EXISTENCE AND UNIQUENESS THEOREM
FOR A HAMMERSTEIN
NONLINEAR INTEGRAL EQUATION

A.Kh. Khachatryan, Kh.A. Khachatryan

Abstract. The existence of a solution, as well as some properties of the obtained solution for a Hammerstein type nonlinear integral equation have been investigated. For a certain class of functions the uniqueness theorem has also been proved.

Keywords: iteration, Wiener-Hopf operator, pointwise convergence, Hammerstein type equation.

Mathematics Subject Classification: 45G05.

1. INTRODUCTION

Let us consider the following class of Hammerstein type nonlinear integral equations

\[ \varphi(x) = \int_{0}^{+\infty} K(x-t)\varphi^\alpha(t)dt, \quad x \in (0, +\infty), \quad \alpha \in (0, 1), \quad (1.1) \]

with respect to an unknown function \( \varphi(x) \geq 0 \). The kernel \( K(x) \geq 0 \) is an integrable function on \(( -\infty, +\infty )\) such that

\[ \int_{-\infty}^{+\infty} K(t)dt = 1, \quad \nu = \nu_+ - \nu_- < 0, \quad (1.2) \]

where \( \nu_+ = \int_{0}^{\infty} tK(t)dt < +\infty \) and \( \nu_- = \int_{-\infty}^{0} tK(-t)dt < +\infty \).

In the present paper we prove the existence of a positive, monotonic increasing and bounded solution \( \varphi(x) \leq 1 \). Moreover, we show that \( \lim_{x \to +\infty} \varphi(x) = 1 \). We also prove that, by putting an additional condition on the kernel, the obtained solution is continuous on \([0, +\infty)\) and unique in a certain class of functions.
2. PRELIMINARIES

Let $E$ be one of the following Banach spaces: $L_p(0, +\infty)$ for $p \geq 1$, $M[0, +\infty)$, $C_M[0, +\infty)$, $C_0[0, +\infty)$, where $M[0, +\infty)$ is the space of bounded functions on $[0, +\infty)$, $C_M[0, +\infty)$ is the space of continuous and bounded functions on $[0, +\infty)$, and finally $C_0[0, +\infty)$ is the space of continuous functions, possessing zero limit at infinity.

We denote by $\mathcal{K}$ the Wiener-Hopf type integral operator with the kernel $K(x)$

$$(Kf)(x) = \int_0^{+\infty} K(x-t)f(t)dt, \quad x > 0, \quad f \in E, \quad \mathcal{K}: E \to E.$$  \hspace{1cm} (2.1)

It is known (see [1, §1, Theorem 1.1]) that given condition (1.2) the operator $I - \mathcal{K}$ permits the following volteryan factorization

$$I - \mathcal{K} = (I - V_-)(I - V_+)$$ \hspace{1cm} (2.2)

as an equality of operators acting in space $E$. Here

$$(V_-f)(x) = \int_x^{+\infty} v_-(t-x)f(t)dt, \quad (V_+f)(x) = \int_0^x v_+(x-t)f(t)dt,$$  \hspace{1cm} (2.3)

where $0 \leq v_\pm \in L_1(0, +\infty)$, and

$$\gamma_- = \int_0^{+\infty} v_-(x)dx = 1, \quad \gamma_+ = \int_0^{+\infty} v_+(x)dx < 1.$$ \hspace{1cm} (2.4)

The existence of the solution of the corresponding linear equation

$$S(x) = \int_0^{+\infty} K(x-t)S(t)dt, \quad x > 0$$ \hspace{1cm} (2.5)

was proved in [3]. Using factorization (2.2), it was proved that the problem (2.5), such that (1.2) holds, has a positive solution, possessing the following properties (see [1, §3, p. 188]):

(a) $1 \leq S(x) \leq (1 - \gamma_+)^{-1}, \quad x > 0,$
(b) $S(x) \uparrow$ by $x$ on $[0, +\infty)$, i.e. $S(x)$ is increasing on $[0, +\infty)$,
(c) $\lim_{x \to +\infty} S(x) = (1 - \gamma_+)^{-1}.$
3. BASIC RESULT

We introduce the following iteration for equation (1.1):
\[ \varphi_{n+1}(x) = \int_{0}^{+\infty} K(x-t)\varphi_n^{\alpha}(t)dt, \quad x > 0, \quad \alpha \in (0, 1), \quad n = 0, 1, 2, \ldots, \]  
\[ \varphi_0(x) \equiv 1, \quad x > 0. \]  
(3.1)

By induction, it is easy to check that the following statements are true:

1. \( \varphi_{n+1}(x) \) is increasing by \( x \),
2. \( \varphi_{n+1}(x) \geq (1 - \gamma_+)S(x), \) \( n = 0, 1, 2, \ldots \)
3. \( \varphi_{n+1}(x) \) is increasing by \( x \) on \( [0, +\infty) \), \( n = 0, 1, 2, \ldots \)

For example, we prove (2) and (3). When \( n = 0 \), inequality (2) immediately follows from the double inequality \( 1 \leq S(x) \leq (1 - \gamma_+)S(x) \). Assuming that \( \varphi_n(x) \geq (1 - \gamma_+)S(x) \), we have

\[ \varphi_{n+1}(x) \geq (1 - \gamma_+)\int_{0}^{+\infty} K(x-t)S^{\alpha}(t)dt \geq (1 - \gamma_+)\int_{0}^{+\infty} K(x-t)S(t)dt = (1 - \gamma_+)S(x), \]

because \( \alpha \in (0, 1) \) and \( 0 < (1 - \gamma_+)S(x) \leq 1 \).

Now we prove statement (3). Let \( x_1, x_2 \in [0, +\infty) \) be arbitrary numbers such that \( x_1 > x_2 \). We may rewrite iteration (3.1) in the following form:

\[ \varphi_{n+1}(x) = \int_{-\infty}^{x} K(\tau)\varphi_n^{\alpha}(x - \tau)d\tau, \quad n = 0, 1, 2, \ldots, \quad \varphi_0(x) \equiv 1, \]

It is obvious that \( \varphi_0(x) \) is increasing by \( x \). Assuming that \( \varphi_n(x) \) is an increasing function by \( x \) we have

\[ \varphi_{n+1}(x_1) - \varphi_{n+1}(x_2) = \int_{-\infty}^{x_1} K(t)\varphi_n^{\alpha}(x_1 - t)dt - \int_{-\infty}^{x_2} K(t)\varphi_n^{\alpha}(x_2 - t)dt \geq \]
\[ \geq \int_{x_1}^{x_2} K(t)\varphi_n^{\alpha}(x_2 - t)dt - \int_{-\infty}^{x_1} K(t)\varphi_n^{\alpha}(x_2 - t)dt = \]

\[ = \int_{x_1}^{x_2} K(t)\varphi_n^{\alpha}(x_2 - t)dt \geq 0. \]

We proved that (3) holds.

It follows from (1) and (2) that the sequence of functions \( \{\varphi_n(x)\}_{n=0}^{\infty} \) has the pointwise limit

\[ \lim_{n \to \infty} \varphi_n(x) = \varphi(x) \leq 1. \]  
(3.2)
From B. Levi’s theorem (see [2]) we deduce that the limit function satisfies equation (1.1). It follows from (3.3) that
\[ \varphi(x) \uparrow \text{ by } x \text{ on } (0, +\infty). \]
(3.3)

Taking into account \( j_2 \) and (3.2) we obtain the following double inequalities:
\[ 1 - \gamma_+ \leq (1 - \gamma_+)S(x) \leq \varphi(x) \leq 1, \]
(3.4)
\[ \lim_{x \to +\infty} \varphi(x) = 1. \]
(3.5)

Now we prove that if
\[ 0 < \gamma_+ < 1 - \frac{1}{e}, \]
(3.6)
then \( \varphi \in C[0, +\infty) \) and a solution of equation (1.1) in the following class of functions
\[ M = \{ f \in M[0, +\infty) : f(x) \geq 1 - \gamma_+ \text{ for all } x \in [0, +\infty) \} \]
(3.7)
is unique.

First we show the continuity of the obtained solution assuming that condition (3.6) is fulfilled. By induction, we show that the following inequality holds
\[ |\varphi_{n+1}(x) - \varphi_n(x)| \leq (\alpha e^{1-\alpha})^n, \quad n = 0, 1, 2, \ldots. \]
(3.8)

In the case of \( n = 0 \) the inequality is obvious, because
\[ |\varphi_1(x) - \varphi_0(x)| = 1 - \int_{-\infty}^{x} K(\tau) d\tau \leq 1. \]

