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A NOTE ON GEODESIC
AND ALMOST GEODESIC MAPPINGS
OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

Abstract. Let $M$ be a differentiable manifold and denote by $\nabla$ and $\tilde{\nabla}$ two linear connections on $M$. $\nabla$ and $\tilde{\nabla}$ are said to be geodesically equivalent if and only if they have the same geodesics. A Riemannian manifold $(M, g)$ is a naturally reductive homogeneous manifold if and only if $\nabla$ and $\tilde{\nabla} = \nabla - T$ are geodesically equivalent, where $T$ is a homogeneous structure on $(M, g)$ ([7]). In the present paper we prove that if it is possible to map geodesically a homogeneous Riemannian manifold $(M, g)$ onto $(M, \tilde{\nabla})$, then the map is affine. If a naturally reductive manifold $(M, g)$ admits a nontrivial geodesic mapping onto a Riemannian manifold $(\tilde{M}, \tilde{g})$ then both manifolds are of constant curvature. We also give some results for almost geodesic mappings $(M, g) \to (M, \tilde{\nabla})$.

Keywords: homogeneous Riemannian manifold, geodesic, almost geodesic, geodesic mapping, almost geodesic mapping.

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1. INTRODUCTION

Let $(M, g)$ be an $n$-dimensional Riemannian manifold of class $C^\infty$. Let $\mathcal{F}(M)$ be the ring of differentiable functions and $\mathcal{X}(M)$ the $\mathcal{F}(M)$–module of differentiable vector fields on $M$. A complete and simply connected manifold $(M, g)$ is homogeneous if there exists a transitive and effective group $G$ of isometries of $M$. Ambrose and Singer proved (see [7]) that a complete and simply connected Riemannian manifold $(M, g)$ is homogeneous if and only if there exists a tensor field $T$ of type $(1, 2)$ such that:

\begin{align}
(i) \quad & g(T_X Y, Z) + g(Y, T_X Z) = 0,
(ii) \quad & (\nabla_X R)_{YZ} = [T_X, R_{YZ}] - RT_X Y Z - RT_Y X Z, \quad (1.1)
(iii) \quad & (\nabla_X T)_Y = [T_X, T_Y] - T_{T_X Y},
\end{align}
for $X, Y, Z \in X(M)$. Here $\nabla$ and $R$ denote the Levi-Civita connection and the Riemannian tensor field, respectively. A tensor field $T$ satisfying the conditions (1.1) on $M$ is called a homogeneous structure on $(M, g)$. It is easy to see that the conditions (1.1) are equivalent to

\[
\tilde{\nabla}_X g(Y, Z) = 0, \quad \tilde{\nabla}_X R(Y, Z) = 0, \quad \tilde{\nabla}_X T(Y, Z) = 0
\]

(1.2)

where $\tilde{\nabla}$ is the connection determined by

\[
\tilde{\nabla}_X Y = \nabla_X Y - T(X, Y).
\]

(1.3)

In [7] F. Tricerri and L. Vanhecke studied the decomposition of the space of all the tensors $T$ satisfying the conditions (1.1) into the irreducible components under the action of orthogonal group. As is well-known, a geodesic in a Riemannian manifold $M$ is a curve of $c: I \to M$ whose tangent vector field $\dot{c}$ is parallel along $c$ ($I$ is an open interval in the real line $\mathbb{R}$). A curve $c$ is almost geodesic in a Riemannian manifold $M$ if there exists a 2– dimensional distribution $E^2$ complanar along $c$, to which the tangent vector $\dot{c}$ of this curve belongs at every point. Let $(\tilde{M}, \tilde{\nabla})$ be a differentiable manifold with a linear symmetric connection $\tilde{\nabla}$. A mapping $f: (M, g) \to (\tilde{M}, \tilde{\nabla})$ is called geodesic or projective if $f$ carries geodesics in $M$ to geodesics in $\tilde{M}$. The mapping $f$ is an almost geodesic mapping if, as a result of $f$, every geodesic in the manifold $M$ passes into an almost geodesic curve in the manifold $\tilde{M}$. If $\tilde{M}$ coincides with $M$ and $f$ is a diffeomorphism, $f$ is called a geodesic or an almost geodesic transformation of $M$.

It is well known, that the identity transformation is geodesic if and only the connection deformation tensor $P(X, Y) = \nabla_X Y - \nabla_Y X$ has the form ([1, 2, 3, 5])

\[
P(X, Y) = \psi(X)Y + \psi(Y)X,
\]

(1.4)

where $\psi$ is a certain 1-form and $\nabla$ denotes the Levi-Civita connection of $(M, g)$, $X, Y \in X(M)$.

