A new approach to the realization problem for fractional discrete-time linear systems

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Abstract. A new approach to the realization problem for fractional discrete-time linear systems is proposed. A procedure for computation of fractional realizations of given transfer matrices is presented and illustrated by numerical examples.

Key words: fractional, linear, discrete-time, system, computation procedure, realization, transfer matrix.

1. Introduction

Determination of the state space equations for given transfer matrices is a classical problem, called the realization problem, which has been addressed in many papers and books [1–8]. An overview of the positive realization problem is given in [1, 2, 6, 9]. The realization problem for positive continuous-time and discrete-time linear systems has been addressed in many papers and books [1–8].

In this paper a new approach to the realization problem for fractional discrete-time linear systems will be proposed. The paper is organized as follows. Some preliminaries and problem formulation are given in Sec. 2. In Sec. 3 the solution to the realization problem for fractional discrete-time linear systems is presented and illustrated by numerical examples. Concluding remarks are given in Sec. 4.

The following notation will be used: \( \mathbb{R} \) – the set of real numbers, \( \mathbb{R}^{n \times m} \) – the set of \( n \times m \) real matrices, \( \mathbb{R}^{n \times m}(w) \) – the set of \( n \times m \) rational matrices in \( w \) with real coefficients, \( \mathbb{Z}_+ \) – the set of nonnegative integers, \( I_n \) – the \( n \times n \) identity matrix.

2. Preliminaries and problem formulation

Consider the fractional discrete-time linear system

\[
\Delta^\alpha x_i = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \tag{1a}
\]

\[
y_i = Cx_i + Du_i, \tag{1b}
\]

where

\[
\Delta^\alpha x_i = \sum_{j=0}^{i} c_j x_{i-j},
\]

\[
c_j = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix}, \tag{1c}
\]

\[
\begin{cases}
1 & \text{for } j = 0 \\
\frac{1}{j!} \alpha(\alpha-1) \cdots (\alpha-j+1) & \text{for } j = 1, 2, \ldots
\end{cases}
\]

\( x_i \in \mathbb{R}^n, u_i \in \mathbb{R}^m, y_i \in \mathbb{R}^p \) are the state, input and output vectors and \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n}, D \in \mathbb{R}^{p \times m}. \)

Using the \( \mathcal{Z} \)-transformation to (1a) and (1b) for zero initial conditions we obtain [6]

\[
Z[\Delta^\alpha x_i] = wX(z) = AX(z) + BU(z), \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\} \tag{2a}
\]

\[
Y(z) = CX(z) + DU(z), \tag{2b}
\]

where

\[
Z[\Delta^\alpha x_i] = (1 - z^{-1})^\alpha X(z) = w(z)X(z) = wX(z),
\]

\[
w = w(z) = (1 - z^{-1})^\alpha = \sum_{i=0}^{\infty} c_i z^{-i}, \tag{2c}
\]

\[
X(z) = Z[x_i] = \sum_{i=0}^{\infty} x_i z^{-i},
\]

\[
U(z) = Z[u_i], \quad Y(z) = Z[y_i].
\]

From (2) we have the transfer matrix

\[
T(w) = C[I_n w - A]^{-1} B + D. \tag{3}
\]

The transfer matrix \( T(z) \) is called proper if and only if

\[
\lim_{w \to \infty} T(w) = D \in \mathbb{R}^{p \times m} \tag{4}
\]

and it is called strictly proper if and only if \( D = 0. \)

From (3) we have

\[
\lim_{w \to \infty} T(w) = D \tag{5}
\]

since \( \lim_{w \to \infty} [I_n w - A]^{-1} = 0. \)
3. Problem solution

3.1. Single-input single-output systems. First the essence of the proposed method is presented for single-input single-output (SISO) fractional discrete-time linear systems with the transfer function

\[ T(w) = \frac{b_n w^n + b_{n-1} w^{n-1} + \ldots + b_1 w + b_0}{w^n + a_{n-1} w^{n-1} + \ldots + a_1 w + a_0}. \]  

(6)

Using (4) for (6) we obtain

\[ D = \lim_{w \to \infty} T(w) = b_n \]  

and

\[ T_{sp}(w) = T(w) - D = \frac{\overline{b}_{n-1} w^{n-1} + \ldots + \overline{b}_1 w + \overline{b}_0}{w^n + a_{n-1} w^{n-1} + \ldots + a_1 w + a_0}. \]  

(8a)

where

\[ \overline{b}_k = b_k - a_k b_n, \quad k = 0, 1, \ldots, n-1. \]  

(8b)

Therefore, the realization problem has been reduced to finding matrices \( A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, C \in \mathbb{R}^{p \times n} \) for given strictly proper transfer function (8a).

