A NEUMANN BOUNDARY VALUE PROBLEM FOR A CLASS OF GRADIENT SYSTEMS

Wen-Wu Pan and Lin Li

Communicated by Vicentiu D. Radulescu

Abstract. In this paper we study a class of two-point boundary value systems. Using very recent critical points theorems, we establish the existence of one non-trivial solution and infinitely many solutions of this problem, respectively.

Keywords: Neumann problems, weak solutions, critical points, \((p_1, \ldots, p_n)\)-Laplacian.

Mathematics Subject Classification: 35J65, 35J60, 47J30, 58E05.

1. INTRODUCTION

In this paper, we study the Neumann boundary value problems:

\[\begin{align*}
-\left(|u_1'(x)|^{p_1-2}u_1'(x)\right)' + |u_1(x)|^{p_1-2}u_1(x) &= \lambda F_{u_1}(x, u_1, \ldots, u_m), \quad x \in (a, b), \\
-\left(|u_2'(x)|^{p_2-2}u_2'(x)\right)' + |u_2(x)|^{p_2-2}u_2(x) &= \lambda F_{u_2}(x, u_1, \ldots, u_m), \quad x \in (a, b), \\
&\ldots \\
-\left(|u_m'(x)|^{p_m-2}u_m'(x)\right)' + |u_m(x)|^{p_m-2}u_m(x) &= \lambda F_{u_m}(x, u_1, \ldots, u_m), \quad x \in (a, b), \\
u_1'(a) = u_1'(b) = 0
\end{align*}\]

where \(p_i > 1\) are constants, for \(1 \leq i \leq m\), \(\lambda\) is a positive parameter, \(F: [a, b] \times \mathbb{R}^m \to \mathbb{R}\) is a function such that \(F(., t_1, \ldots, t_m)\) is measurable in \([a, b]\) for all \((t_1, \ldots, t_m) \in \mathbb{R}^m\), \(F(x, \ldots, .)\) is \(C^1\) in \(\mathbb{R}^m\) for every \(x \in [a, b]\) and for every \(\rho > 0\),

\[\sup_{|(t_1, \ldots, t_m)| \leq \rho} \sum_{i=1}^m |F_i(x, t_1, \ldots, t_m)| \in L^1([a, b]),\]

and \(F_{u_i}\) denotes the partial derivative of \(F\) with respect to \(u_i\) for \(1 \leq i \leq m\).

In the last decade or so, many authors applied variational methods to study the existence or multiplicity solutions of the Neumann problem of its variations; see, for
example, [6, 7, 9–13] and the references therein. We note that the main tools in these cited papers are several critical point theorems due to Bonanno [3], Bonanno and Bisci [4], Bonanno and Marano [8]. A Neumann boundary value problem for a class of gradient systems has already been studied by Afrouzi, Hadjian and Heidarkhani [1] and Hedarkhani and Tian [14] in the ODE case and Afrouzi, Heidarkhani and O’Regan [2] in the PDE case. In that papers at least three solutions are established. The aim of this article is to prove the existence of at least one non-trivial solution and infinitely many solutions for \((P_\lambda)\) for appropriate values of the parameter \(\lambda\) belonging to a precise real interval. Our motivation comes from the recent paper [4, 10]. We want to systematically study a class of gradient systems under a Neumann boundary using Bonanno’s critical point theorems. For basic notation and definitions, and also for a thorough account of the subject, we refer the reader to [15, 16].

2. PRELIMINARIES AND BASIC NOTATION

First we recall Bonanno’s critical point theorems which is our main tool to transfer the question of existence of weak solutions of \((P_\lambda)\) to the existence of critical points of the Euler functional.

