Dual model for classic transportation problem as a tool for dynamizing management in a logistics company

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Abstract. Each primary model of the linear programming problem has a corresponding dual model. It is widely accepted that the simplex method, in addition to determining the optimal solution for the original problem, also allows specifying a solution to the dual problem. So far, the dual problem solution has mainly served the post-optimization procedure, i.e. the analysis of modification of the primary model [20, 21, 27, 28]. However, the dual model itself is not generally subject to a deeper study and no conclusions are drawn from its full analysis. The lasting and prominent place that the classic transportation model takes, requires also to be complemented through the full development of its dual problem interpretation, including post-optimization problems. This paper presents and, for the first time, widely interprets the dual model for the classic model of the transportation problem. Moreover, potential possibilities connected with the use of ambiguities of the obtained solutions to the dual problem have been shown. It has been pointed out how these capabilities can be applied to a flexible financial policy of a logistics company.

Keywords: transport logistics, transport model, primary and dual problems, cost, revenue

INTRODUCTION

The process of turning inside out is mainly related to tailoring and means altering a garment by turning the fabric inside out. It allows the owner of the garment to obtain additional benefits existing within the fabric of the clothing. This concept has its counterpart in problems of linear programming (LP) in the form of a pair of primary and dual problems. Both of these issues also have their economic sources. F.L. Hitchcock [9] formulated the transportation problem, and G. Stigler [24] distinguished the mathematical problem of diet related directly to the war and food rationing. J. von Neumann was familiar with the duality principle as early as in 1947, but directly after G. Dantzig [2] formulated the general LP problem covering transport and diet, von Neumann proposed his duality theorem pertaining to the LP problem. Later, Dantzig [3] introduced the simplex algorithm for LP. Since then, different methods of operational research have been widely developed, including the problem of algorithmization of LP [12; 22, 23; 8; 30].

Creating a dual problem consists in:
1. converting a maximization problem into one of minimization or vice-versa;
2. creating decision variables (DecVar) for the dual model in a number equal to the number of the limiting conditions (LC) of the original model;
3. transporting the matrix of the limiting conditions;
4. exchanging the roles between the parameters b of the right sides of the limiting conditions and the coefficients c of the objective function (OF);
5. the appropriate change in the relations between the left and right side of the limiting conditions;
6. proper modification of the boundary conditions (BC).

Strict rules formulating the principles of creating the dual problem are known in the literature [4, 5; 8; 13; 17]. Following the creator of the dual problem for the issues concerning the linear programming, it should be noted that the aforementioned operations, originating from optimization problems of the revenues generated from production, lead to the dual problem model, where the competition intends to buy the company for the cheapest price possible. At the same time, maximizing the revenues from the selection range of production is replaced by the minimization of the purchase costs of the means of production from the manufacturer [7, 14, 29]. Hence, the dual problem has found its permanent place in planning business activities. It manifests itself through the use of dual models for developing the business plans with post-optimization methods and is intended to provide stability to a business venture when changing the planned parameters describing the situation of the enterprise and its environment. Instability of parameters of the transportation model corresponds to the dynamic economic conditions and the difficulties involved in their
strict definition [1, 15], to the changes which result from the introduction of innovative solutions in the enterprise [11] as well as to the variable legal situation [26]. Moreover, MacKenzie, et al. [16], Schrijver [19], Sab [18] and Glaeser [6] point out the sensitivity of the transport systems to natural disasters, military operations and terrorist threats.

The ambiguity of the dual solution to the classic transportation problem has neither been properly studied yet, nor included in the literature, and its exhaustive interpretation has not been presented. This is despite the permanent place of the transportation models in solving economic issues. This paper attempts to provide a possibly broad interpretation of the economic advantages inherent to this ambiguity. It is presented in the following example.

