NEIGHBOURHOOD TOTAL DOMINATION
IN GRAPHS

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Abstract. Let $G = (V, E)$ be a graph without isolated vertices. A dominating set $S$ of $G$ is called a neighbourhood total dominating set (ntd-set) if the induced subgraph $\langle N(S) \rangle$ has no isolated vertices. The minimum cardinality of a ntd-set of $G$ is called the neighbourhood total domination number of $G$ and is denoted by $\gamma_{nt}(G)$. The maximum order of a partition of $V$ into ntd-sets is called the neighbourhood total domatic number of $G$ and is denoted by $d_{nt}(G)$. In this paper we initiate a study of these parameters.

Keywords: neighbourhood total domination, total domination, connected domination, paired domination, neighbourhood total domatic number.

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1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of $G$ are denoted by $n$ and $m$ respectively. For graph theoretic terminology we refer to Chartrand and Lesniak [3].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighbourhood and the closed neighbourhood of $v$ are denoted by $N(v)$ and $N[v] = N(v) \cup \{v\}$ respectively. If $S \subseteq V$, then $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$. If $S \subseteq V$ and $u \in S$, then the private neighbour set of $u$ with respect to $S$ is defined by $pn[u, S] = \{v : N[v] \cap S = \{u\}\}$.

A subset $S$ of $V$ is called a dominating set of $G$ if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of $G$ is called the domination number (upper domination number) of $G$ and is denoted by $\gamma(G)$ ($\Gamma(G)$). An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [6]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [7].

Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the Appendix of Haynes et al. [6].
Sampathkumar and Walikar [9] introduced the concept of connected domination in graphs. A dominating set $S$ of a connected graph $G$ is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set of $G$ is called the connected domination number of $G$ and is denoted by $\gamma_c(G)$. Cockayne et al. [4] introduced the concept of total domination in graphs. A dominating set $S$ of a graph $G$ without isolated vertices is called a total dominating set of $G$ if $\langle S \rangle$ has no isolated vertices. The minimum cardinality of a total dominating set of $G$ is called the total domination number of $G$ and is denoted by $\gamma_t(G)$. Haynes and Slater [5] introduced the concept of paired domination in graphs. A dominating set $S$ of a graph $G$ without isolated vertices is called a paired dominating set if $\langle S \rangle$ has a perfect matching. The minimum cardinality of a paired dominating set of $G$ is called the paired domination number of $G$ and is denoted by $\gamma_{pr}(G)$.

For a dominating set $S$ of $G$ it is natural to look at how $N(S)$ behaves. For example, for the cycle $C_6 = (v_1, v_2, v_3, v_4, v_5, v_6, v_1)$, $S_1 = \{v_1, v_4\}$ and $S_2 = \{v_1, v_2, v_4\}$ are dominating sets, $\langle N(S_1) \rangle$ is not connected and $\langle N(S_2) \rangle$ is connected. Motivated by this example, in [1] we have introduced the concept of neighbourhood connected domination in graphs.

**Definition 1.1 ([1]).** A dominating set $S$ of a connected graph $G$ is called a neighbourhood connected dominating set (ncd-set) if the induced subgraph $\langle N(S) \rangle$ is connected. A ncd-set $S$ is said to be minimal if no proper subset of $S$ is a ncd-set. The minimum cardinality of a ncd-set of $G$ is called the neighbourhood connected domination number of $G$ and is denoted by $\gamma_{nc}(G)$.

For the path $P_{10} = (v_1, v_2, \ldots, v_{10})$, $S_1 = \{v_2, v_5, v_7, v_9\}$ and $S_2 = \{v_1, v_4, v_6, v_7, v_{10}\}$ are dominating sets, $\langle N(S_1) \rangle$ has isolates and $\langle N(S_2) \rangle$ has no isolates. Motivated by this example, in this paper we introduce the concept of neighbourhood total domination and initiate a study of neighbourhood total domination number and neighbourhood total domatic number.

We need the following theorems.

**Theorem 1.2 ([8]).** Let $G$ be a nontrivial connected graph. Then $\gamma_c(G) + \kappa(G) = n$ if and only if $G = C_n$ or $K_n$ or $K_{2a} - X$ where $a \geq 3$ and $X$ is a 1-factor of $K_{2a}$.

**Theorem 1.3 ([1]).** Let $G$ be any graph such that both $G$ and $\overline{G}$ are connected. Then

$$\gamma_{nc}(G) + \gamma_{nc}(\overline{G}) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2 & \text{if diam } G \geq 3, \\ \left\lceil \frac{n}{2} \right\rceil + 3 & \text{if diam } G = 2. \end{cases}$$

**Theorem 1.4 ([1]).** Let $T$ be any tree with $n > 2$. Then $\gamma_{nc}(T) = n - \Delta$ if and only if $T$ can be obtained from a star by subdividing $k$ of its edges, $k \geq 1$, once or by subdividing exactly one edge twice.
2. MAIN RESULTS

We assume throughout that $G$ is a graph without isolated vertices.

**Definition 2.1.** A dominating set $S$ of a graph $G$ is called a neighbourhood total dominating set (ntd-set) if the induced subgraph $\langle N(S) \rangle$ contains no isolated vertices. A ntd-set $S$ is said to be minimal if no proper subset of $S$ is a ntd-set. The minimum cardinality of a ntd-set of $G$ is called the neighbourhood total domination number of $G$ and is denoted by $\gamma_{nt}(G)$.