For a given non-empty set \(X\), and two functionals \(\Phi, \Psi : X \to \mathbb{R}\), we define the following two functions:

\[
\beta(r_1, r_2) = \inf_{v \in \Phi^{-1}((r_1, r_2))} \frac{\sup_{u \in \Phi^{-1}((r_1, r_2))} \Psi(u) - \Psi(v)}{r_2 - \Phi(v)},
\]

\[
\rho(r_1, r_2) = \sup_{v \in \Phi^{-1}((r_1, r_2))} \frac{\Psi(v) - \sup_{u \in \Phi^{-1}((\mathbb{R}, r_1))} \Psi(u)}{\Phi(v) - r_1}
\]

for all \(r_1, r_2 \in \mathbb{R}, r_1 < r_2\).

**Theorem 2.1** ([3, Theorem 5.1]). Let \(X\) be a reflexive real Banach space, \(\Phi : X \to \mathbb{R}\) be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on \(X^*\) and \(\Psi : X \to \mathbb{R}\) be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Put \(I_\lambda = \Phi - \lambda \Psi\) and assume that there are \(r_1, r_2 \in \mathbb{R}, r_1 < r_2\), such that

\[
\beta(r_1, r_2) < \rho(r_1, r_2).
\]

Then, for each \(\lambda \in \left(\frac{1}{\rho(r_1, r_2)} - \frac{1}{\beta(r_1, r_2)}\right)\) there is \(u_{0, \lambda} \in \Phi^{-1}((r_1, r_2))\) such that \(I_\lambda(u_{0, \lambda}) \leq I_\lambda(u)\) for each \(u \in \Phi^{-1}((r_1, r_2))\) and \(I_\lambda'(u_{0, \lambda}) = 0\).

**Theorem 2.2** ([4, Theorem 2.1]). Let \(X\) be a reflexive real Banach space, let \(\Phi, \Psi : X \to \mathbb{R}\) be two Gâteaux differentiable functionals such that \(\Phi\) is sequentially weakly lower semicontinuous and coercive and \(\Psi\) is sequentially weakly upper semicontinuous. For every \(r > \inf_X \Phi\), let us put

\[
\varphi(r) := \inf_{u \in \Phi^{-1}((\mathbb{R}, r))} \frac{\left(\sup_{v \in \Phi^{-1}((\mathbb{R}, r))} \Psi(v)\right) - \Psi(u)}{r - \Phi(u)}
\]
A Neumann boundary value problem for a class of gradient systems

\[ \gamma := \liminf_{r \to +\infty} \varphi(r). \]

Under the above assumptions if \( \gamma < +\infty \) then, for each \( \lambda \in \left(0, \frac{1}{\gamma}\right)\), the following alternative holds:

either

(b₁) \( I_\lambda \) possesses a global minimum,

or

(b₂) there is a sequence \( \{u_n\} \) of critical points (local minima) of \( I_\lambda \) such that \( \lim_{n \to +\infty} \Phi(u_n) = +\infty \).

Let us introduce notation that will be used later. Let \( Y_i \) be the Sobolev space \( W^{1,p_i}(\mathbb{R}) \) endowed with the norm

\[ \|u\|_{p_i} := \left( \int_a^b |u'(x)|^{p_i} \, dx + \int_a^b |u(x)|^{p_i} \, dx \right)^{1/p_i}, \]

and let

\[ k_i = 2^{(p_i - 1)/p_i} \max\{(b - a)^{-1/p_i}, (b - a)^{(p_i - 1)/p_i}\}, \]

we recall the following inequality which we use in the sequel

\[ |u(x)| \leq k_i \|u\|_{p_i}, \tag{2.1} \]

for all \( u \in Y_i \), and for all \( x \in [a, b] \). Let \( K = \max\{k_i\} \), for \( 1 \leq i \leq m \). Here and in the sequel, \( X := Y_1 \times \cdots \times Y_m \).