In this case, $\nabla$ and $\tilde{\nabla}$ are said to be geodesically (or projectively) equivalent or geodesically (projectively) related. Two such connections define the same system of geodesics. Obviously $\sim$ is an equivalence relation and an equivalence class $[\nabla]$ containing $\nabla$ is called a projective structure on $M$.

Sinyukow [5] defined three kinds of almost geodesic mappings, namely $\pi_1$, $\pi_2$ and $\pi_3$ which are characterized, respectively, by the conditions:

\[
\begin{align*}
\pi_1: & \quad \mathcal{S}_{X, Y, Z} [(\nabla_X P)(Y, Z) + P(P(X, Y), Z) - a(X, Y)Z - P(X, Y)b(Z)] = 0 \\
& \quad (\mathcal{S} \text{ is cyclic sum}),
\end{align*}
\]

(1.5)
\[ P(X,Y) = \psi(X)Y + \psi(Y)X + F(X)\varphi(Y) + F(Y)\varphi(X), \]
\[ (\nabla_X F)(Y) + (\nabla_Y F)(X) + F(F(X))\varphi(Y) + F(F(Y))\varphi(X) = \mu(X)F(Y) + \mu(Y)F(X) + \rho(X)Y + \rho(Y)X; \]
\[ P(X,Y) = \psi(X)Y + \psi(Y)X + a(X,Y)\nu, \]
\[ \nabla_X \nu = \theta(X)\nu + \lambda X, \quad \lambda \in \mathfrak{F}(M); \quad X,Y,Z \in \mathfrak{X}(M), \]
\[ \pi_2: \quad P(X,Y) = \psi(X)Y + \psi(Y)X + F(X)\varphi(Y) + F(Y)\varphi(X), \]
\[ (\nabla_X F)(Y) + (\nabla_Y F)(X) + F(F(X))\varphi(Y) + F(F(Y))\varphi(X) = \mu(X)F(Y) + \mu(Y)F(X) + \rho(X)Y + \rho(Y)X; \]
\[ \pi_3: \quad P(X,Y) = \psi(X)Y + \psi(Y)X + a(X,Y)\nu, \]
\[ \nabla_X \nu = \theta(X)\nu + \lambda X, \quad \lambda \in \mathfrak{F}(M); \quad X,Y,Z \in \mathfrak{X}(M), \]
\[ \pi_1: \quad P(X,Y) = \psi(X)Y + \psi(Y)X + a(X,Y)\nu, \]
\[ \nabla_X \nu = \theta(X)\nu + \lambda X, \quad \lambda \in \mathfrak{F}(M); \quad X,Y,Z \in \mathfrak{X}(M), \]
where \( P(X,Y) = \nabla_X Y - \nabla_Y X \) is the connection deformation tensor and \( \varphi, \psi, b, \theta, \rho, \nu, a, F \) are tensors of the corresponding types.

In the present paper we shall study a geodesic and an almost geodesic related connections \( \nabla \) and \( \tilde{\nabla} = \nabla - T \), where \( T \) is a homogeneous structure on \( (M,g) \).

2. GEODESIC MAPPINGS OF HOMOGENEOUS RIEMANNIAN MANIFOLDS

By [7], Theorem 6.8, a complete and simply connected Riemannian manifold \( (M,g) \) is naturally reductive homogeneous manifold if and only if there exists a tensor field \( T \) of type \( (1,2) \) satisfying the conditions (1.1) and such that \( \tilde{\nabla} \) and \( \nabla \) are geodesically equivalent.

Now we shall prove

**Lemma 2.1.** If it is possible to map geodesically a homogeneous Riemannian manifold \( (M,g) \) onto a manifold \( (M,\tilde{\nabla}) \), then the map is affine.

**Proof.** The connections \( \nabla \) and \( \tilde{\nabla} \) are geodesically equivalent if and only if the connection deformation \( D \) have the form
\[ D(X,Y) = -T(X,Y) = \psi(X)Y + \psi(Y)X + S(X,Y) \]
where \( \psi \) is a 1-form and the tensor field \( S \) satisfies
\[ S(X,Y) + S(Y,X) = 0. \]

We put
\[ \tilde{P}(X,Y) = \frac{1}{2}(T(X,Y) + T(Y,X)) \]
and
\[ \tilde{P}(X,Y,Z) = \tilde{g}(\tilde{P}(X,Y),Z). \]

From (1.1 (i)) we obtain
\[ \tilde{g}_{X,Y,Z} \tilde{P}(X,Y,Z) = 0. \]

Hence and from (2.1) we have
\[ \psi(X)\tilde{g}(Y,Z) + \psi(Y)\tilde{g}(X,Z) + \psi(Z)\tilde{g}(X,Y) = 0. \]

Therefore \( \psi(X) = 0 \) for all \( X \in \mathfrak{X}(M) \). This completes the proof.
If $T = 0$, then (1.1) implies that $(M, g)$ is a symmetric manifold. In view of the Sinyukov theorem we obtain: if it is possible to map geodesically a complete and simply connected Riemannian manifold with the homogeneous structure $T = 0$ into a manifold $(\overline{M}, \overline{g})$ then both manifolds are of constant sectional curvature.

