Abstract. In this paper we prove that the extended spectrum $\Sigma(x)$, defined by W. Żelazko, of an element $x$ of a pseudo-complete locally convex unital complex algebra $A$ is a subset of the spectrum $\sigma_A(x)$, defined by G.R. Allan. Furthermore, we prove that they coincide when $\Sigma(x)$ is closed. We also establish some order relations between several topological radii of $x$, among which are the topological spectral radius $R_t(x)$ and the topological radius of boundedness $\beta_t(x)$.

Keywords: topological algebra, bounded element, spectrum, pseudocomplete algebra, topologically invertible element, extended spectral radius, topological spectral radius.

Mathematics Subject Classification: 46H05.

1. INTRODUCTION

A complex algebra $A$ with a topology $\tau$ is a locally convex algebra if it is a Hausdorff locally convex space and its multiplication $(x, y) \mapsto xy$ is jointly continuous. The topology of $A$ can be given by the family of all continuous seminorms on $A$.

Throughout this paper $A = (A, \tau)$ will be a locally convex complex algebra with unit $e$, $A'$ its topological dual and $\{\|\cdot\|_\alpha : \alpha \in A\}$ the family of all continuous seminorms on $A$.

An element $x \in A$ is called bounded if for some non-zero complex number $\lambda$, the set $\{(\lambda x)^n : n = 1, 2, \ldots\}$ is a bounded set of $A$. The set of all bounded elements of $A$ is denoted by $A_0$.

For $x \in A$ define the radius of boundedness $\beta(x)$ of $x$ by

$$\beta(x) = \inf \left\{ \lambda > 0 : \left\{ \left( \frac{x}{\lambda} \right)^n : n \geq 1 \right\} \text{ is bounded} \right\}$$

adopting the usual convention that $\inf \emptyset = \infty$. Henceforth we shall use this convention without further mention.
Notice that $\lambda_0 > 0$ and $\{\left(\frac{x}{\lambda_0}\right)^n : n \geq 1\}$ bounded imply $\|\left(\frac{x}{\lambda}\right)^n\|_\alpha \to 0$ for all $|\lambda| > \lambda_0$ and $\alpha \in \Lambda$. Using this fact it is easy to see that $\beta(x) = \beta_0(x)$, where

$$\beta_0(x) = \inf \left\{ \lambda > 0 : \lim_{n \to \infty} \left(\frac{x}{\lambda}\right)^n = 0 \right\}.$$  

In [1], by $B_1$ it is denoted the collection of all subsets $B$ of $A$ such that:

(i) $B$ is absolutely convex and $B^2 \subset B$,

(ii) $B$ is bounded and closed.

For any $B \in B_1$, let $A(B)$ be the subalgebra of $A$ generated by $B$. From (i) we get

$$A(B) = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}.$$  

The formula

$$\|x\|_B = \inf \{\lambda > 0 : x \in \lambda B\}$$  

defines a norm in $A(B)$, which makes it a normed algebra. It will always be assumed that $A(B)$ carries the topology induced by this norm. Since $B$ is bounded in $(A, \tau)$, the norm topology on $A(B)$ is finer than its topology as a subspace of $(A, \tau)$.

The algebra $A$ is called pseudo-complete if each of the normed algebras $A(B)$, for $B \in B_1$, is a Banach algebra. It is proved in [1, Proposition 2.6] that if $A$ is sequentially complete, then $A$ is pseudo-complete.

In [1], it is also introduced by G. R. Allan the spectrum $\sigma_A(x)$ of $x \in A$ as the subset of the Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ defined as follows:

(a) for $\lambda \neq \infty$, $\lambda \in \sigma_A(x)$ if and only if $\lambda e - x$ has no inverse belonging to $A_0$,

(b) $\infty \in \sigma_A(x)$ if and only if $x \notin A_0$.

In [1, Corollary 3.9] it is proved that $\sigma_A(x) \neq \emptyset$ for all $x$. We shall call $\sigma_A(x)$ the Allan spectrum.

The Allan spectral radius $r_A(x)$ of $x$ is defined by

$$r_A(x) = \sup \{ |\lambda| : \lambda \in \sigma_A(x) \},$$  

where $|\infty| = \infty$.

On the other hand, W. Żelazko defined in [4] the concept of extended spectrum of $x \in A$ in the way that we now recall.