MATERIAL AND METHODS - A MATHEMATICAL MODEL OF THE CLASSIC TRANSPORTATION PROBLEM

I The primary problem: Company A mediates transport between two warehouses: $M_1$ and $M_2$ and three stores: $S_1$, $S_2$, $S_3$. The quantity of goods in the warehouses is 40 and 60 tons, respectively. The stores' requirements are 35, 35 and 30 tons, respectively. The established unit costs of transport borne by the logistics company A in €/ton are presented in Table 1 below. The problem consists in establishing the transport plan, i.e., which transport routes should be taken to minimize the total transport costs?

Table 1. The unit transport costs $c_{ij}$ between warehouses $M_i$ and stores $S_j$

<table>
<thead>
<tr>
<th>$c_{ij}$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M_1$</td>
<td>7.3</td>
<td>8.2</td>
<td>6.5</td>
</tr>
<tr>
<td>$M_2$</td>
<td>8.0</td>
<td>6.9</td>
<td>7.4</td>
</tr>
</tbody>
</table>

Since the total supply is equal to the total demand, the classic primary ZZT1 model was applied to the balanced transportation problem in the following form:

$$A = \begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 
\end{bmatrix}, \quad b = \begin{bmatrix}
40[t] \\
60[t] \\
35[t] \\
35[t] \\
30[t] 
\end{bmatrix}$$

DecVar_P: $x_{ij}$ - stock quantity [t] transported from the $i$-th warehouse to the $j$-th shop store $i \in \{1, 2\}, j \in \{1, 2, 3\}$;

$OF_P$: $f_1(x) = c^T \cdot x \rightarrow \min$.

$LC_P$: $A \cdot x = b$;

$BC_P$: $x_{ij} \geq 0; i=1, 2; j=1, 2, 3$;

The limiting conditions are to ensure the implementation of the supply and demand. Note, that for $w' = [w_1][i\in\{1, \ldots, 5\}] = [1; 1; -1; -1; -1]$ we have $w' \cdot A = 0$. Hence, we get the balance condition of total supply and total demand in the form of $w' \cdot b = 0$. The only vectors orthogonal towards $A$ in the form of $s \cdot w$, where $s \in R$ are the balancing vectors which describe the linear dependence of the ranks of the matrix $A$. The existence of the vector $w$ causes the rank of the matrix $A$ to equal 4, i.e. the number of warehouses (2) plus the number of stores (3) minus 1.

II The dual problem: The logistic company A decided to support the transport from the warehouses $M_1$ and $M_2$ to the stores $S_1$, $S_2$, $S_3$. It was established that from the warehouses $M_1$ and $M_2$ 40 and 60 tonnes, respectively, would be delivered. And to the stores $S_1$, $S_2$, $S_3$, 35, 35 and 30 tonnes, respectively, would be delivered. It is known that the unit revenue [€/t] from the transport of 1 tonne of goods on a specified route does not exceed the values contained in Table 1.

The problem is, then, what revenues should the company A generate while receiving goods from different warehouses, and what revenues should it generate while providing for each store to maximize the total revenue?

The dual model for ZZT1 is the following:

DecVar_D: $y_i$ - revenue (loss) per unit [€/t] obtained for: the acceptance of one tonne of goods from the $i$-th warehouse for $i \in \{1, 2\}$; or at the delivery of one tonne of goods to the ($i$-2)-th stores for $i \in \{3, 4, 5\}$ ; $y = [y_i][i\in\{1, \ldots, 5\}]$;

$OF_D$: $f_2(y) = b' \cdot y \rightarrow \max$.

$LC_D$: $A' \cdot y \leq c$;

$BC_D$: $y_i \in R; i=1, \ldots, 5$.

Since the limiting conditions are to ensure the return of the unit transport costs on the routes where the cargo transport will take place, we have the following equations in $LC_D$ [20].