**Remark 2.2.** (i) Let $S$ be a ntd-set of $G$. Since $\langle N(S) \rangle$ has no isolated vertices, it follows that $|N(S)| \geq 2$.

(ii) Clearly $\gamma_{nt} \geq \gamma$. Further if $S$ is a total dominating set or a paired dominating set or a connected dominating set with $|S| > 1$, then $N(S) = V$ and hence $\gamma_{nt} \leq \gamma_r, \gamma_{nt} \leq \gamma_{pr}$ and $\gamma_{nt} \leq \gamma_c$ if $\gamma_c > 1$.

(iii) For any connected graph $G, \gamma_{nt} = 1$ if and only if there exists a vertex $v \in V(G)$ such that $deg v = n - 1$ and $G - v$ has no isolated vertices.

**Theorem 2.3.** For any connected graph $G$, $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_{nc}(G) \leq 2\gamma(G)$. Further given three positive integers $a, b$ and $c$ with $a \leq b \leq c \leq 2a$, there exists a graph $G$ with $\gamma(G) = a, \gamma_{nt}(G) = b$ and $\gamma_{nc}(G) = c$.

**Proof.** We have $\gamma(G) \leq \gamma_{nt}(G) \leq \gamma_{nc}(G) \leq \gamma_{pr}(G) \leq 2\gamma(G)$. Now, let $a, b$ and $c$ be positive integers with $a \leq b \leq c \leq 2a$. Let $b = a + r, 0 \leq r \leq a, c = a + k, r \leq k \leq 2a - r$. Consider the corona $K_a \circ K_1$ with $V(K_a) = \{v_1, v_2, \ldots, v_n\}$ and let $u_i$ be the pendant vertex adjacent to $v_i$. Take $r$ copies $H_1, H_2, \ldots, H_r$ of $K_2$ and $k - r$ copies $G_{r+1}, G_{r+2}, \ldots, G_k$ of $P_4$. Let $G$ be the graph obtained from $K_a \circ K_1$ by joining $u_i$ to all the vertices of $H_i$ where $1 \leq i \leq r$ and by joining $u_{r+j}$ to all the vertices of $G_{r+j}$ where $1 \leq j \leq k - r$. Then $\gamma(G) = a, \gamma_{nt}(G) = a + r = b$ and $\gamma_{nc}(G) = a + k = c$. □

**Theorem 2.4.** For the path $P_n$,

$$\gamma_{nt}(P_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1 \pmod{3}, \\
\left\lceil \frac{n}{3} \right\rceil + 1 & \text{otherwise}. \end{cases}$$

**Proof.** Let $P_n = (v_1, v_2, \ldots, v_n)$. If $n \equiv 1 \pmod{3}$, then $S = \{v_i : i = 3k + 1, k = 0,1,2,\ldots\}$ is a ntd-set of $P_n$. If $n \equiv 2 \pmod{3}$, then $S \cup \{v_n\}$ is a ntd-set of $P_n$. If $n \equiv 0 \pmod{3}$, then $S \cup \{v_{n-1}\}$ is a ntd-set of $P_n$. Hence

$$\gamma_{nt}(P_n) \leq \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \equiv 1 \pmod{3}, \\
\left\lceil \frac{n}{3} \right\rceil + 1 & \text{otherwise}. \end{cases}$$

Now, $\gamma_{nt}(P_n) \geq \gamma(P_n) = \left\lceil \frac{n}{3} \right\rceil$. Further if $n \not\equiv 1 \pmod{3}$, then for any $\gamma$-set $S$ of $P_n$, $\langle N(S) \rangle$ has at least one isolated vertex and hence $\gamma_{nt}(P_n) \geq \left\lceil \frac{n}{3} \right\rceil + 1$. Hence the result follows. □
Corollary 2.5. For any nontrivial path $P_n$,

(i) $\gamma_{nt}(P_n) = \gamma(P_n)$ if and only if $n \equiv 1 \pmod{3}$.
(ii) $\gamma_{nt}(P_n) = \gamma_c(P_n)$ if and only if $n = 4$ or $5$.
(iii) $\gamma_{nt}(P_n) = \gamma_t(P_n)$ if and only if $n = 2, 3, 4, 5$ or $8$.
(iv) $\gamma_{nt}(P_n) = \gamma_{nc}(P_n)$ if and only if $n = 3, 4, 5, 6$ or $8$.

Proof. Since $\gamma(P_n) = \lceil \frac{n}{3} \rceil$, $\gamma_c(P_n) = n - 2$,

$$\gamma_t(P_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 0 \pmod{4}, \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{otherwise} \end{cases}$$

and $\gamma_{nc}(P_n) = \left\lceil \frac{n}{3} \right\rceil$ the corollary follows.

Theorem 2.6. For the cycle $C_n$,

$$\gamma_{nt}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 2 \pmod{3}, \\ \left\lceil \frac{n}{3} \right\rceil & \text{otherwise}. \end{cases}$$

Proof. Let $C_n = (v_1, v_2, \ldots, v_n, v_1)$ and $n = 3k + r$, where $0 \leq r \leq 2$.

Let $S = \{v_i : i = 3j + 1, 0 \leq j \leq k\}$.