We say that \( u = (u_1, \ldots, u_m) \) is a weak solution to the \((P_\lambda)\) if

\[ \sum_{i=1}^m \int_a^b \left( |u_i'(x)|^{p_i-2} u_i'(x)v_i'(x) + |u_i(x)|^{p_i-2} u_i(x)v_i(x) \right) \, dx - \lambda \sum_{i=1}^m \int_a^b F_{u_i}(x, u_1(x), \ldots, u_m(x))v_i(x) \, dx = 0 \]

for every \( v = (v_1, \ldots, v_m) \in X \). For \( \gamma > 0 \) we denote the set

\[ \Theta(\gamma) = \left\{ (t_1, \ldots, t_m) \in \mathbb{R}^m : \sum_{i=1}^m \frac{|t_i|^{p_i}}{p_i} \leq \frac{\gamma}{\Pi_{i=1}^m p_i} \right\}. \tag{2.2} \]

Let

\[ \Phi(u) = \sum_{i=1}^m \frac{\|u_i\|_{p_i}^{p_i}}{p_i}, \tag{2.3} \]

\[ \Psi(u) = \int_a^b F(x, u_1(x), \ldots, u_m(x)) \, dx. \tag{2.4} \]
It is well known that $\Phi$ and $\Psi$ are well defined and continuously differentiable functionals whose derivatives at the point $u = (u_1, \ldots, u_m) \in X$ are the functionals $\Phi'(u), \Psi'(u) \in X^*$, given by

$$
\Phi'(u)(v) = \sum_{i=1}^{m} \int_{a}^{b} (|u_i'(x)|^{p_i-2}u_i'(x)v_i'(x) + |u_i(x)|^{p_i-2}u_i(x)v_i(x)) \, dx,
$$

$$
\Psi'(u)(v) = \int_{a}^{b} \sum_{i=1}^{m} F_{u_i}(x, u_1(x), \ldots, u_m(x))v_i(x) \, dx
$$

for every $v = (v_1, \ldots, v_m) \in X$, respectively. Moreover, $\Phi$ is sequentially weakly lower semicontinuous, $\Phi'$ admits a continuous inverse on $X^*$ as well as $\Psi$ is sequentially weakly upper semicontinuous. Furthermore, $\Psi' : X \to X^*$ is a compact operator. Indeed, it is enough to show that $\Psi'$ is strongly continuous on $X$. For this, for fixed $(u_1, \ldots, u_m) \in X$, let $(u_{1n}, \ldots, u_{mn}) \to (u_1, \ldots, u_m)$ weakly in $X$ as $n \to +\infty$, then we have $(u_{1n}, \ldots, u_{mn})$ converges uniformly to $(u_1, \ldots, u_m)$ on $[a, b]$ as $n \to +\infty$ (see [16]). Since $F(x, \ldots)$ is $C^1$ in $\mathbb{R}^m$ for every $x \in [a, b]$, the derivatives of $F$ are continuous in $\mathbb{R}^m$ for every $x \in [a, b]$, so for $1 \leq i \leq m$, $F_{u_i}(x, u_{1n}, \ldots, u_{mn}) \to F_{u_i}(x, u_1, \ldots, u_m)$ strongly as $n \to +\infty$ which follows $\Psi'(u_{1n}, \ldots, u_{mn}) \to \Psi'(u_1, \ldots, u_m)$ strongly as $n \to +\infty$. Thus we proved that $\Psi'$ is strongly continuous on $X$, which implies that $\Psi'$ is a compact operator by Proposition 26.2 of [16].

3. RESULTS

Before our proof, we first list nonlinear term $F$ which satisfies the following hypotheses, where $\mu_1$, $\mu_2$ and $\nu$ are some constants.

(H1) $F(x, 0, \ldots, 0) = 0$ for a.e. $x \in [a, b]$,
(H2) $a_\nu(\mu_2) < a_\nu(\mu_1)$, where

$$
a_\nu(\mu) := K \frac{\int_{a}^{b} \sup_{(t_1, \ldots, t_m) \in \Theta(\mu)} F(x, t_1, \ldots, t_m) \, dx - \int_{a}^{b} F(x, \nu, \ldots, \nu) \, dx}{\mu - K \sum_{i=1}^{m} (\prod_{j=1,j \neq i}^{m} \Pi_{p_j}^{\nu_j}) \nu_{p_i}},
$$