As usual

$$\sigma(x) = \{ \lambda \in \mathbb{C} : \lambda e - x \notin G(A) \},$$  

where $G(A)$ is the set of all invertible elements of $A$. The resolvent

$$\lambda \to R(\lambda, x) = (\lambda e - x)^{-1}$$  

is then defined on $\mathbb{C} \setminus \sigma(x)$, but it is not always a continuous map. Put

$$\sigma_d(x) = \{ \lambda_0 \in \mathbb{C} \setminus \sigma(x) : R(\lambda, x) \text{ is discontinuous at } \lambda = \lambda_0 \}$$  

and

\[ \sigma_\infty(x) = \begin{cases} \emptyset & \text{if } \lambda \to R(1, \lambda x) \text{ is continuous at } \lambda = 0, \\ \infty & \text{otherwise.} \end{cases} \]

Then the extended spectrum of \( x \) is the set

\[ \Sigma(x) = \sigma(x) \cup \sigma_d(x) \cup \sigma_\infty(x). \]

It is proved in [4, Theorem 15.2] that if \( A \) is complete, then \( \Sigma(x) \) is a non empty set of \( \mathbb{C}_\infty \) for every \( x \), and the extended spectral radius \( R(x) \) is defined by

\[ R(x) = \sup \{ |\lambda| : \lambda \in \Sigma(x) \}. \]

We shall not assume that \( A \) is complete. Nevertheless, from now on we assume that \( \Sigma(x) \) is a non empty set of \( \mathbb{C}_\infty \) for every \( x \in A \).

2. COMPARISON OF \( \Sigma(x) \) AND \( \sigma_A(x) \)

**Theorem 2.1.** If \( A \) is pseudo-complete, then \( \Sigma(x) \subset \sigma_A(x) \) for any \( x \in A \).

**Proof.** Let \( \lambda \notin \sigma_A(x) \) with \( \lambda \neq \infty \), then \( \lambda \notin \sigma(x) \) and \( R(\lambda, x) \) is bounded. Hence \( R(\lambda, x) \in A(B) \) for some \( B \in B_1 \) ([1, Proposition 2.4]).

For any \( \mu \in \mathbb{C} \), we have that \( (\mu e - x) = (\lambda e - x) + (\mu - \lambda) e \). Let \( 0 < \gamma < \| R(\lambda, x) \|^{-1} \), then for \( |\mu - \lambda| < \gamma \), the formula

\[ S_n(\mu) = R(\lambda, x) - (\mu - \lambda) R(\lambda, x)^2 + (\mu - \lambda)^2 R(\lambda, x)^3 - \ldots + (-1)^n (\mu - \lambda)^n R(\lambda, x)^{n+1}, \]

defines a Cauchy sequence in the Banach algebra \( A(B) \). Therefore, it converges in \( A(B) \) to \( R(\mu, x) \).

Given \( \varepsilon > 0 \), there exists \( 0 < \delta < \gamma \) such that

\[ \| S_n(\mu) - R(\lambda, x) \|_B \leq |\mu - \lambda| \| R(\lambda, x) \|_B \left( \frac{1}{1 - \gamma \| R(\lambda, x) \|_B} \right) < \varepsilon \]

for all \( n \) if \( |\lambda - \mu| < \delta \), which implies that \( \| R(\mu, x) - R(\lambda, x) \| \leq \varepsilon \) if \( |\lambda - \mu| < \delta \).

Hence \( R(\mu, x) \to R(\lambda, x) \) as \( \mu \to \lambda \), in \( A(B) \) and also in \( (A, \tau) \), therefore \( \lambda \notin \sigma_A(x) \).

Thus, \( \lambda \notin \Sigma(x) \).

If \( \infty \notin \sigma_A(x) \), then \( x \) is bounded and there exists \( r > 0 \) such that the idempotent set \( \{ (\xi)^n : n \geq 1 \} \) is bounded. The closed absolutely convex hull \( B \) of \( \{ (\xi)^n : n \geq 1 \} \) belongs to \( B_1 \). Consider the Banach algebra \( A(B) \). Since \( \| \frac{x}{\beta} \|_B < 1 \) for every \( |\beta| > r \), we obtain

\[ R\left(1, \frac{x}{\beta}\right) = e + \frac{x}{\beta} + \left(\frac{x}{\beta}\right)^2 + \ldots \]

in the Banach algebra \( A(B) \).
Since
\[ \left\| R \left( \frac{1}{1 + \beta} \right) - e \right\|_B \to 0 \]
as \( |\beta| \to \infty \), we have that \( R (1, tx) \to e \) as \( t \to 0 \), in \( A (B) \) and hence in \( (A, \tau) \) as well. Thus \( R (1, tx) \) is continuous in \( t = 0 \) and \( \infty \notin \Sigma(x) \). \qed