Let $S_1 = \begin{cases} S \cup \{v_n\} & \text{if } n \equiv 2 \pmod{3}, \\ S & \text{otherwise}. \end{cases}$

Then $S_1$ is a ndt-set of $C_n$ and hence

$$\gamma_{nt}(C_n) \leq \begin{cases} \left\lceil \frac{n}{3} \right\rceil + 1 & \text{if } n \equiv 2 \pmod{3}, \\ \left\lceil \frac{n}{3} \right\rceil & \text{otherwise}. \end{cases}$$

Now, $\gamma_{nt}(C_n) \geq \gamma(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$. Further if $n \equiv 2 \pmod{3}$, then for any $\gamma$-set of $S$ of $C_n$, $\langle N(S) \rangle$ has at least one isolated vertex and hence $\gamma_{nt}(C_n) \geq \left\lceil \frac{n}{3} \right\rceil + 1$. Hence the result follows.

Corollary 2.7. (i) $\gamma_{nt}(C_n) = \gamma(C_n)$ if and only if $n \not\equiv 2 \pmod{3}$.
(ii) $\gamma_{nt}(C_n) = \gamma_c(C_n)$ if and only if $n = 3, 4$ or $5$.
(iii) $\gamma_{nt}(C_n) = \gamma_t(C_n)$ if and only if $n = 4, 5$ or $8$.
(iv) $\gamma_{nt}(C_n) = \gamma_{nc}(C_n)$ if and only if $n = 3, 4, 5$ or $7$.

Proof. Since $\gamma(C_n) = \left\lceil \frac{n}{3} \right\rceil$, $\gamma_c(C_n) = n - 2$,

$$\gamma_t(C_n) = \begin{cases} \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4}, \\ \left\lfloor \frac{n}{2} \right\rfloor & \text{otherwise}, \end{cases}$$

and

$$\gamma_{nc}(C_n) = \begin{cases} \left\lceil \frac{n}{3} \right\rceil & \text{if } n \not\equiv 3 \pmod{4}, \\ \left\lceil \frac{n}{4} \right\rceil & \text{if } n \equiv 3 \pmod{4} \end{cases}$$

the result follows.
We now proceed to obtain a characterization of minimal ntd-sets.

**Lemma 2.8.** A superset of a ntd-set is a ntd-set.

**Proof.** Let $S$ be a ntd-set of a graph $G$ and let $S_1 = S \cup \{v\}$, where $v \in V - S$. Clearly, $v \in N(S_1)$ and $S_1$ is a dominating set of $G$. Suppose there exists an isolated vertex $y$ in $N(S_1)$. Then $N(y) \subseteq S - N(S)$ and hence $y$ is an isolated vertex in $N(S)$, which is a contradiction. Hence $N(S_1)$ has no isolated vertices and $S_1$ is a ntd-set.

**Theorem 2.9.** A ntd-set $S$ of a graph $G$ is a minimal ntd-set if and only if for every $u \in S$, one of the following holds:

1. $pm[u, S] \neq \emptyset$.
2. There exists a vertex $x \in N(S - \{u\})$ such that $N(x) \cap N(S - \{u\}) = \emptyset$.

**Proof.** Let $S$ be a minimal ntd-set of $G$. Let $u \in S$. Then either $S - \{u\}$ is not a dominating set of $G$ or $S - \{u\}$ is a dominating set and $N(S - \{u\})$ has an isolated vertex. If $S - \{u\}$ is not a dominating set of $G$, then $pm[u, S] \neq \emptyset$. If $S - \{u\}$ is a dominating set and if $x \in N(S - \{u\})$ is an isolated vertex in $N(S - \{u\})$, then $N(x) \cap N(S - \{u\}) = \emptyset$. Conversely, if $S$ is a ntd-set of $G$ satisfying the conditions of the theorem, then $S$ is a 1-minimal ntd-set and hence the result follows from Lemma 2.8.

**Remark 2.10.** Let $G$ be a graph with $\Delta = n - 1$. Then $\gamma_{nt}(G) = 1$ or 2. Further $\gamma_{nt}(G) = 2$ if and only if $G$ has exactly one vertex $v$ with $\deg v = n - 1$ and $v$ is adjacent to a vertex of degree 1. (A vertex which is adjacent to a vertex of degree 1 is called a support vertex).

**Remark 2.11.** Since any ntd-set of a spanning subgraph $H$ of a graph $G$ is a ntd-set of $G$, we have $\gamma_{nt}(G) \leq \gamma_{nt}(H)$.

**Remark 2.12.** If $G$ is a disconnected graph with $k$ components $G_1, G_2, \ldots, G_k$ then $\gamma_{nt}(G) = \gamma_{nt}(G_1) + \gamma_{nt}(G_2) + \cdots + \gamma_{nt}(G_k)$.

We now proceed to obtain bounds for $\gamma_{nt}$.

**Observation 2.13.** For any graph $G$, $\gamma_{nt}(G) = n$ if and only if $G = mK_2$.

**Theorem 2.14.** For any graph $G$, $\gamma_{nt}(G) \leq n - \Delta + 1$. Further, $\gamma_{nt}(G) = n - \Delta + 1$ if and only if $G$ is isomorphic to $H$ or $sK_2 \cup H$ where $H$ is any graph having a support vertex $v$ with $\deg v = |V(H)| - 1$.

**Proof.** Let $v \in V(G)$ and $\deg v = \Delta$. Let $S = N(v) - \{u\}$ where $u \in N(v)$. Then $V - S$ is a ntd-set of $G$ and hence $\gamma_{nt}(G) \leq n - \Delta + 1$.

