.130235782084677 + j0.7551994054009926, \\
\bar{p}_4 &= \bar{p}_3^*
\end{align*}$$

where (*) denotes the complex conjugate. The unit step
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Applying now a diminishing factor $\nu_1 = 0.99$ to all coefficients (39), i.e., taking $\nu_1 \bar{p}_j$, $j = 1, 2, 3, 4$, we get an asymptotically stable system. For an increasing factor $\nu_2 = 1.01$, the system (39) with $\nu_2 \bar{p}_j$, $j = 1, 2, 3, 4$ is unstable. Stable and unstable system responses are presented in Figs. 8 and 9 respectively.

Finally, one can mention that the systems considered above are FO first and second order systems, due to the highest orders of the difference equations (15).

5. Conclusions

The transformation (26) proposed in this paper allows quick and precise graphical and numerical evaluation of the stability domains of FO linear discrete systems. One should note that an approximation of the FO discrete system by an integer order system may be inadequate, especially when the system is on the stability limit. Its
simplicity and visibility may be useful in a robust digital controller in FO closed-loop control system synthesis due to plant parameter changes leading to different closed-loop system poles configurations. The presented transient responses of FO stable systems revealing new shapes of waves may be helpful in the FO generator or digital filter synthesis.

References


Piotr Ostalczyk received an M.Sc. degree in electrical engineering from the Faculty of Electrical Engineering of the Technical University of Łódź in 1976. There, in 1981, he obtained a Ph.D. degree. In 1991 he received a D.Sc. degree and in 2008 a professorial title. Since 1994 his main field of interest has been the application of fractional calculus in discrete-time control and dynamic system identification using fractional-order difference equations.

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