**Lemma 2.2.** Suppose \( A \) is pseudo-complete and let \( x \in A \) be such that the extended spectral radius \( R(x) < \infty \). Then for each \( f \in A' \) the function \( F(\lambda) = f (R (1, \lambda x)) \) is holomorphic in the open disc \( D(0, \delta) \), with \( \delta = \frac{1}{R(1)} \), where \( D(0, \delta) = \mathbb{C} \) when \( R(x) = 0 \). Furthermore,
\[ F^{(n)} (\lambda) = n! f \left( R (1, \lambda x)^{n+1} x^n \right) \tag{2.1} \]
for every \( \lambda \in D(0, \delta) \) and \( n = 0, 1, 2, \ldots \). In particular,
\[ F^{(n)} (0) = n! f(x^n) \]
for all \( n \geq 0 \).

**Proof.** We have that \( \lambda \notin \Sigma(x) \) whenever \( |\lambda| > R(x) \). This implies that the function
\[ \lambda \to R (1, \lambda x) \]
is continuous in the open disc \( D = D(0, \delta) \). By definition \( F^{(0)} (0) = f (e) \) and \( F(\lambda) = f (R (1, \lambda x)) \) is holomorphic in \( D \) since
\[ F^{(r)} (\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{f (R (1, \lambda x)) - f (R (1, \lambda_0 x))}{\lambda - \lambda_0} = \lim_{\lambda \to \lambda_0} f \left( R (1, \lambda x) R (1, \lambda_0 x) \frac{\lambda - \lambda_0}{\lambda - \lambda_0} \right) = f \left( R (1, \lambda_0 x)^2 x \right) \]
for every \( \lambda_0 \in D \).

It is easy to obtain (2.1) by induction. \qed

**Theorem 2.3.** If \( A \) is pseudo-complete, then for any \( x \in A \) we have that \( \Sigma(x) = \sigma_A(x) \) if \( \Sigma(x) \) is closed in \( \mathbb{C}_\infty \).

**Proof.** Let \( x \in A \) and assume that \( \Sigma(x) \) is closed, then by Theorem 2.1 we only have to prove that \( \lambda_0 \notin \Sigma(x) \) implies \( \lambda_0 \notin \sigma_A(x) \).

Let \( \lambda_0 \notin \Sigma(x) \), with \( \lambda_0 \neq \infty \), then \( \lambda_0 e - x \in G(A) \). We shall show that \( (\lambda_0 e - x)^{-1} \) is bounded. Since \( \Sigma(x) \) is closed, then there exists an open disc \( D(\lambda_0) \) around \( \lambda_0 \) such that \( \lambda e - x \in G(A) \) if \( \lambda \in D(\lambda_0) \) and \( R(\lambda, x) \) is continuous at \( \lambda = \lambda_0 \). Using the identity
\[ (\lambda e - x)^{-1} - (\lambda_0 e - x)^{-1} = (\lambda_0 - \lambda)(\lambda e - x)^{-1}(\lambda e - x)^{-1}, \]
we obtain
\[ \lim_{\lambda \to \lambda_0} \frac{R(\lambda, x) - R(\lambda_0, x)}{\lambda - \lambda_0} = -R(\lambda_0, x)^2. \]

Then for any \( f \in A' \) we get
\[ \lim_{\lambda \to \lambda_0} \frac{f(R(\lambda, x)) - f(R(\lambda_0, x))}{\lambda - \lambda_0} = -f(R(\lambda_0, x)^2), \]

which implies that \( R(\lambda, x) \) is weakly holomorphic in \( \lambda = \lambda_0 \). By [1, Theorem 3.8 (i)] we obtain that \((\lambda_0e - x)^{-1}\) is bounded in \( A \). Therefore, \( \lambda_0 \notin \sigma_A(x) \).