Now, let $G$ be any graph with $\gamma_{nt}(G) = n - \Delta + 1$. Case i. $G$ is connected.

If $\Delta < n - 1$, then $V - S$ where $S = (N(v) - \{u\}) \cup \{w\}$, $u \in N(v)$, $w \notin N[v]$, is a ntd-set of $G$ with $|V - S| = n - \Delta$ which is a contradiction. Hence $\Delta = n - 1$ and $\deg v = n - 1$. If $n = 2$, then $H = K_2$. Suppose $n \geq 3$. If $\deg u \geq 2$ for all $u \in N(v)$,
Theorem 2.18. Let $G$ be disconnected.

Let $G_1, G_2, \ldots, G_k$ be the components of $G$ and let $|V(G_i)| = n_i$. If $\Delta = 1$, then $\gamma_{nt}(G) = n = \gamma_{nt}(G_i)$ is isomorphic to $K_2$. Suppose $\Delta \geq 2$. Let $v \in V(G_1)$ be such that $\deg v = \Delta$. Since $\gamma_{nt}(G) = n - \Delta + 1$ it follows that $\gamma_{nt}(G_1) = n_1 - \Delta + 1$ and $\gamma_{nt}(G_i) = n_i$ for all $i \geq 2$. Hence by Case i, $G_1$ is isomorphic to $K_2$ and $H$ is any graph having a support vertex $v$ with $\deg v = |V(H)| - 1$ and $G_i$ is isomorphic to $K_2$ for all $i \geq 2$.

Theorem 2.15. Let $G$ be a connected graph with $\Delta < n - 1$. Then $\gamma_{nt}(G) \leq n - \Delta$. Further, for a tree $T$ with $\Delta < n - 1$ the following are equivalent.

(i) $\gamma_{nt}(T) = n - \Delta$.
(ii) $\gamma_{nc}(T) = n - \Delta$.
(iii) $T$ can be obtained from a star by subdividing $k$ of its edges, $k \geq 1$ once or by subdividing exactly one edge twice.

Proof. Let $v \in V(G)$ and $\deg v = \Delta$. Since $G$ is connected and $\Delta < n - 1$, there exist two adjacent vertices $u$ and $w$ such that $u \in N(v)$ and $w \notin N[v]$. Let $S = (N(v) - \{u\}) \cup \{w\}$. Then $V - S$ is a ntd-set of $G$ and hence $\gamma_{nt}(G) \leq n - \Delta$.

Now, let $T$ be a tree with $\Delta < n - 1$. Suppose $\gamma_{nt}(T) = n - \Delta$. Then $n - \Delta = \gamma_{nt}(T) \leq \gamma_{nc}(T) \leq n - \Delta$. Hence $\gamma_{nc}(T) = n - \Delta$, so that (i) implies (ii).

It follows from Theorem 1.4 that (ii) implies (iii). We now prove (iii) implies (i). Consider the star $K_{1, \Delta}$, where $V(K_{1, \Delta}) = \{v, v_1, v_2, \ldots, v_\Delta\}$ with $\deg v = \Delta$.

Case i. $T$ is obtained from $K_{1, \Delta}$ by subdividing the $k$ edges $vv_1, vv_2, \ldots, vv_k$. Let $u_i$ be the vertex subdividing $vv_i$, $1 \leq i \leq k$. Clearly, $n - \Delta = k + 1$. Also any ntd-set $S$ of $T$ contains either $u_i$ or $v_i$ for each $i, 1 \leq i \leq k$ and also contains the vertex $v$. Hence it follows that $|S| \geq k + 1 = n - \Delta$ and $\gamma_{nt}(T) = n - \Delta$.

Case ii. $T$ is obtained from $K_{1, \Delta}$ by subdividing the edge $vv_1$ twice.

Let $u_1, u_2$ be the vertices subdividing $vv_1$. Then $n - \Delta = 3$ and $S = \{v, u_1, u_2\}$ is a minimum ntd-set of $T$. Thus $\gamma_{nt}(T) = n - \Delta$.

Corollary 2.16. For a forest $G$, $\gamma_{nt}(G) = n - \Delta$ if and only if $G$ is isomorphic to $K_2 \cup T$, where $T$ is a tree with $\gamma_{nt}(T) = |V(T)| - \Delta(T)$.

Theorem 2.17. For each $\gamma_{nt}$-set $S$ of a connected graph $G$, let $t_S$ denote the number of vertices $v$ such that $v$ is not a pendant vertex of $G$ and $v$ is isolated in $\langle S \rangle$. Let $t = \min\{t_S : S$ is a $\gamma_{nt}$-set of $G\}$. Then $\gamma_{nc}(G) \leq \gamma_{nt}(G) + t$.

Proof. Let $S$ be a $\gamma_{nt}$-set of $G$ such that the number of vertices in $S$ which are non-adjacent vertices of $G$ and are isolated in $\langle S \rangle$ is $t$.

Let $X = \{v \in S : d(v) = 0 \text{ in } \langle S \rangle \text{ and } d(v) > 1 \text{ in } G\}$ so that $|X| = t$. For each $v \in X$, choose a vertex $f(v) \in V(G)$ which is adjacent to $v$. Then $S_1 = S \cup \{f(v) : v \in X\}$ is a ncd-set of $G$ and hence $\gamma_{nc}(G) \leq |S_1| \leq \gamma_{nt}(G) + t$.