(H3)

$$
\frac{\int_{a}^{b} \sup_{(t_1, \ldots, t_m) \in \Theta(\mu)} F(x, t_1, \ldots, t_m) \, dx}{\mu} \leq \frac{\int_{a}^{b} F(x, \nu, \ldots, \nu) \, dx}{K \sum_{i=1}^{m} (\prod_{j=1,j \neq i}^{m} \Pi_{p_j}^{\nu_j}) \nu_{p_i}},
$$

(H4)

$$
\liminf_{\mu \to +\infty} \frac{\int_{a}^{b} \sup_{(t_1, \ldots, t_m) \in \Theta(\mu)} F(x, t_1, \ldots, t_m) \, dx}{\mu} < \frac{1}{K \prod_{i=1}^{m} p_i (b - a)} \limsup_{|t_1| \to +\infty, \ldots, |t_m| \to +\infty} \frac{\int_{a}^{b} F(x, t_1, \ldots, t_m) \, dx}{\sum_{i=1}^{m} \frac{|t_i|^p}{p_i}}.
$$
3.1. ONE NONTRIVIAL SOLUTION

We formulate our main result as follows:

**Theorem 3.1.** Assume that there exist a non-negative constant $c_1$ and two positive constants $c_2$ and $d$ with

$$
c_1 < K(b-a) \sum_{i=1}^{m} (\Pi_{j=1,j\neq i}^{m} |p_j|) d^{p_i} < c_2
$$

such that (H1) and (H2) are satisfied. Then, for each $\lambda \in \left(\frac{1}{a_d(c_1)}, \frac{1}{a_d(c_2)}\right)$, system $(P_\lambda)$ admits at least one non-trivial weak solution $u_0 = (u_{01}, \ldots, u_{0m}) \in X$ such that

$$
\frac{c_1}{K \Pi_{i=1}^{m} \|u_i\|^p_{p_i}} < \sum_{i=1}^{m} \frac{\|u_{0i}\|^p_{p_i}}{p_i} < \frac{c_2}{K \Pi_{i=1}^{m} \|u_i\|^p_{p_i}}.
$$

**Proof.** To apply Theorem 2.1 to our problem, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ for each $u = (u_1, \ldots, u_m) \in X$, as (2.3) and (2.4). Moreover, $\Phi$ is sequentially weakly lower semicontinuous, $\Phi'$ admits a continuous inverse on $X^*$ as well as $\Psi' : X \to X^*$ is a compact operator. Set $w(x) = (w_1(x), \ldots, w_m(x))$ such that for $1 \leq i \leq m$,

$$
w_i(x) = d
$$

$r_1 = \frac{c_1}{K \Pi_{i=1}^{m} \|u_i\|^p_{p_i}}$ and $r_2 = \frac{c_2}{K \Pi_{i=1}^{m} \|u_i\|^p_{p_i}}$. It is easy to verify that $w = (w_1, \ldots, w_m) \in X$, and in particular, one has

$$
\|w_i\|^p_{p_i} = (b-a)d_i
$$

for $1 \leq i \leq m$. So, from the definition of $\Phi$, we have

$$
\Phi(w) = (b-a) \sum_{i=1}^{m} d_i.
$$

From the conditions $c_1 < K \sum_{i=1}^{m} (\Pi_{j=1,j\neq i}^{m} |p_j|)(b-a) d_i < c_2$, we obtain

$$
r_1 < \Phi(w) < r_2.
$$

Moreover, from (2.1) one has

$$
\sup_{x \in [a,b]} |u_i(x)|^p_{p_i} \leq K_i \|u_i\|^p_{p_i}
$$

and

$$
\sup_{x \in [a,b]} |u_i(x)|^p_{p_i} \leq K \|u_i\|^p_{p_i}
$$

for each $u = (u_1, \ldots, u_m) \in X$, so from the definition of $\Phi$, we observe that

$$
\Phi^{-1}((-\infty, r_2)) = \{(u_1, \ldots, u_n) \in X : \Phi(u_1, \ldots, u_n) < r_2\} =
$$