If \( \infty \notin \Sigma(x) \), then some neighborhood of \( \infty \) does not intersect \( \Sigma(x) \) and we have that \( R(x) < \infty \). Let \( f \in A' \). By Lemma 2.2, the Taylor expansion of \( F(\lambda) = f(R(1, \lambda x)) \) around 0 is
\[ F(\lambda) = f(\epsilon) + \lambda f(x) + \frac{2\lambda^2}{2!} f(x^2) + \ldots \]
for \( |\lambda| < \frac{1}{R(\epsilon, x)} \). In particular, \( \lim_{n \to \infty} f(\lambda_0^n x^n) = 0 \) for some \( \lambda_0 > 0 \) and then \( \{f(\lambda_0^n x^n) : n \geq 1\} \) is bounded; therefore \( \{(\lambda_0 x)^n : n \geq 1\} \) is bounded. Thus \( x \in A_0 \) and \( \infty \notin \sigma_A(x) \).

\[ \Box \]

3. COMPARISON BETWEEN TOPOLOGICAL RADI

Let \( x \in A \), we say that \( x \) is **topologically invertible** if \( xA = Ax = A \), i.e. for each neighborhood \( V \) of \( e \) there exist \( a_\lambda, a_\lambda' \in A \) such that \( x a_\lambda e = x \in V \) and \( a_\lambda' x \in V \).

The **topological spectrum** \( \sigma_t(x) \) of \( x \) is the set
\[ \sigma_t(x) = \{ \lambda \in \mathbb{C} : \lambda e - x \text{ is not topological invertible} \}. \]

The **topological spectral radius** \( R_t(x) \) is defined by
\[ R_t(x) = \sup \{|\lambda| : \lambda \in \sigma_t(x)\}. \]

Having in mind the definition of \( \beta_0(x) \) we define the **topological radius of boundedness** \( \beta_t(x) \) of \( x \) by
\[ \beta_t(x) = \inf \left\{ \lambda > 0 : \liminf_n \left\| \left( \frac{x}{\lambda} \right)^n \right\| = 0 \text{ for all } \alpha \in \Lambda \right\}. \]

In [2] the first author defined the **lower extended spectral radius** of \( x \) by
\[ R_*(x) = \sup_{\alpha \in \Lambda} \liminf_n \sqrt[n]{\left\| x^n \right\|_\alpha} \]
and in [3] it is proved that if \( A \) is a complete locally convex unital algebra, then for any \( x \in A \) we have \( R_*(x) \leq r_0(x) \), and \( R_*(x) = r_0(x) \) if \( A \) is a unital \( B_0 \)-algebra (metrizable complete locally convex algebra), where
\[ r_0(x) = \inf \{ 0 < r \leq \infty : \text{there exists } (a_n)_{n=0}^\infty, a_n \in \mathbb{C}, \text{ such that } \sum_{n=0}^\infty a_n \lambda^n \text{ has radius of convergence } r \text{ and } \sum_{n=0}^\infty a_n x^n \text{ converges in } A \} \]
(In [3] this radius is denoted by \( r_0(x) \).)
Here we have the following result.

**Proposition 3.1.** Let \( x \in A \). Then

\[
R_\ell(x) \leq \beta_\ell(x) = R_\ast(x) \leq \beta(x) \leq r_A(x).
\]

**Proof.** The first inequality is obvious if \( \beta_\ell(x) = \infty \), therefore let \( \beta_\ell(x) < \infty \). Given \( \lambda > \beta_\ell(x) \) and \( \alpha \in \Lambda \), there exists a subsequence \((n_k)_k = (n_k(\alpha))_k \) of the natural sequence \((n)_n \) such that \( \lim_{k \to \infty} \| \left( \frac{x}{\lambda} \right)^{n_k} \|_\alpha = 0 \). Then

\[
\lim_{k \to \infty} \left\| \left( e + \frac{x}{\lambda^2} + \ldots + \frac{x^{n_k-1}}{\lambda^{n_k}} \right) (\lambda e - x) - e \right\|_\alpha = 0.
\]

Hence \( \lambda e - x \) is topologically invertible for any such \( \lambda \) and it follows that \( R_\ell(x) \leq \beta_\ell(x) \).

If \( R_\ast(x) = \infty \), then \( \beta_\ell(x) \leq R_\ast(x) \). Now suppose \( R_\ast(x) < \mu < \lambda < \infty \). Then given \( \alpha \in \Lambda \) there exists a subsequence \((n_k)_k = (n_k(\alpha))_k \) of \((n)_n \) such that \( \| x^n \|_\alpha < \mu \) \( \lambda < \lambda \), which implies that \( \left\| \left( \frac{x}{\lambda} \right)^{n_k} \right\|_\alpha < \left( \frac{\mu}{\lambda} \right)^{n_k} \). Therefore, \( \beta_\ell(x) \leq \lambda \) and we have \( \beta_\ell(x) \leq R_\ast(x) \).