Multiplying the numerator and the denominator of (8a) by \( w^{-n} \) we obtain

\[ T_{sp}(w) = \frac{Y}{U} = \frac{\overline{b}_{n-1} w^{-1} + \ldots + \overline{b}_1 w^{-n} + \overline{b}_0 w^{-n}}{1 + a_{n-1} w^{-1} + \ldots + a_1 w^{-n} + a_0 w^{-n}}. \]  

(9)

where \( Y \) and \( U \) are the Z-transforms of \( y_i \) and \( u_i \), respectively.

Define

\[ E = \frac{U}{1 + a_{n-1} w^{-1} + \ldots + a_1 w^{-n} + a_0 w^{-n}}. \]  

(10)

From (9) and (10) we have

\[ E = U - (a_{n-1} w^{-1} + \ldots + a_1 w^{-n} + a_0 w^{-n})E, \]  

(11a)

\[ Y = (\overline{b}_{n-1} w^{-1} + \ldots + \overline{b}_1 w^{-n} + \overline{b}_0 w^{-n})E. \]  

(11b)

From (11) follows the block diagram shown in Fig. 1.

Assuming as the state variables \( x_{1,i}, x_{2,i}, \ldots, x_{n,i} \) the outputs of the delay elements we may write the equations

\[ \Delta^\alpha x_{1,i} = x_{2,i}, \]  

(12a)

\[ \Delta^\alpha x_{2,i} = x_{3,i}, \]  

\[ \vdots \]  

\[ \Delta^\alpha x_{n-1,i} = x_{n,i}, \]  

\[ \Delta^\alpha x_{n,i} = -a_0 x_{1,i} - a_1 x_{2,i} - \ldots - a_{n-1} x_{n,i} + u_i \]  

and

\[ y_i = \overline{b}_0 x_{1,i} + \overline{b}_1 x_{2,i} + \ldots + \overline{b}_{n-1} x_{n,i}. \]  

(12b)

The Eq. (12) can be written in the form

\[ \Delta^\alpha x_i = Ax_i + Bu_i, \]  

(13a)

\[ y_i = C x_i, \]  

(13b)

where

\[ x_i = [x_{1,i}, x_{2,i}, \ldots, x_{n,i}]^T, \quad i \in \mathbb{Z}_+, \]

\[ A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \]  

(14)

\[ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad C = [\overline{b}_0 \ \overline{b}_1 \ \ldots \ \overline{b}_{n-1}]. \]
Remark 1. If we choose the state variables so that \( x_k = x_{n-k+1} \) for \( k = 1, \ldots, n \) then the realization of (8) has the form

\[
A_1 = \begin{bmatrix}
-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0 \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 1 & -a_{n-1}
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
1 \\
0 \\
\vdots \\
0
\end{bmatrix}, \quad C_1 = [ \overline{b}_{n-1} \; \overline{b}_{n-2} \; \cdots \; \overline{b}_0 ].
\]

Remark 2. Note that the transposition (denoted by \( T \)) of the transfer function does change it, i.e., \( [T_{sp}(w)]^T = T_{sp}(w) = [C[I_n w - A]^{-1} B]^T = B^T [I_n w - A]^T^{-1} C^T \) and the matrices

\[
A_2 = A^T = \begin{bmatrix}
0 & 0 & \cdots & 0 & -a_0 \\
1 & 0 & \cdots & 0 & -a_1 \\
0 & 1 & \cdots & 0 & -a_2 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & -a_{n-1}
\end{bmatrix},
\]

\[
B_2 = C^T = \begin{bmatrix}
\overline{b}_0 \\
\overline{b}_1 \\
\overline{b}_2 \\
\vdots \\
\overline{b}_{n-1}
\end{bmatrix},
\]

\[
C_2 = B^T = [ 0 \; \cdots \; 0 \; 1 ]
\]

and

\[
A_3 = A_1^T = \begin{bmatrix}
-a_{n-1} & 1 & 0 & \cdots & 0 \\
-a_{n-2} & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-a_1 & 0 & 0 & \cdots & 1 \\
-a_0 & 0 & 0 & \cdots & 0
\end{bmatrix},
\]

\[
B_3 = C_1^T = \begin{bmatrix}
\overline{b}_{n-1} \\
\overline{b}_{n-2} \\
\vdots \\
\overline{b}_1 \\
\overline{b}_0
\end{bmatrix},
\]

\[
C_3 = B_1^T = [ 1 \; 0 \; \cdots \; 0 ]
\]

are also the realizations of the transfer function (8).