In the primary problem, the minimization of the total costs of transport is associated with the needs of senders and recipients. In the dual model it is represented by a logistics company's pursuit to obtain the maximum revenue from the transport services. What can be observed here is the opposite convergence to the common optimum of total costs and revenues amounting to [€]709.50. While creating this dual model, we can note that it is necessary to change the way of indexation of the decision variables in the dual problem from double to single. Consequently, the transport organisation in a transport company is divided into inbound transport (with the corresponding indexes from 1 to 2) and outbound transport (with the corresponding indexes from 3 to 5). In
the dual model, the direct routes connecting senders and recipients disappear. The company performs any potential
RESULTS AND DISCUSSION

Ambiguity of the dual problem solution. Despite the fact that the objective function value amounting to $\langle E \rangle 709.50 is common for the primary and dual problems, in this case the unambiguous, integer, non-negative solution to the primary problem in the form of the vector $x^* = [x_{1;1}; x_{1;2}; x_{1;3}; x_{2;1}; x_{2;2}; x_{2;3}] = [10; 0; 30; 25; 35; 0] is accompanied by the one-dimensional space $J$ of real solutions to the dual problem. The manner of its representation is not unambiguous either. For example, it may be the following: $y_i = \gamma_i + t \cdot (\gamma_i - \gamma_i')$, where $\gamma_i' = [y_1; y_2; y_3; y_4; y_5] = [7.3; 8.0; 0; -1.1; -0.8]$, $\gamma_i = [y_1; y_2; y_3; y_4; y_5] = [6.2; 6.9; 1.1; 0; 0.3]$, $t \in R$. Hence, alternatively, the space $J$ can be also formulated as $y_i = \gamma_i + s \cdot w = y_{1;1-1} \cdot w$ where $w$ is a pre-defined balancing vector of the classic transportation problem, and $R \ni s = 1.1 \cdot t$ (cf. Fig. 1). Note that for each $s \in R$ the ambiguity of the solution of the dual problem stems from the orthogonality of the balancing vector $w$ against the matrix $A$ and the vector $b$ because: $f_0(y_2) = b' \cdot y_2 = b' \cdot (\gamma + s \cdot w) = b' \cdot \gamma + s \cdot (w' \cdot b)' = b' \cdot \gamma = 709.50[E]$

Table 2. The values of the parameter $s$ changing the character of the optimal dual variables

<table>
<thead>
<tr>
<th>$s$</th>
<th>$y_{i;5} &lt; 0$</th>
<th>$y_{i;5} = 0$</th>
<th>$y_{i;5} &gt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y_{5;5}$</td>
<td>$s &lt; -7.3$</td>
<td>$-7.3$</td>
<td>$s &gt; -7.3$</td>
</tr>
<tr>
<td>$y_{5;5}$</td>
<td>$s &lt; -1.1$</td>
<td>$-1.1$</td>
<td>$s &gt; -1.1$</td>
</tr>
<tr>
<td>$y_{5;5}$</td>
<td>$s &gt; -0.8$</td>
<td>$-0.8$</td>
<td>$s &gt; -0.8$</td>
</tr>
</tbody>
</table>

Interpretation of the limiting conditions of the dual problem. Interestingly, $LC1_D$ means that the revenue obtained from the inbound transport of the goods from warehouse 1 integrated with the separate revenue from the outbound transport to store 1 cannot exceed $7.3[E/t]$ which is the previously established unit cost of transport of 1 [t] of cargo on this route. The other limiting conditions must be interpreted similarly. Since the values of $y_i$ are not limited by the boundary conditions, $y_i$ can be shaped freely so that $y_i$ belong to the space $J$. At the same time, it is observed that this surprising limitlessness in boundary conditions results also in the possibility of negative dual values of decision variables, however, it has its economic justification presented below.

An exemplary solution of the dual problem: 1) $y_{-2;2} = [5.1; 5.8; 2.2; 1.1; 1.4]$ for $s = -2.2$ is interpreted as achieving positive revenues from all inbound and outbound transport procedures; 2) For $s \in [-4.55; -3.65]$ all the counterparties provide the company with quite similar inbound and outbound transport revenues (see Fig. 1); 3) For $\gamma_i = y_{-1;1} = [6.2; 6.9; 1.1; 0; 0.3]'$ distribution to store 2 does not yield revenue; 4) For $\gamma_i = y_{0;0} = [7.3; 8.0; 0; -1.1; -0.8]'$ distribution to store 1 does not yield any revenue and outbound transport to stores 2 and 3 yields losses balanced by the increased revenue obtained only from the delivery from warehouses 1 and 2. For $\gamma_i$, the revenue of the logistics company is shaped only by the revenue obtained from warehouses. This means that the total transport costs are shifted onto only some of the participants in the goods circulation. Distribution to stores 2 and 3 is associated with losses on the part of the logistics company.