$$
= \{(u_1, \ldots, u_n) \in X : \sum_{i=1}^{m} \frac{\|u_i\|^p_{p_i}}{p_i} < r_2\} \subseteq
$$

$$
\subseteq \left\{(u_1, \ldots, u_n) \in X : \sum_{i=1}^{m} \frac{|u_i(x)|^p_{p_i}}{p_i} \leq \frac{c_2}{\Pi_{i=1}^{m} p_i} \text{ for all } x \in [a, b]\right\},
$$

respectively.
from which it follows
\[
\sup_{(u_1, \ldots, u_m) \in \Phi^{-1}((-\infty, r_2))} \Psi(u) = \sup_{(u_1, \ldots, u_m) \in \Phi^{-1}((-\infty, r_2))} \int_a^b F(x, u_1(x), \ldots, u_m(x)) dx \leq \int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m) dx.
\]

Since for \(1 \leq i \leq m\), for each \(x \in [a, b]\), the condition (A1) ensures that
\[
\beta(r_1, r_2) \leq \sup_{u \in \Phi^{-1}((-\infty, r_2))} \Psi(u) - \Psi(w) \leq \frac{\int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m) dx - \Psi(w)}{r_2 - \Phi(w)} \leq a_d(c_2).
\]

On the other hand, by similar reasoning as before, one has
\[
\rho(r_1, r_2) \geq \frac{\Psi(w) - \sup_{u \in \Phi^{-1}((-\infty, r_1))} \Psi(u)}{\Phi(w) - r_1} \geq \frac{\Psi(w) - \int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m) dx}{\Phi(w) - r_1} \geq a_d(c_1).
\]

Hence, from Assumption (A2), one has \(\beta(r_1, r_2) < \rho(r_1, r_2)\). Therefore, from Theorem 2.1, taking into account that the weak solutions of the system \((P_\lambda)\) are exactly the solutions of the equation \(\Phi'(u) - \lambda \Psi'(u) = 0\), we have the conclusion.

Now we point out the following consequence of Theorem 3.1.

**Theorem 3.2.** Suppose that there exist two positive constants \(c\) and \(d\) with
\[
c > K(b - a) \prod_{i=1}^m (\Pi_{j=1, j \neq i}^m P_j)^{d_i}
\]
such that (H1) and (H3) hold. Then, for each
\[
\lambda \in \left( \frac{K(b - a) \sum_{i=1}^m (\Pi_{j=1, j \neq i}^m P_j)^{d_i}}{\int_a^b F(x, d, \ldots, d) dx}, \frac{c}{\int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m) dx} \right),
\]
system \((P_\lambda)\) admits at least one non-trivial weak solution \(u_0 = (u_{01}, \ldots, u_{0n}) \in X\) such that
\[
\sum_{i=1}^m \frac{\|u_0\|_{p_i}}{p_i} < \frac{c}{K \prod_{i=1}^m P_i}.
\]
Proof. The conclusion follows from Theorem 3.1, by taking $c_1 = 0$ and $c_2 = c$. Indeed, owing to our assumptions, one has

$$a_d(c_2) = \frac{\int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m)dx - \int_a^b F(x, d, \ldots, d)dx}{c - K(b - a) \sum_{i=1}^m (\Pi_{j=1, j \neq i}^m p_j) d_{pi}} \leq K \prod_{i=1}^m p_i,$$

$$\leq \frac{\int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m)dx}{c - K(b - a) \sum_{i=1}^m (\Pi_{j=1, j \neq i}^m p_j) d_{pi}} = \frac{\int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m)dx}{c}.$$

On the other hand, taking Assumption (A1) into account, one has

$$\frac{\int_a^b F(x, d, \ldots, d)dx}{K(b - a) \sum_{i=1}^m (\Pi_{j=1, j \neq i}^m p_j) d_{pi}} = a_d(c_1).$$

Moreover, since

$$\sup_{x \in [a, b]} |u_i(x)|^{p_i} \leq K \|u_i\|_{p_i}^{p_i},$$

for each $u = (u_1, \ldots, u_m) \in X$, an easy computation ensures that

$$\sum_{i=1}^m \frac{\|u_0\|_{p_i}}{p_i} < \frac{c}{K \prod_{i=1}^m p_i},$$

whenever $\Phi(u) < r_2$. Now, owing to Assumption (A3), it is sufficient to invoke Theorem 3.1 to conclude the proof.