Assume that \( \beta_\ell(x) < R_\ast(x) \), then there exist \( \lambda > 0 \) and \( \alpha_0 \in \Lambda \) such that \( \beta_\ell(x) < \lambda < R_\ast(x) \) and \( \lambda < \liminf_n \sqrt[n]{\| x^n \|_{\alpha_0}} \). Hence \( \liminf_n \sqrt[n]{\| \left( \frac{x}{\lambda} \right)^{n} \|_{\alpha_0}} > 1 \). On the other hand, \( \lambda > \beta_\ell(x) \) implies that \( \liminf_n \sqrt[n]{\| \left( \frac{x}{\lambda} \right)^{n} \|_{\alpha_0}} = 0 \), which contradicts the previous statement. Thus, \( \beta_\ell(x) = R_\ast(x) \).

Since \( \beta(x) = \beta_\ell(x) \) it is clear that \( \beta_\ell(x) \leq \beta(x) \). Finally, \( \beta(x) \leq r_A(x) \) by [1, Theorem 3.12].

**Corollary 3.2.** If \( A \) is pseudo-complete, then

\[
R_\ell(x) \leq R_\ast(x) = \beta_\ell(x) \leq \beta(x) = r_A(x) \leq R(x)
\]

for every \( x \in A \).

**Proof.** Let \( x \in A \). We have by [1, Theorem 3.12] that \( \beta(x) = r_A(x) \). Thus we only have to prove that \( \beta(x) \leq R(x) \). This is obvious if \( R(x) = \infty \), so assume that \( R(x) < \infty \), therefore \( \infty \notin \Sigma(x) \). Applying Lemma 2.2 we obtain that the Taylor expansion about \( 0 \) of \( F(\lambda) = f(R(1, \lambda x)) \) is

\[
F(\lambda) = f(e) + \lambda f(x) + \frac{2\lambda^2}{2!} f_2(x^2) + \ldots
\]

for \( f \in A' \) and \( |\lambda| < \frac{1}{R(x)} \).

Then \( \lim_{n \to \infty} \sqrt[n]{\| (\lambda x)^n \|_{\alpha_0}} = 0 \) for any \( 0 < \lambda < \frac{1}{R(x)} \) and \( f \in A' \). In particular, for any such \( \lambda \) the set \( \{ (\lambda x)^n : n \geq 1 \} \) is weakly bounded and therefore \( \{ (\lambda x)^n : n \geq 1 \} \) is bounded in \( A \). It follows that \( \lambda \geq \beta(x) \) for every \( \lambda > R(x) \) and then \( \beta(x) \leq R(x) \). □

**Proposition 3.3.** If \( A \) is complete, then \( r_A(x) = \beta(x) = R(x) \) for all \( x \in A \).
Proof. It remains to prove that \( R(x) \leq \beta(x) \). We can assume that \( \beta(x) < \infty \). Let \( r > \beta(x) \), then we have that \( f \left( \left( \frac{x}{r} \right)^n \right) \to 0 \) for every \( f \in A' \), therefore

\[
\limsup_n \sqrt[n]{|f(x^n)|} \leq r
\]

for every \( f \in A' \). We get from [4, Theorem 15.6] that

\[
R(x) = R_2(x) = \sup_{f \in A'} \limsup_n \sqrt[n]{|f(x^n)|} \leq r.
\]

Therefore, \( R(x) \leq \beta(x) \).

Remark 3.4. In [2] it is constructed a unital \( B_0 \)-algebra \( A \) in which there is an element \( x \) such \( R_*(x) = 1 \) and \( R(x) = \infty \). On the other hand, if we consider the non-complete algebra \( A = (\mathcal{P}(t), ||\cdot||) \) of all complex polynomials with the norm \( ||p(t)|| = \max_{0 \leq t \leq 1} |p(t)| \), then for every \( \lambda \neq 0 \) we have that \( \left\| \left( \frac{1}{\lambda} \right)^n \right\| = \frac{1}{|\lambda|^n} \). Therefore \( \beta(t) = 1 \), nevertheless \( R(t) = \infty \) since \( \lambda - t \) does not have an inverse for all \( \lambda \in \mathbb{C} \).

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