Theorem 2.18. Let $G$ be a connected graph with $\text{diam} G = 2$. Then $\gamma_{nt}(G) \leq 1 + \delta(G)$ and the bound is sharp.
Theorem 2.19. Let $G$ be a connected graph with $diam G = 2$ and $\gamma_{nt}(G) = 1 + \delta(G)$. Then for every vertex $v \in V(G)$ with $deg v = \delta(G)$, $N(v)$ is an independent set and for all $u \in N(v)$ there exists a vertex $w \notin N(v)$ such that $w$ is adjacent only to $u$.

Proof. Let $S_1 = N(v)$. Clearly $S_1$ is a dominating set of $G$. Now, suppose $N(v)$ is not an independent set. Then $\langle N(v) \rangle$ contains an edge $e = xy$. Hence $v$ is not isolated in $\langle N(S_1) \rangle$ and since $diam G = 2$, every vertex $w \notin N[v]$ is adjacent to either $x$ or a neighbour of $x$. Thus $w$ is not isolated in $\langle N(S_1) \rangle$. Hence $S_1$ is a ntd-set of $G$ and $\gamma_{nt}(G) \leq \delta(G)$ which is a contradiction. Thus $N(v)$ is an independent set.

Now, suppose there exists a vertex $u \in N(v)$ such that $u$ has no private neighbour in $V - N[v]$. Then $N[v] - \{u\}$ is a ntd-set of $G$ with cardinality $\delta(G)$ which is a contradiction. Hence the result follows.

Remark 2.20. The converse of Theorem 2.19 is not true. Consider the graph $G$ given in Figure 1.

![Fig. 1](image)

Here $\delta(G) = 2$ and $\gamma_{nt}(G) = 2$. However, the unique vertex $v$ with $deg v = \delta = 2$ satisfies the conditions given in Theorem 2.19.

Theorem 2.21. Let $G$ be a graph such that both $G$ and $\overline{G}$ have no isolated vertices. Then $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 2$. Further, equality holds if and only if $G$ or $\overline{G}$ is isomorphic to $sK_2$, where $s > 1$.

Proof. If $G$ and $\overline{G}$ are both connected, then $\gamma_{nt}(G) \leq \gamma_{nt}(\overline{G}) \leq \left\lceil \frac{n}{2} \right\rceil$ and $\gamma_{nt}(\overline{G}) \leq \left\lceil \frac{n}{2} \right\rceil$, so that $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 1$.

If $G$ is disconnected, then $\gamma_{nt}(G) = 2$ and hence $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq n + 2$.

Now, let $G$ be any graph with $\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) = n + 2$. Then $G$ or $\overline{G}$ is disconnected. Suppose $G$ is disconnected. Then $\gamma_{nt}(G) = n$ and $\gamma_{nt}(\overline{G}) = 2$ and hence $G$ is isomorphic to $sK_2$ where $s > 1$. The converse is obvious.

The bound given by Theorem 2.21 can be substantially improved when $G$ and $\overline{G}$ are both connected, as shown in the following theorem.

Theorem 2.22. Let $G$ be any graph such that both $G$ and $\overline{G}$ are connected. Then

$$\gamma_{nt}(G) + \gamma_{nt}(\overline{G}) \leq \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 2 & \text{if } diam G \geq 3, \\ \left\lceil \frac{n}{2} \right\rceil + 3 & \text{if } diam G = 2. \end{cases}$$
Proof. Since $\gamma_{nt} \leq \gamma_{nc}$ the result follows from Theorem 1.3.

Remark 2.23. The bounds given in Theorem 2.22 are sharp. The graph $G = C_5$ has diameter 2: $\gamma_{nt}(G) = \gamma_{nt}(G) = 3$ and $\gamma_{nt}(G) + \gamma_{nt}(G) = 6 = \left\lceil \frac{n}{2} \right\rceil + 3$. For the graph $G = C_k \circ K_1$ diam $G \geq 3$ and $\gamma_{nt}(G) + \gamma_{nt}(G) = \left\lceil \frac{n}{2} \right\rceil + 2$.

Problem 2.24. Characterize graphs which attain the bounds given in Theorem 2.22.

Theorem 2.25. For any connected graph $G$, $\gamma_{nt}(G) + \kappa(G) \leq n - \Delta + \delta + 1$ and equality holds if and only if $G$ contains a support vertex $v$ with $\deg v = n - 1$.

Proof. We have $\gamma_{nt} \leq n - \Delta + 1$ and $\kappa \leq \delta$. Hence $\gamma_{nt} + \kappa \leq n - \Delta + \delta + 1$.

Let $G$ be a connected graph and let $\gamma_{nt}(G) + \kappa(G) = n - \Delta + \delta + 1$. Then $\gamma_{nt}(G) = n - \Delta + 1$ and $\kappa = \delta$ and the result follows from Theorem 2.14.

Theorem 2.26. For any graph $G$, $\gamma_{nt}(G) + \kappa(G) = n$ if and only if $G$ is isomorphic to one of the graphs $sK_2$, $s > 1$, $P_3$ or $C_5$ or $K_n$ or $K_{2a} - X$, $a \geq 3$ and $X$ is a $1$-factor of $K_{2a}$.

Proof. Let $G$ be a graph with $\gamma_{nt}(G) + \kappa(G) = n$.

Case i. $G$ is connected.