Fig. 1. The optimal values of the dual variables depending on the parameters
The unit costs of transport estimated in the primary problem do not indicate business entities that are actually supposed to participate in these costs. The fourth selected sample vector \( y \) breaks down the counterparties of the logistics company into revenue-generating, loss-generating and neutral entities. Fig. 1 enables the reading of other sample values of the optimal dual variables according to parameter \( s \).

One should notice that zeros for the \( i \)-th vector coordinate of the optimal dual prices \( y_i = [y_{1,i}, y_{2,i}, y_{3,i}, y_{4,i}, y_{5,i}] \) appear when \( s = w_i \cdot (y_{i}^+), \) as illustrated in the 3rd column of Table 2. And the return of inequalities in column 2 and 4 of Table 2 results from the signs of the coordinates of the vector \( w \) and is presented in Fig.1. For the logistics company, the \( w_i \cdot (y_{i}^+) \) values of the parameter \( s \) have

<table>
<thead>
<tr>
<th>( y_{1,s} - y_{j,s} )</th>
<th>( y_{1,i} )</th>
<th>( y_{2,s} )</th>
<th>( y_{3,i} )</th>
<th>( y_{4,s} )</th>
<th>( y_{5,s} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_{1,s} )</td>
<td>0</td>
<td>-0.7</td>
<td>7.3+2s</td>
<td>8.4+2s</td>
<td>8.1+2s</td>
</tr>
<tr>
<td>( y_{2,s} )</td>
<td>0.7</td>
<td>0</td>
<td>8+2s</td>
<td>9.1+2s</td>
<td>8.8+2s</td>
</tr>
<tr>
<td>( y_{3,s} )</td>
<td>-7.3-2s</td>
<td>-8-2s</td>
<td>0</td>
<td>1.1</td>
<td>0.8</td>
</tr>
<tr>
<td>( y_{4,s} )</td>
<td>-8.4-2s</td>
<td>-9.1-2s</td>
<td>-1.1</td>
<td>0</td>
<td>-0.3</td>
</tr>
<tr>
<td>( y_{5,s} )</td>
<td>-8.1-2s</td>
<td>-8.8-2s</td>
<td>-0.8</td>
<td>0.3</td>
<td>0</td>
</tr>
</tbody>
</table>

The capability to freely shape these three groups of counterparties within the space \( J \) allows the logistics company to conduct a fairly flexible financial policy. This means that it is possible to shift the transport costs to selected entities only, while introducing special offers for other participants of the distribution system. An example of such logistics economy can be a non-profit collection of recyclable materials, such as waste paper, clothing, debris and metals, from which profits are obtained only from recipients of these materials. When the waste inbound transport costs are not matched by relevant revenue from their recipients, they must be balanced and even surpassed in the form of a fixed fee charged on individual property managers with the possible subsidization from the city or municipality budget or other external funding.

The freedom to shape the financial policy, which is reflected by the dual variable values, enables also a competitive behaviour on the market. In a clash with smaller, local companies, a company that operates in many areas of the market may periodically flexibly shape its financial relations and take over areas of smaller, less flexible business entities. Therefore, businesses can take over a particular area of the market by offering special offers or buying goods at higher prices and deploying clever economy. One of the tools used for this purpose is the use of exclusivity clauses to acquire particular goods. Each business entity must be aware of the fact that the described seemingly favourable substitution of single revenue is also one of the key factors threatening the attractiveness and stability of each sector of economy. Any business entity which aggressively applies the principle of substitutability of revenue is seen as the one which impairs the market stability and might face a massive and effective retaliation of even seemingly insignificant entities. This model indicates that it is the current transport process where the possibilities for running a flexible financial policy occur, and they do not necessarily require expenditures from previously accumulated funds or loans drawn to this end.