Let $(M, g)$ be a connected Riemannian manifold and suppose $M$ admits a non-trivial homogeneous structure by

$$T(X, Y, Z) + T(Y, X, Z) = 0,$$

where $X, Y, Z \in \mathfrak{X}(M)$.

From (1.1) and (2.2) we get easily.

**Lemma 2.2.** Let $(M, g)$ be a connected Riemannian manifold with the homogeneous structure of type (2.2). Then Ricci tensor on $M$ satisfies

$$\bar{\nabla}_{X,Y,Z} (\nabla_X Ric)(Y, Z) = 0.$$  

(2.3)

Now we shall prove

**Theorem 2.1.** If it is possible to map geodesically on $(M, g)$ satisfying (2.3) onto a manifold $(\overline{M}, \overline{g})$, then both manifolds are of constant curvature.

**Proof.** As is well-known a manifold $(M, g)$ admits a geodesic mapping if and only if there exists a function $\phi \in \overline{\mathfrak{G}}(M)$ and a symmetric non-singular bilinear form $a$ on $M$ satisfying

$$\nabla_X a(Y, Z) = (Y \phi)g(X, Z) + (Z \phi)g(X, Y)$$

(2.4)

for all $X, Y, Z \in \mathfrak{X}(M)$ ([5]).

Let $p \in M$ be such that $d\phi \neq 0$ and (2.4) hold at $p$. Choose a local coordinate system $(U, x)$ so that $p \in U$. By $R^t_{ijkl}, R^t_{ik}, g_{ik}, a_{ik}, \phi_{ik}$ we denote the components of the tensors $R, Ric, g, a$ and the Hessian $H \phi$ of $\phi$ in this coordinate system. Differentiating covariantly (2.4) and applying the Ricci identity we get

$$a_{it}R^i_{tjkl} + a_{tj}R^i_{tikl} = \phi_{it}g_{jk} + \phi_{ij}g_{ik} - \phi_{ki}g_{jl} - \phi_{kj}g_{il}.$$  

(2.5)

Differentiating covariantly (2.5) with respect to $x^m$, contracting with $g^{jk}$ and applying the Ricci identity, by (2.4) and (2.3), we obtain

$$4 \phi_{it}R^l_{jkl} = 3R^l_{kij}g_{ji} - 4 \phi_k R^l_{jik} + 4 \phi_i R^l_{ijk} - 3g_{jk}R^l_{ij}a_{it} + a_{it}g_{jk} - a_{jk}g_{it},$$

(2.6)

where $a_i = \nabla_{x^i} \phi_{it}g^{is}$. Transvecting (2.6) with $g^{jk}$ we get $R^l_{ij} \phi_t = \rho \phi_i$, $\rho \in \mathfrak{G}(U)$. Following considerations made in [6] we get

$$a^t_i \phi_t = \tau \phi_i, \quad \phi^t_i \phi_t = \lambda \phi_i, \quad \tau, \lambda \in \mathfrak{G}(U),$$

and finally we obtain

$$H \phi(X, Y) = \Phi(\phi)g(X, Y)$$

(2.7)

where $H \phi$ is the Hessian of $\phi$ and $\Phi \in \mathfrak{G}(M)$. 

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By [7] if a complete and simply connected manifold with homogeneous structure
$T$ admits condition (2.7), then the manifold $(M, g)$ is of constant curvature. This
completes the proof.

From Lemmas 2.1 and 2.2 and Theorem 2.1 we obtain

**Theorem 2.2.** On a homogeneous manifold the geodesic of $\nabla$ and $\tilde{\nabla} = \nabla - T$ are the same if and only if $M$ is naturally reductive. The geodesic mapping $(M, g) \to (M, \tilde{\nabla})$ is affine. If a naturally reductive manifold $(M, g)$ admits a non-trivial geodesic mapping onto a Riemannian manifold $(\tilde{M}, \tilde{g})$, then both manifolds are of constant curvature.