Suppose $\Delta = n - 1$. Then $\gamma_{nt} = 1$ or 2. If $\gamma_{nt} = 1$, then $\kappa = n - 1$ and hence $G$ is isomorphic to $K_n$. If $\gamma_{nt} = 2$ then $G$ contains a support vertex of degree $n - 1$ and hence $\kappa = n$, $n = 3$. Hence $G$ is isomorphic to $P_3$.

Suppose $\Delta < n - 1$. Then $\gamma_{nt} \leq \gamma_c$ and $\gamma_{nt} + \kappa \leq \gamma_c + \kappa$ so that $\gamma_c + \kappa \geq n$. Since $\gamma_c + \kappa \leq n$ we get $\gamma_c + \kappa = n$ and $\gamma_{nt} = \gamma_c$. Therefore by Theorem 1.2 $G$ is isomorphic to $C_5$ or $K_{2a} - X$ where $X$ is a 1-factor in $K_{2a}$.

Case ii. $G$ is disconnected.

Then $\kappa = 0$. Hence $\gamma_{nt} = n$ so that $G$ is isomorphic to $sK_2$, $s > 1$. The converse is obvious.

3. NEIGHBOURHOOD TOTAL DOMATIC NUMBER

The maximum order of a partition of the vertex set $V$ of a graph $G$ into dominating sets is called the domatic number of $G$ and is denoted by $d(G)$. For a survey of results on domatic number and their variants we refer to Zelinka [10]. In [2] we have initiated a study of the neighbourhood connected domatic number of a graph. In this section we present a few basic results on the neighbourhood total domatic number of a graph.

Definition 3.1. Let $G$ be a graph without isolated vertices. A neighbourhood total domatic partition (nt-domatic partition) of $G$ is a partition $\{V_1, V_2, \ldots, V_k\}$ of $V(G)$ in which each $V_i$ is a ndt-set of $G$. The maximum order of an nt-domatic partition of $G$ is called the neighbourhood total domatic number (nt-domatic number) of $G$ and is denoted by $d_{nt}(G)$.

Observation 3.2. Since any domatic partition of $K_n$, where $n \geq 3$, is also an nt-domatic partition, we have $d_{nt}(K_n) = d(K_n) = n$. Similarly $d_{nt}(K_{r,s}) = d(K_{r,s}) = \min\{r, s\}$. Also for the wheel $W_n$, $d_{nt}(W_n) = d(W_n) = \begin{cases} 4 & \text{if } n \equiv 1(\text{mod } 3), \\ 3 & \text{otherwise}. \end{cases}$
**Observation 3.3.** Since any total domatic partition of $G$ is a nt-domatic partition and any nc-domatic partition is a nt-domatic partition, we have $d_t(G) \leq d_{nc}(G) \leq d_{nt}(G) \leq d(G)$.

**Observation 3.4.** Let $v \in V(G)$ and $\deg v = \delta$. Since any ntd-set of $G$ must contain either $v$ or a neighbour of $v$, it follows that $d_{nt}(G) \leq \delta(G) + 1$.

**Definition 3.5.** A graph $G$ is called nt-dominically full if $d_{nt}(G) = \delta(G) + 1$.

**Example 3.6.** The graph $G$ given in Figure 2 is nt-dominically full. In fact $\{\{v_1\}, \{v_2, v_4, v_6, v_8\}, \{v_3, v_5, v_7, v_9\}\}$ is a nt-domatic partition of $G$ of maximum order and $d_{nt}(G) = 3 = 1 + \delta(G)$.

![Fig. 2. nt-dominically full graph](image)

**Observation 3.7.** Given two positive integers $n$ and $k$ with $n \geq 4$ and $1 \leq k \leq n$, there exists a graph $G$ with $n$ vertices such that $d_{nt}(G) = k$. We take

$$G = \begin{cases} K_n & \text{if } k = n, n \geq 3, \\ K_{1,n-1} & \text{if } k = 1, \\ B(n_1, n - 2 - n_1) & \text{if } k = 2, \\ K_{k-1} + K_{n-k+1} & \text{otherwise.} \end{cases}$$

**Theorem 3.8.** For the path $P_n$, $n \geq 2$, we have

$$d_{nt}(P_n) = \begin{cases} 1 & \text{if } n = 2, 3 \text{ or } 5, \\ 2 & \text{otherwise.} \end{cases}$$

**Proof.** Let $P_n = (v_1, v_2, \ldots, v_n)$. The result is trivial for $n = 2, 3$ or 5. Suppose $n \neq 2, 3, 5$. It follows from Observation 3.4 that $d_{nt}(P_n) \leq 2$.

Now let $S = \{v_i : i \equiv 1(\text{mod } 3)\}$ and let

$$V_1 = \begin{cases} S & \text{if } n \equiv 1(\text{mod } 3), \\ S \cup \{v_{n-2}\} & \text{if } n \equiv 2(\text{mod } 3), \\ S \cup \{v_{n-1}\} & \text{if } n \equiv 0(\text{mod } 3). \end{cases}$$
Then \( \{V_1, V - V_1\} \) is a nt-domatic partition of \( P_n \) and hence \( d_{nt}(P_n) = 2 \).