Order of examination of the primary and dual models. It is usually assumed that it is mainly the primary model that is created, while the dual model is used optionally to accelerate the primary or secondary optimization procedure resulting from a change in some of the model’s parameters or to draw simple post-optimization conclusions based on the sensitivity analysis [cf. 10, 25]. But in this case, other surprising conclusions come to mind. The above examples demonstrate the need for simultaneous analysis of both models. And, for the classic transportation problem that is specifically discussed in this paper, even a need for prior consideration of the dual model is recognised. It can be described as follows.

First, the logistics company determines for itself the actual transport costs on individual routes. By complementing them with e.g. a diverse percentage margin that generates the profit, the amounts of unit revenues from individual routes, in this case 6, are achieved. The amounts planned in such a manner are on the right side of the limiting conditions of the dual problem and they constitute a starting point for linear
programming in the dual model. As a result, by finding space \( J \) of the solutions to the dual problem, the logistics company forms the above-described policy towards its counterparties. After the conclusion of relevant agreements with them, it proceeds to the solution of the so-called primary problem, as a result of which it finds an optimal route for the distribution of goods. Finally, it performs the transport of goods.

If the primary model describes the relationship repeated cyclically and, at the same time, there are changes in the parameters of this model, a need for post-optimizing associated with the use of the dual model also occurs. In total, this indicates a need for interchangeable application of both models (primary and dual) in the entire cycle of management of a logistics company.

**Alternative versions of unit revenues.** The need to use both the primary and dual model is thus apparent. This leads to several effects: first, the determination of the right sides of the limiting conditions of the dual problem gives a logistic company a point of reference as to the possibility (expressed by the space \( J \)). As it is evident in the above dependency and Figure 1, upon designation of any solution of the dual problem e.g. \( y \), an analyst adjusting the parameter \( s \) sees the alternative versions of the vector of unit revenue (loss) \( y_s \) achieved from individual counterparties, from which they choose a version that is the most convenient and possible to implement. While choosing versions that are possible to implement, they should note that the parameter \( s \) multiplies the vector \( w \) consisting of ones and minus ones. This means that all versions of the solutions proposed by the space \( J \) are the result of simultaneous increase or decrease of the initial \( y \) revenue (loss) in the two opposing groups of counterparties (namely warehouses and stores) by values equal to \( \Delta s \). And so the unit increase of revenue (loss) \([E/\ell]\) in each of the two warehouses is balanced by the unit increase of loss (revenue) \([E/\ell]\) in each of the three stores. This remark indicates the easiness with which the logistics company’s decision-maker, making arrangements with counterparties, finds the final implemented value of the solution to the dual problem. The finally designated \( y_s \) solution divides the counterparties into three groups described above, and its value determines the level of revenue (loss) negotiated with the counterparties by the logistics company. Thus, it seems that using the original model to determine transportation routes is only an additional problem which an analyst of a logistics company must face.

Started in 1947 by J. von Neumann, the process of using the dual models in various technical and economic problems leads to increasingly more precise descriptions of the encountered problems, thus contributing to the development and higher dynamics of various business activities.

**CONCLUSIONS**

1. This paper shows that in the transportation processes it is justifiable and necessary to use both the primary and the dual model.
2. The dual model allows a logistics company to undertake the planned operations of the flexible policy in relation to both the business partners and local competitors;
3. Flexible policy of the logistics company requires the existence of a relation between the revenues or losses, expressed by the space \( J \) of the optimal solutions to the dual model. It allows to fulfil the pre-determined limiting conditions;
4. Both the economic benefits and risks associated with the principle of subsidiarity of the revenue have been indicated.
5. This dual model is the basis for a problem that has not been discussed here, namely the post-optimization resulting from changes in the parameters of the primary model. Post-optimization problems for the transport models have not been fully studied, and some recent results in this area can be found in the studies [10, 25, 26, 27, 28].

**REFERENCES**