Example 1. Find the fractional realization of the transfer function

\[
T(w) = \frac{2w^2 + 11w + 10}{w^2 + 3w + 4}.
\]

Using (7) we obtain

\[
D = \lim_{w \to \infty} T(w) = 2
\]

and

\[
T_{sp}(w) = T(w) - D = \frac{5w + 2}{w^2 + 3w + 4} = \frac{5w^{-1} + 2w^{-2}}{1 + 3w^{-1} + 4w^{-2}}.
\]

In this case we have

\[
E = \frac{U}{1 + 3w^{-1} + 4w^{-2}}
\]

and

\[
E = U - (3w^{-1} + 4w^{-2})E, \quad Y = (5w^{-1} + 2w^{-2})E.
\]

The block diagram corresponding to (22) is shown in Fig. 2.

For the choice of the state variables shown in Fig. 2 we obtain the equations

\[
\Delta^a x_{1,i} = x_{2,i}, \quad \Delta^a x_{2,i} = -4x_{1,i} - 3x_{2,i} + u, \quad y_i = 2x_{1,i} + 5x_{2,i}
\]

and the realization

\[
A = \begin{bmatrix}
0 & 1 \\
-4 & -3
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
1
\end{bmatrix}, \quad C = [ 2 \; 5 ].
\]

3.2. Multi-input multi-output systems. Consider a strictly proper transfer matrix \( T_{sp}(w) \in \mathbb{R}^{p \times m}(w) \). Let

\[
D_i(w) = w^{d_i} - (a_{i,d_i-1}w^{d_i-1} + \ldots + a_{i,1}w + a_{i,0}), \quad i = 1, \ldots, m
\]

be the least common denominator of all entries of the \( i \)-th column of \( T_{sp}(w) \).

Using (25) we may write \( T_{sp}(w) \) in the form

\[
T_{sp}(w) = \begin{bmatrix}
N_{11}(w) & \cdots & N_{1m}(w) \\
D_1(w) & \cdots & D_m(w) \\
\vdots & \ddots & \vdots \\
N_{p1}(w) & \cdots & N_{pm}(w) \\
D_1(w) & \cdots & D_m(w)
\end{bmatrix} = N(w)D^{-1}(w),
\]

where

\[
N(w) = \begin{bmatrix}
N_{11}(w) & \cdots & N_{1m}(w) \\
\vdots & \ddots & \vdots \\
N_{p1}(w) & \cdots & N_{pm}(w)
\end{bmatrix}, \quad D(w) = \text{diag}[D_1(w) \; \cdots \; D_m(w)].
\]
Step 1. Using (4) find the matrix $D$ and the strictly proper transfer matrix $T_{sp}(w)$.

Step 2. Find the least common denominators $D_1(w), \ldots, D_m(w)$ and write $T_{sp}(w)$ in the form (26).

Step 3. Knowing $D(w)$ find the indices $d_1, \ldots, d_m$ and the matrices $W$ and $\overline{A}_m$.

Step 4. Knowing $N(w)$ find the matrix $C$ defined by (28c).

Step 5. Using (29) find the matrices $A$ and $B$.

Remark 3. Similar results can be obtained for the least common denominator of all entries of the $j$-th row of $T_{sp}(w)$.

Example 2. Find the fractional realization of the transfer matrix

$$T(w) = \begin{bmatrix} 2w + 1 & w + 3 \\ w & w + 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

Using Procedure 1 and (33) we obtain the following:

Step 1. Using (4) and (33) we obtain

$$D = \lim_{w \to \infty} T(w)$$

and

$$T_{sp}(w) = T(w) - D = \begin{bmatrix} 1 & \frac{2}{w} \\ \frac{2}{w+2} & 1 \end{bmatrix}$$

Step 2. From (35) we have $D_1(w) = w(w + 2)$, $D_2(w) = (w + 1)(w + 2)$ and

$$T_{sp}(w) = N(w)D^{-1}(w),$$

where

$$N(w) = \begin{bmatrix} w + 2 & 2(w + 2) \\ 2w & w + 1 \end{bmatrix}$$

$$D(w) = \begin{bmatrix} w(w + 2) & 0 \\ 0 & (w + 1)(w + 2) \end{bmatrix}.$$
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Step 4. Using (36b) we obtain

\[ N(w) = \begin{bmatrix} w + 2 & 2w + 4 \\ w & w + 1 \end{bmatrix} = CW \]

and

\[ C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

(39a)

(39b)

Step 5. Using (29) and (37) we obtain

\[ A = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

(40)

The desired fractional realization of (33) is given by (40), (39b) and (34).

4. Concluding remarks

A new approach to finding fractional realizations of given transfer matrices of discrete-time linear systems has been proposed. It has been shown that for any given proper transfer matrix there exist always many fractional realizations. A procedure for computation a fractional realization of a given transfer matrix has been proposed. The effectiveness of the procedure has been demonstrated on numerical examples. The classical Gilbert method [29] can also be applied to compute the fractional realizations of the given transfer matrices of discrete-time linear systems.

The presented method can be easily extended to positive fractional linear discrete-time systems without and with delays.

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REFERENCES