**Theorem 3.9.** For the cycle \( C_n \) with \( n \geq 4 \) we have

\[
d_{nt}(C_n) = \begin{cases} 1 & \text{if } n = 5, \\ 3 & \text{if } n \equiv 0 \pmod{3}, \\ 2 & \text{otherwise}. \end{cases}
\]

**Proof.** Let \( C_n = (v_0, v_1, \ldots, v_{n-1}, v_0) \). The result is trivial for \( n = 5 \). Suppose \( n \neq 5 \).

It follows from Observation 3.4 that \( d_{nt}(C_n) \leq 3 \). If \( n \equiv 0 \pmod{3} \), let \( n = 3k \) and let \( S_i = \{v_j : 0 \leq j \leq n - 1 \text{ and } j \equiv i \pmod{3}\}, i = 0, 1, 2 \). Then \( \{S_0, S_1, S_2\} \) is a nt-domatic partition of \( C_n \) and hence \( d_{nt}(C_n) = 3 \). Now, suppose \( n \not\equiv 0 \pmod{3} \). Let \( n = 3k + r \) where \( r = 1 \) or \( 2 \).

Let \( S_1 = \{v_i : i \equiv 1 \pmod{3}\} \) if \( n \equiv 1 \pmod{3} \), \( S_2 = \{v_i : i \equiv 2 \pmod{4}\} \) if \( n \equiv 2 \pmod{3} \).

Then \( \{S_1, V - S_1\} \) is a nt-domatic partition of \( C_n \) and hence \( d_{nt}(C_n) \geq 2 \). Also it follows from Theorem 2.6 that \( d_{nt}(C_n) \leq 2 \) and hence \( d_{nt}(C_n) = 2 \).

**Observation 3.10.** If \( \{V_1, V_2, \ldots, V_{d_{nt}}\} \) is a nt-domatic partition of \( G \), then \( |V_i| \geq \gamma_{nt}(G) \) for each \( i \) and hence \( \gamma_{nt}(G)d_{nt}(G) \leq n \).

**Example 3.11.** (i) If \( G \cong sK_r \), \( r \geq 3, s \geq 1 \), then \( d_{nt}(G) = r \) and \( \gamma_{nt}(G) = s \) and hence \( d_{nt}(G)\gamma_{nt}(G) = sr = n \).

(ii) If \( G \cong sK_{r,r} \), \( r \geq 2, s \geq 1 \), then \( d_{nt}(G) = r \), \( \gamma_{nt}(G) = 2s \) and hence \( d_{nt}(G)\gamma_{nt}(G) = 2sr = n \).

(iii) If \( G \cong G_1 \circ K_1 \) where \( G_1 \) is any connected graph, then \( d_{nt}(G) = 2 \) and \( \gamma_{nt}(G) = \frac{n}{2} \) and hence \( d_{nt}(G)\gamma_{nt}(G) = n \).

**Problem 3.12.** Characterize the class of graphs for which \( d_{nt}(G)\gamma_{nt}(G) = n \).

**Theorem 3.13.** Let \( G \) be a graph of order \( n \geq 5 \) with \( \Delta = n - 1 \) and let \( k \) denote the number of vertices of degree \( n - 1 \). Then \( d_{nt}(G) \leq \frac{1}{2}(n + k) \). Further \( d_{nt}(G) = \frac{1}{2}(n + k) \) if and only if \( G \) is isomorphic to \( 2K_{\frac{n-k}{2}} \) or \( H \) is a connected graph with \( V(H) = X_1 \cup X_2 \cup \cdots \cup X_r, r = \frac{n-k}{2}, |X_i| = 2, X_i \cap X_j = \emptyset \) for all \( i \neq j \) and the subgraph induced by the edges of \( H \) with one end in \( X_i \) and the other end in \( X_j \) has a perfect matching.

**Proof.** Let \( \{V_1, V_2, \ldots, V_s\} \) be any nt-domatic partition of \( G \) with \( |V_i| = 1, 1 \leq i \leq k \).

Since \( |V_j| \geq 2 \) for all \( j \) with \( k + 1 \leq j \leq s \), it follows that \( s \leq k + \frac{n-k}{2} = \frac{n+k}{2} \). Hence \( d_{nt}(G) \leq \frac{1}{2}(n + k) \).

Now, let \( G \) be a graph with \( d_{nt}(G) = \frac{1}{2}(n + k) \). Then there exists a nt-domatic partition \( \{V_1, V_2, \ldots, V_k, V_{k+1}, \ldots, V_{\frac{n-k}{2}}\} \) such that \( |V_i| = 1 \) if \( 1 \leq i \leq k \) and \( |V_j| = 2 \) if \( k + 1 \leq j \leq \frac{n-k}{2} \). Clearly, \( \{V_1 \cup V_2 \cup \cdots \cup V_k\} \cong K_k \). Let \( H = \left\langle V_{k+1} \cup \cdots \cup V_{\frac{n-k}{2}} \right\rangle \).

Case i. \( H \) is disconnected.

Since \( |V_j| = 2 \) for all \( j \) with \( k + 1 \leq j \leq \frac{n-k}{2} \), it follows that \( H \) has exactly two components \( H_1, H_2 \) and each \( V_j \) contains one vertex from \( H_1 \) and one vertex from \( H_2 \). Since \( V_j \) is a ntd-set of \( G \), it follows that \( H_1 \) and \( H_2 \) are complete graphs and
Since both $H_1$ and $H_2$ must contain at least two vertices. Hence $n \geq 5$. 

Case ii. $H$ is connected.