3.2. INFINITY MANY SOLUTIONS

**Theorem 3.3.** Assume that (H1) and (H4) hold. Then, for every $\lambda \in \Lambda := (\lambda_1, \lambda_2)$, where

$$\lambda_1 = \frac{(b - a)}{\limsup_{|t_1| \to +\infty, \ldots, |t_m| \to +\infty} \frac{\int_a^b F(x, t_1, \ldots, t_m)dx}{\sum_{i=1}^m |t_i|^p_{pi}^p}},$$

and

$$\lambda_2 = \frac{1}{K \prod_{i=1}^m \liminf_{\mu \to +\infty} \frac{\int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c)} F(x, t_1, \ldots, t_m)dx}{\mu}},$$

the problem ($P_\lambda$) admits an unbounded sequence of weak solutions which is unbounded in $X$. 


Proof. Our goal is to apply Theorem 2.2. Now, as has been pointed out before, the functionals $\Phi$ and $\Psi$ satisfy the regularity assumptions required in Theorem 2.2. Let $\{c_n\}$ be a real sequence such that $\lim_{n \to +\infty} c_n = +\infty$ and
\[
\liminf_{n \to +\infty} \frac{b}{c_n} \int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c_n)} F(x, t_1, \ldots, t_m) \, dx = A.
\] (3.1)

Taking into account (2.1) for every $u \in X$ one has
\[
|u(x)| \leq K \|u\|_{p_i}.
\]

Also note
\[
\sum_{i=1}^m \frac{|u_i(x)|^{p_i}}{p_i} \leq K \left( \sum_{i=1}^m \frac{\|u_i(x)\|_{p_i}^{p_i}}{p_i} \right).
\]

Hence, an easy computation ensures that $\sum_{i=1}^m u_i \leq c_n$ whenever $u \in \Phi^{-1}((-\infty, r_n))$, where
\[
r_n = \frac{1}{K} \prod_{i=1}^m \frac{c_n}{p_i}.
\]

Taking into account $\|u_i^0\|_{p_i} = 0$ (where $u_i^0(x) = 0$ for every $x \in [a, b]$) and that $\int_a^b F(t, 0, \ldots, 0) \, dx = 0$ for all $x \in [a, b]$, for every $n$ large enough, one has
\[
\varphi(r_n) = \inf_{u \in \Phi^{-1}((-\infty, r_n))} \left( \sup_{v \in \Phi^{-1}((-\infty, r_n))} \Psi(v) - \Psi(u) \right) = \\
= \sup_{u \in \Phi^{-1}((-\infty, r_n))} \left( \int_a^b F(t, v_1(x), \ldots, v_m(x)) \, dx - \int_a^b F(t, u_1(x), \ldots, u_m(x)) \, dx \right) \leq \\
\leq \frac{\sum_{i=1}^m \frac{|v_i|^{p_i}}{p_i} < r_n}{r_n} \int_a^b F(t, v_1(x), \ldots, v_m(x)) \, dx \leq \\
\leq K \prod_{i=1}^m p_i \liminf_{n \to +\infty} \frac{b}{c_n} \int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(c_n)} F(x, t_1, \ldots, t_m) \, dx.
\]

Therefore, since from assumption (H4) one has $\mathcal{A} < +\infty$, we obtain
\[
\gamma = \liminf_{n \to +\infty} \varphi(r_n) \leq K \prod_{i=1}^m p_i \mathcal{A} < +\infty.
\] (3.2)
A Neumann boundary value problem for a class of gradient systems