Assume that (3.8) is true for any \( n = p \in \mathbb{N} \). Taking into account the inequality
\[ |x_1^\alpha - x_2^\alpha| \leq \alpha \left( \frac{1}{1 - \gamma_+} \right)^{1-\alpha} |x_1 - x_2| \text{ for all } x_1, x_2 \in [1 - \gamma_+, +\infty) \]
(3.9)
we obtain from (3.1) that
\[ |\varphi_{p+2}(x) - \varphi_{p+1}(x)| \leq \int_{0}^{+\infty} K(x-t)|\varphi_{p+1}^\alpha(t) - \varphi_p^\alpha(t)|dt \leq \]
\[ \leq \alpha \left( \frac{1}{1 - \gamma_+} \right)^{1-\alpha} \int_{0}^{+\infty} K(x-t)|\varphi_{p+1}(t) - \varphi_p(t)|dt \leq \]
\[ \leq \alpha \left( \frac{1}{1 - \gamma_+} \right)^{1-\alpha} \alpha^p e^{-\alpha p} \int_{-\infty}^{x} K(\tau) d\tau \leq \alpha^{(p+1)} e^{(1-\alpha)(p+1)}. \]
As $e^{\alpha-1} > \alpha$, $\alpha \in (0, 1)$, then $q = \alpha e^{1-\alpha} \in (0, 1)$. Therefore, in accordance with the Weierstrass theorem, from (3.8) it follows that the convergence of sequences of functions $\{\varphi_n(x)\}_{n=0}^\infty$ is uniform. By induction, the reader may easily convince himself that $\varphi_n(x) \in C[0, +\infty)$. Thus, from the Dini inverse theorem it follows that the limit function $\varphi$ is continuous.

Now we prove uniqueness of a solution of equation (1.1) in the class $\mathfrak{M}$. We assume that equation (1.1) has two different solutions $\varphi$ and $\varphi^*$, which belong to $\mathfrak{M}$. Then from (1.1), (3.6) and (3.9) we have

$$|\varphi(x) - \varphi^*(x)| \leq \alpha e^{1-\alpha} \int_0^\infty K(x - t)|\varphi(t) - \varphi^*(t)|dt.$$  \hspace{1cm} (3.10)

We set 

$$\delta = \sup_{x \in \mathbb{R}^+} |\varphi(x) - \varphi^*(x)|$$

Then from (3.10) we infer that $\delta \leq q\delta$ or $\delta = 0$. Therefore, $\varphi(x) = \varphi^*(x)$. In this way we prove that the following theorem holds.

**Theorem 3.1.** Assume that condition (1.2) is fulfilled. Then equation (1.1) has a positive, monotonic increasing and bounded solution $\varphi(x)$ such that $\lim_{x \to +\infty} \varphi(x) = 1$. Moreover, if condition (3.6) holds then the obtained solution is continuous and unique in the class $\mathfrak{M}$.

**Example 3.2.** Assume that $K(x)$ has the following form:

$$K(x) = \begin{cases} \beta e^{-x}, & x > 0 \\ (1 - \beta)e^x, & x < 0 \end{cases}, \quad \beta \in \left(0, \frac{1}{2}\right).$$  \hspace{1cm} (3.11)

Opening brackets in (2.2), from operator equality we come to Yengibaryan’s nonlinear factorization equation (see [1]).

$$v_\pm(x) = K(\pm x) + \int_0^\infty v_\pm(t)v_\pm(x + t)dt, \quad x > 0.$$  \hspace{1cm} (3.12)

From (3.11) and (3.12) it follows that $v_+ = 2\beta e^{-x}$ ($x > 0$), $v_- = e^x$ ($x < 0$), i.e. $\gamma_+ = 2\beta$, $\gamma_- = 1$. If $\beta \in \left(0, \frac{1}{2}\left(1 - \frac{1}{\alpha}\right)\right)$, then both conditions (1.2) and (3.6) are fulfilled. Equation (1.1) with kernel (3.11) is reduced to the following ordinary differential equation

$$\varphi''(x) + (1 - 2\beta)\alpha\varphi^{a-1}(x)\varphi'(x) - \varphi(x) = 0.$$  \hspace{1cm} (3.13)

From the proof it follows that equation (3.13) possesses positive, bounded and monotonic increasing solution, which tends to 1 when $x \to +\infty$.

**Remark 3.3.** It should be noted that if we assume a weaker condition $0 < \gamma_+ < (1 - \frac{1}{\alpha})^{-\frac{1}{\alpha}}$ instead of (3.6) then the assertion of the theorem remains true.
Acknowledgments
We express our deep gratitude to professor N.B. Yengibaryan for discussion and referee for useful remarks.

REFERENCES


Aghavard Khachatryan
Aghavard@hotbox.ru

National Academy of Sciences Republic of Armenia
Institute of Mathematics
Marshal Bagramyan str. 24b
0019 Yerevan, Republic of Armenia

Khachatur Khachatryan
Khach82@rambler.ru

National Academy of Sciences Republic of Armenia
Institute of Mathematics
Marshal Bagramyan str. 24b
0019 Yerevan, Republic of Armenia

Received: May 6, 2010.
Accepted: October 4, 2010.