Let $X_i = V_{k+1}, 1 \leq i \leq \frac{n-k}{2}$. Then $V(H) = X_1 \cup X_2 \cup \cdots \cup X_r$ and $X_i \cap X_j = \emptyset$ when $i \neq j$. Now, since each $X_i$ is a dominating set of $G$, it follows that the subgraph induced by the edges of $H$ with one end in $X_i$ and the other end in $X_j$ has a perfect matching.

Conversely, suppose $G$ is of the form given in the theorem. Let $u_1, u_2, \ldots, u_k$ be the vertices of $G$ with $\deg u_i = n - 1, 1 \leq i \leq k$.

Suppose $G = K_k + H$ where $H$ is isomorphic to $2K_{\frac{n-k}{2}}$ with $n \geq 5$ when $k = 1$.

Let $H_1$ and $H_2$ be the two components of $H$ with $V(H_1) = \{x_i : k + 1 \leq i \leq \frac{n-k}{2}\}$ and $V(H_2) = \{y_i : k + 1 \leq i \leq \frac{n-k}{2}\}$. Let

$$V_i = \begin{cases} \{u_i\} & \text{if } 1 \leq i \leq k, \\ \{x_i, y_i\} & \text{where } x_i \in V(H_1) \text{ and } y_i \in V(H_2), \text{ if } k + 1 \leq i \leq \frac{n-k}{2}. \end{cases}$$

Then $\{V_1, V_2, \ldots, V_{\frac{n-k}{2}}\}$ is a nt-domatic partition of $G$. Also if $G = K_k + H$, where $H$ is a connected graph satisfying the conditions stated in the theorem, then $\{\{u_1\}, \{u_2\}, \ldots, \{u_k\}, X_1, X_2, \ldots, X_r\}$ is a nt-domatic partition of $G$. Thus $d_{nt}(G) \geq k + r = \frac{n-k}{2}$ and hence $d_{nt}(G) = \frac{n-k}{2}$. \hfill \Box

**Corollary 3.14.** Let $G$ be a graph with $\Delta < n - 1$. Then $d_{nt}(G) \leq \frac{n}{2}$. Further $d_{nt}(G) = \frac{n}{2}$ if and only if $V = X_1 \cup X_2 \cup \cdots \cup X_\frac{n}{2}$, where $|X_i| = 2$ for all $i$, $X_i \cap X_j = \emptyset$ if $i \neq j$, the subgraph induced by the edges of $G$ with one end in $X_i$ and the other end in $X_j$ has a perfect matching and $(V - X_1)$ has no isolated vertex if $X_1$ is independent.

**Theorem 3.15.** Let $G$ be any graph such that both $G$ and $\overline{G}$ are connected. Then $\overline{d_{nt}(G)} + d_{nt}(\overline{G}) \leq n$. Further equality holds if and only if $V(G) = X_1 \cup X_2 \cup \cdots \cup X_\frac{n}{2}$, where $X_i \cap X_j = \emptyset$ and $(X_i \cup X_j)$ is $C_4$ or $P_4$ or $2K_2$ for all $i \neq j$.

**Proof.** Since both $G$ and $\overline{G}$ are connected, it follows that $\Delta < n - 1$. Hence $d_{nt}(G) \leq \frac{n}{2}$ and $d_{nt}(\overline{G}) \leq \frac{n}{2}$, so that $d_{nt}(G) + d_{nt}(\overline{G}) \leq n$.

Now, suppose $d_{nt}(G) + d_{nt}(\overline{G}) = n$. Then $d_{nt}(G) = \frac{n}{2}$ and $d_{nt}(\overline{G}) = \frac{n}{2}$. Since $d_{nt}(G) \leq \delta(G) + 1$, it follows that $\delta(G) \geq \frac{n}{2} - 1$ and $\delta(\overline{G}) \geq \frac{n}{2} - 1$ and hence $\deg v = \frac{n}{2} - 1$ or $\frac{n}{2}$ for all $v \in V(G)$.

Now, let $V = X_1 \cup X_2 \cup \cdots \cup X_\frac{n}{2}$ be a nt-domatic partition of $G$. Then the subgraph induced by the edges of $G$ with one end in $X_i$ and the other end in $X_j$ has a perfect matching. Further, if $(X_i \cup X_j)$ has more than four edges, then at least one vertex $v$ of $(X_i \cup X_j)$ has degree at least 3. Since there are $\frac{n}{2} - 2$ nt-sets other than $X_1$ and $X_\frac{n}{2}$, $\deg v \geq \frac{n}{2} + 1$ which is a contradiction. Thus $(X_i \cup X_j)$ contains at most four edges and hence is isomorphic to $C_4$ or $P_4$ or $2K_2$. The converse is obvious. \hfill \Box
4. CONCLUSION AND SCOPE

In this paper we have introduced a new type of domination, namely, neighbourhood total domination. We have also discussed the corresponding neighbour total domatic partition. The following are some interesting problems for further investigation.

**Problem 4.1.** Characterize the class of graphs for which $\gamma_{nt}(G) = n - \Delta$.

**Problem 4.2.** Characterize graphs for which $\gamma_{nt}(G) = \left\lceil \frac{n}{2} \right\rceil$.

**Problem 4.3.** Characterize the class of graphs for which $\gamma_{nt}(G) = n - 1$ or $n - 2$.

**Problem 4.4.** Characterize nt-domatically full graphs.

**Problem 4.5.** Characterize graphs for which $d_{nt}(G) = d(G)$.

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