3. ALMOST GEODESIC MAPPINGS OF HOMOGENEOUS MANIFOLDS

On the basis [7] the most general form of the structure tensor $T$ is following

$$T(X, Y) = g(X, Y)\Phi - g(\Phi, Y)X + \frac{2}{2}T(X, Y)$$

(3.1)

where $\Phi$ is a given vector field on $(M, g)$ and $\frac{2}{2}T$ is a tensor field such that

$$g(\tilde{T}(X, Y), Z) + g(Y, \tilde{T}(X, Z)) = 0, \quad \tilde{\nabla}^2\tilde{T} = 0,$$

$$C_{12}(\tilde{T}) = \sum_{i=1}^{n} \frac{2}{2}T(X_i, X_i) = 0,$$

where $X_i$ is the base vector of the natural frame.

We put

$$\frac{1}{2}P(X, Y) = \frac{1}{2}(\psi(X)Y + \psi(Y)X) - g(X, Y),$$

$$\frac{1}{2}S(X, Y) = \frac{1}{2}(\psi(X)Y - \psi(Y)X),$$

$$\frac{2}{2}P(X, Y) = -\frac{1}{2}\left(\frac{2}{2}T(X, Y) + \frac{2}{2}T(Y, X)\right),$$

$$\frac{2}{2}S(X, Y) = -\frac{1}{2}\left(\frac{2}{2}T(X, Y) - \frac{2}{2}T(Y, X)\right),$$

(3.3)

and

$$P(X, Y) = \frac{1}{2}P(X, Y) + \frac{2}{2}P(X, Y)$$

$$S(X, Y) = \frac{1}{2}S(X, Y) + \frac{2}{2}S(X, Y)$$

where $\psi(X) = g(X, \Phi)$.

Here $P$ denotes the symmetric part of the tensor field $T$ and $S$ – the skew-symmetric one.
Then we have
\[ \tilde{\nabla}_X Y = \nabla_X Y + P(X,Y) + S(X,Y). \]  
(3.4)
and the connection deformation tensor \( D \) have the form
\[ D(X,Y) = P(X,Y) + S(X,Y) \]  
(3.5)
for all \( X,Y \in \mathfrak{X}(M) \).

We shall prove

**Theorem 3.1.** On the homogeneous Riemannian manifold the connections \( \nabla \) and \( \tilde{\nabla} \) defined by (3.3) and (3.4) are almost geodesically related if and only if the tensor fields \( \tilde{P} \) and \( \tilde{S} \) satisfy the relations
\[
\begin{align*}
\mathcal{E}_{X,Y,Z} & \left[ (\nabla_X \tilde{P})(Y,Z) + \tilde{P}(P(X,Y),Z) - \tilde{P}(X,Y)b(Z) - \tilde{P}(X,\psi)g(Y,Z) + \right. \\
& \quad - \tilde{P}(\tilde{S}(X,Y),Z) - \tilde{S}(X,\psi)g(Y,Z) + \\
& \quad - h(X,Y)\nabla_Z \psi + k(X,Y,Z)\psi + q(X,Y)Z \right] = 0,
\end{align*}
\]  
(3.6)
where: \( b, d, h, k, q \) are tensors of the corresponding types.

**Proof.** By [5] the mapping \( \nabla \to \tilde{\nabla} \) is almost geodesic if and only if the connection deformation tensor \( D \) satisfies the relations
\[
\left( \nabla_\gamma D^h_{\alpha\beta} + D^h_{\gamma\alpha}D^\beta_\rho \right) \lambda^\alpha \lambda^\beta \lambda^\gamma = bD^h_{\alpha\beta} \lambda^\alpha \lambda^\beta + a \lambda^h \]  
(3.7)
where \( \lambda' = \frac{dc}{dt} \) denotes the vector tangent to the geodesic \( c(t) = (c^i(t)) \). We conclude from (3.3), (3.4), (3.5), (3.7) that (3.6) holds. This proves the theorem. \( \square \)

**Corollary 3.1.** If \( \tilde{P} = 0 \) and \( \tilde{S} = 0 \) then the almost geodesic mapping is of the kind (1.7).

**Corollary 3.2.** If \( \tilde{P} = 0 \) and \( \tilde{S} = 0 \) then a homogeneous Riemannian manifold is a manifold of constant curvature (see [7]).

**Corollary 3.3.** If \( g(X,Y)\nabla_Z \psi + g(X,Y)\tilde{P}(Z,\psi) + g(X,Y)\tilde{S}(Z,\psi) + k(X,Y,Z)\psi = 0 \) then the almost geodesic mapping (3.6) is of the kind (1.5).

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