Now, fix \( \lambda \in (\lambda_1, \lambda_2) \) and let us verify that the functional \( I_\lambda \) is unbounded from below. Let \( \{ \xi_{i,n} \} \) be positive real sequences such that \( \lim_{n \to +\infty} \sqrt{\sum_{i=1}^{m} \xi_{i,n}^2} = +\infty \), and

\[
\lim_{n \to +\infty} \frac{\int_{a}^{b} F(x, \xi_{1,n}, \ldots, \xi_{m,n}) \, dx}{\sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i}} = B. \tag{3.3}
\]

For each \( n \in \mathbb{N} \) define

\[ w_{i,n}(x) := \xi_{i,n} \]

and put \( w_n := (w_{1,n}, \ldots, w_{m,n}) \).

We easily get that

\[ \| w_{i,n} \|_{p_i} = (b-a)|\xi_{i,n}|^{p_i}. \]

At this point, bearing in mind (i), we infer

\[
\Phi(w_n) - \lambda \Psi(w_n) = \sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda \int_{a}^{b} F(x, \xi_{1,n}, \ldots, \xi_{m,n}) \, dx, \quad n \in \mathbb{N}.
\]

If \( B < +\infty \), let \( \epsilon \in \left( \frac{1}{\lambda B}, 1 \right) \). By (3.3), there exists \( v_\epsilon \) such that

\[
\int_{a}^{b} F(x, \xi_{1,n}, \ldots, \xi_{m,n}) \, dx > \epsilon B \sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.
\]

Moreover,

\[
\Phi(w_n) - \lambda \Psi(w_n) \leq \sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda \epsilon B \sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.
\]

Taking into account the choice of \( \epsilon \), one has

\[
\lim_{n \to +\infty} \left[ \Phi(w_n) - \Psi(w_n) \right] = -\infty.
\]

If \( B = +\infty \), let us consider \( M > \frac{1}{\lambda} \). By (3.3), there exist \( v_m \) such that

\[
\int_{a}^{b} F(x, \xi_{1,n}, \ldots, \xi_{m,n}) \, dx > M \sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_m.
\]

Moreover,

\[
\Phi(w_n) - \lambda \Psi(w_n) \leq \sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i} - \lambda M \sum_{i=1}^{m} \frac{|\xi_{i,n}|^{p_i}}{p_i}, \quad n > v_\epsilon.
\]
Taking into account the choice of $M$, also in this case, one has

\[\lim_{n \to +\infty} [\Phi(w_n) - \Psi(w_n)] = -\infty.\]

Applying Theorem 2.2, we deduce that the functional $\Phi - \lambda \Psi$ admits a sequence of critical points which is unbounded in $X$. Hence, our claim is proved and the conclusion is achieved.

Remark 3.4. If

\[\liminf_{\mu \to +\infty} \int_a^b \sup_{(t_1, \ldots, t_m) \in \Theta(\mu)} F(x, t_1, \ldots, t_m)dx = 0\]

and

\[\limsup_{|t_1| \to +\infty, \ldots, |t_m| \to +\infty} \frac{\int_a^b F(x, t_1, \ldots, t_m)dx}{\sum_{i=1}^m \frac{|t_i|^p}{p_i}} = +\infty,\]

clearly, hypothesis (H4) is verified and Theorem 3.3 guarantees the existence of infinitely many weak solutions for problem $(P_\lambda)$, for every $\lambda \in (0, +\infty)$, the main result ensures the existence of infinitely many weak solutions for problem $(P_\lambda)$.

Acknowledgments

The work is supported by Scientific Research Fund of SUSE (No. 2011KY03) and Scientific Research Fund of SiChuan Provincial Education Department (No. 12ZB081).

REFERENCES


A Neumann boundary value problem for a class of gradient systems


Wen-Wu Pan
23973445@qq.com

Sichuan University of Science and Engineering
Department of Science
Zigong 643000, PR China

Lin Li
lilin420@gmail.com

Southwest University
School of Mathematics and Statistics
Chongqing 400715, PR China

Received: January 16, 2013.
Revised: June 24, 2013.
Accepted: June 28, 2013.